For All Hyperoperations, $2^4 = 4^2$ is The Only Whole Number Pair Exception to their Anticommutativity

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Abstract

All higher-order hyperoperations beyond multiplication are anticommutative, featuring a pair of distinct input values being the base and the power, such as x^y. Using real whole numbers, other than the infinite trivial examples where $x = y$, it has been proven that $2^4 = 4^2$ is the only exception to the anticommutativity property of the hyperoperation exponentiation. This proof shows that for all higher-order hyperoperations, including tetration, pentation, and beyond, that singular exception, $H_3(2, 4) = H_3(4, 2)$, remains the sole example of "anti"-anticommutativity using real whole number inputs.

1 Introduction

The hyperoperation sequence, beginning with the successor function, or zeration, features the most fundamental arithmetic operations in mathematics. The commutative property of binary operations creates a basis for many proofs in mathematics, and with arithmetic functions obligatorily featuring either commutativity or anticommutativity as a property, any exceptions to this quality must be understood before generalizations can be made.

Exponentiation is the third function in the hyperoperation sequence, and it is the first hyperoperation which is anticommutative. Beginning with exponentiation, all higher-order hyperoperations feature a base value and a power value, such as x^y, with each carrying noninterchangeable properties, hence eliminating commutativity. However, outside of infinite, trivial examples when $x = y$, a single pair of whole number values, $(2, 4)$, can be input into either the base value or the power value and have the statement remain true[1]. Considering that $2^4 = 4^2$ is an exception to the anticommutative property of an otherwise anticommutative operation, this could be referred to as "anti"-anticommutative. One method is shown to prove this and is relevant for later generalization.

The hyperoperation sequence continues recursively and infinitely, and as of the time of authorship, the presence of any exceptions to the anticommutative property in tetration, pentation, and beyond has never been explored. Considering the simplicity of the concepts of the commutative property, the motivation for this proof is to better exemplify and understand the uniqueness of common arithmetic operations, and to give greater knowledge of the relatively unexplored hyperoperations of tetration and beyond. The applications of higher-order hyperoperations are more limited, and as a result, their specific qualities are far less explored, so this proof adds to this relatively neglected realm of number theory. For a fundamental property of the most fundamental sequence of operations in mathematics, understanding their commutativity or anticommutativity should be valued for the purposes of pure mathematics.

2 Basis in Exponentiation

The exponentiation equation $x^y = y^x$ can be manipulated by taking the natural logarithm of each side, $ln(x^y) = ln(y^x)$. Then, using the laws of logarithms with exponents and dividing each side by xy results in the following.

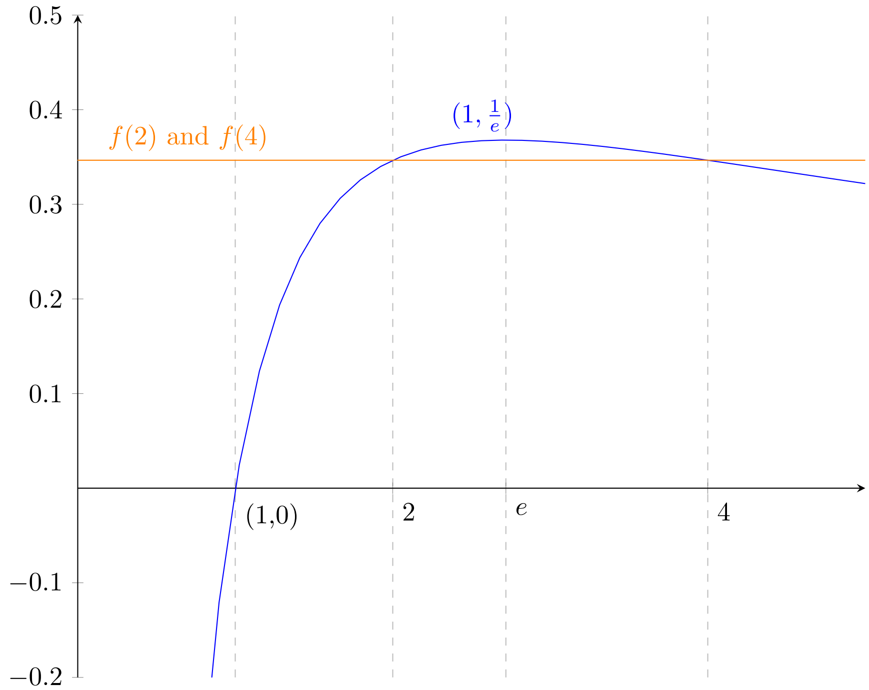
$$ln(x^y) = ln(y^x) \rightarrow yln(x) = xln(y) \rightarrow \frac{yln(x)}{xy} = \frac{xln(y)}{xy} \rightarrow \frac{\cancel{y}ln(x)}{x\cancel{y}} = \frac{\cancel{x}ln(y)}{\cancel{x}y} \rightarrow \frac{ln(x)}{x} = \frac{ln(y)}{y}$$

Both sides of this equation are equivalent other than their single respective variable, so a single function can be observed.

$$f(x) = \frac{ln(x)}{x}$$

With this function, if any two distinct whole number values for x result in the same whole number value for $f(x)$, then those two values could be input into the original equation, $x^y = y^x$ and a solution pair (x, y) would be found[1]. The function in Figure 1 begins with the interval $(0, e)$ with an asymptote from $-\infty$ and is strictly increasing to e. The function is strictly decreasing in the interval (e, ∞) with an asymptote approaching 0. The function crosses the x-axis at $(1, 0)$ and its critical point is at $(1, e)$.

Figure 1. Plot of $f(x) = \frac{ln(x)}{x}$



Given the shape of this function, for a whole number pair of inputs to be eligible as an anti-anticommutative exception, two $f(x)$ value outputs would need to be on each side of the critical point[2]. The only whole numbers within the interval $(0, e)$ are 1 and 2. The value for $f(1) = 0$, and while zero is a whole number, $f(0)$ is undefined and therefore not equal to 1, so this pair cannot satisfy the equation. Therefore, the only eligible x value on the to the left of the critical point is 2.

$$f(2) = \frac{ln(2)}{2} \approx 0.34657359028$$

Following the horizontal line of $f(2)$ reveals the corresponding intersection beyond the critical point when $x = 4$. Inputting these values for the variables x and y,

$$x^y = y^x \rightarrow 2^4 = 4^2 \rightarrow 16 = 16$$

As the interval beyond 4 is strictly descending, no whole number input greater than 4 could have an equal value to $f(2)$. Thus, the only real, whole number pair solution for the original equation $x^y = y^x$ is $(2, 4)$.

3 Extension of Proof to Tetration

Higher-order hyperoperations beyond exponentiation have multiple notations, without a unified, standard style. For simplicity across various higher-order hyperoperations, the notation used will be $H_n(x, y)$ with x being the base value, y being the power value, and n being the placement in the hyperoperation sequence. For instance, H_2 represents multiplication, H_3 represents exponentiation, which is repeated iterations of multiplication, while H_4 represents tetration, a higher-order hyperoperation of repeated iterations of exponentiation[4]. Demonstrating the anticommutativity of tetration,

$$H_4(2,4) = 2^{2^{2^2}} = 2^{2^4} = 2^{16} = 65536, \text{ and its inverse, } H_4(4,2) = 4^4 = 256$$

The equation $H_4(x, y) = H_4(y, x)$ features tetrational superpowers in the place of exponential powers. The superpower of 4 indicates the number of repetitions that the exponentiation of 2 is to be applied to the base of same value. The superlogarithm function, which is defined as $slog_x z = y$ if and only if $z = H_4(x, y)$, is an inverse operation to tetration in the same manner as logarithms to exponentiation, representing repeated iterations of a logarithm until a result of 1 is obtained[3]. When the superlogarithm has a base of e, a super-natural-logarithm, $sln(x)$, and is applied to the equation $H_4(x, y) = H_4(y, x)$, the resultant equation is found.

$$sln(H_4(x, y)) = sln(H_4(y, x))$$

By definition, superlogarithms feature comparable algebraic properties with tetration as compared with logarithms with exponentiation[6]. Using the same type of manipulation in this equation produces a similarly useful equation as the previous proof in exponentiation.

$$sln(H_4(x, y)) = sln(H_4(y, x)) \to ysln(x) = xsln(y) \to$$

$$\frac{ysln(x)}{xy} = \frac{xsln(y)}{xy} \to \frac{\cancel{y}sln(x)}{x\cancel{y}} = \frac{\cancel{x}sln(y)}{\cancel{x}y} \to \frac{sln(x)}{x} = \frac{sln(y)}{y}$$

This equation is again equivalent on both sides, so a singular function can be utilized to find two x value inputs with identical $f(x)$ outputs.

$$f(x) = \frac{sln(x)}{x}$$

When observing this function, there are unapproximated values at key points, with all examples greater than 1 being at whole number superpowers of e, shown in Table 1.

Table 1. Key values of $f(x) = \frac{sln(x)}{x}$	
x	$f(x)$
0	Undefined
1	e
$H_4(e,1) = e$	$\frac{1}{e}$
$H_4(e,2) = e^e$	$\frac{2}{e^e}$
$H_4(e,3) = e^{e^e}$	$\frac{3}{e^{e^e}}$
$H_4(e,4) = e^{e^{e^e}}$	$\frac{4}{e^{e^{e^e}}}$
...	...

Notably, this function features a strikingly similar shape as the lower-order version seen in Figure 1. The critical point is again found at $(e, \frac{1}{e})$, following the same strictly increasing interval of $(0, e)$ and strictly decreasing interval of (e, ∞) asymptomatically approaching zero. Therefore, the only whole numbers to the left of the critical point are again 1 and 2. Considering that again $f(1) = 0$, an x value of 1 is not eligible for pairing as $f(0)$ is again undefined and not equal to 1. Therefore, 2 is the only eligible whole number less than e to pair with a solution for the original equation.

The value of the super-natural-logarithm of 2 calculated to ten decimal places is 0.7015456018. Entering this value into the function,

$$f(2) = \frac{sln(2)}{2} \approx \frac{0.7015456018}{2} \approx 0.3507728009$$

To identify another x value input beyond the critical point in the interval (e, ∞) with an equal $f(x)$ value output to $f(2)$, we can compare to the unapproximated values of the function $f(x) = \frac{sln(x)}{x}$. In the interval beyond the critical point at $f(e)$, the value of $f(2)$ is greater in value than $f(e^e)$, as

$\frac{1}{e} > \frac{sln(2)}{2} > \frac{2}{e^e}$, or their approximate values, $0.3678794412 > 0.3507728009 > 0.1319760717$

With this knowledge, the x value input to be paired with 2 are the whole numbers between e and e^e, which are 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, and 15. Therefore, these are the values that could plausibly be matched with 2 to establish a pair (x, y) to satisfy $H_4(x, y) = H_4(y, x)$. Using $H_4(x, y) = H_4(y, x)$, we can observe their outputs as pairs.

Table 2: Values for $H_4(x, y) = H_4(y, x)$

$H_4(2, y)$	$f(H_4(2, y))$	$H_4(x, 2)$	$f(H_4(x, 2))$
$H_4(2, 3)$	16	$H_4(3, 2)$	27
$H_4(2, 4)$	$65,536$	$H_4(4, 2)$	256
$H_4(2, 5)$	$2.00352993\text{x}10^{19,728}$	$H_4(5, 2)$	$3,125$
$H_4(2, 6)$	$> 10^{60^{19,727}}$	$H_4(6, 2)$	$46,656$
$H_4(2, 7)$	(exceedingly large)	$H_4(7, 2)$	$823,543$
$H_4(2, 8)$	$\ggg$	$H_4(8, 2)$	$16,777,216$
$H_4(2, 9)$	$\ggg$	$H_4(9, 2)$	$387,420,489$
$H_4(2, 10)$	$\ggg$	$H_4(10, 2)$	$10,000,000,000$
$H_4(2, 11)$	$\ggg$	$H_4(11, 2)$	$285,311,670,611$
$H_4(2, 12)$	$\ggg$	$H_4(12, 2)$	$8,916,100,448,256$
$H_4(2, 13)$	$\ggg$	$H_4(13, 2)$	$302,875,106,592,253$
$H_4(2, 14)$	$\ggg$	$H_4(14, 2)$	$1.11120068\text{x}10^{16}$
$H_4(2, 15)$	$\ggg$	$H_4(15, 2)$	$4.37893890\text{x}10^{17}$

With the given inputs for the equation $H_4(x, y) = H_4(y, x)$, many of these values can be calculated, but the sheer enormity of values produced by tetration is quickly noted. However, this is not a limiting obstacle towards this proof, as all values of $f(H_4(x, 2))$ up to and including $x = 15$ do not exceed some computational limit. When compared to relevant values that can be calculated of $f(H_4(2, x))$, it is possible to calculate greater than $f(H_4(2, 5)$. Importantly, this serves the proof conveniently well, as $H_4(2, 4) < H_4(15, 2) < H_4(2, 5)$, so no further computations are required for this rapidly growing function.

When there is a base of 2 with an incrementing superpower, the value explosively exceeds the value of the inverse using a variable base and a superpower of 2. The only exception to the this is at the early values of $H_4(2, 3) < H_4(3, 2)$. This can also be noted in the values of the function $f(x) = \frac{sln(x)}{x}$ with $\frac{sln(3)}{3}$ and $\frac{sln(4)}{4}$ calculated to ten digits at 0.36274971204 and 0.33023644672, respectively. This shows that $f(4) > f(2) > f(3)$. This is reflected in Table 2, which will be valuable later in the generalization of the proof to all higher-order hyperoperations.

As the Table 2 also shows, no values are shared between the two columns. Considering that this set of numbers has been shown to include all eligible values of x and y for this purpose, this exhaustively shows that a pair of real, whole number values cannot satisfy the equation $H_4(x, y) = H_4(y, x)$ when $x \neq y$, and therefore tetration is an anticommutative operation without an anti-anticommutative exception.

4 Generalization to Higher-Order Hyperoperations

With all higher-order hyperoperations beyond tetration, this method can be generalized. Every hyperoperation will feature a higher-order logarithm equivalent beyond superlogarithms with the same algebraic qualities[5].

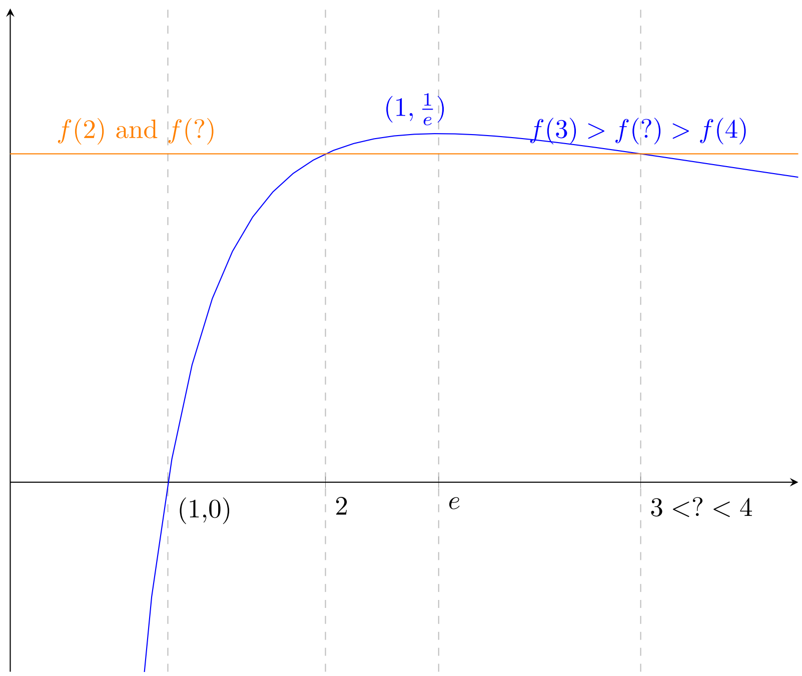
$$hyper_n log_x z = y$$

With the same algebraic manipulations being available and using e as a base for all hyperlogarithms, a useful family of functions can be identified,

$$f(x) = \frac{hyper_n ln(x)}{x}$$

As with the prior example in tetration, this family of functions will therefore always feature a critical point at $(e, \frac{1}{e})$ as well as crossing the x-axis at $(1, 0)$, along with the same asymptotes present. Secondary to this continuation of the general shape for this family of functions, 2 is once again obligatorily one of the pairing values if an anti-anticommutative pair were to be found.

Figure 2. Generalized plot of $f(x) = \frac{hyper_n ln(x)}{x}$, $n > 3$



Using this generalized function and the the horizontal line of $f(2)$ to find a corresponding intersection beyond the critical point, whole number inputs for x which are greater than e can be assessed for their equivalence. As discussed above in H_3 and H_4, using base and power inputs of 2 and 3, respectively, always results in the inequality of $f(2) < f(3)$, and in general, when $n \geq 3$, then $H_n(2, 3) < H_n(3, 2)$. For hyperoperations 1 through 6, addition through hexation, example values are shown below. The symbols $\ll$ and $\gg$ will be used in the place of $<$ and $>$ when a higher-order hyperoperation is required to even reasonably describe one of the values in comparison to the other.

Table 3: Values for $H_n(2, 3) = H_n(3, 2)$

n	$f(n) = H_n(2, 3)$	$\stackrel{?}{=}$	$f(n) = H_n(3, 2)$
1	5	$=$	5
2	6	$=$	6
3	8	$<$	9
4	16	$<$	27
5	$65,536$	$<$	$7,625,597,484,987$
6	$H_4(2, 65536)$	$\ll$	$H_5(3, 7625597484987)$
...	...	$\ll$	...

As shown above in Table 3, the values are equal in commutative operations of addition and multiplication, but beyond this, beginning with exponentiation, the values with a base of 3 become explosively larger than those with a base of 2. This holds true in all further hyperoperations, with the gap between only growing more extreme. Despite the stronger power value in $H_n(2, 3)$, this is offset by the weakness of 2 as a base value. When 2 and 4 are paired together, the following values are observed in Table 4.

Table 4: Values for $H_n(2,4) = H_n(4,2)$			
n	$f(n) = H_n(2,4)$	$\stackrel{?}{=}$	$f(n) = H_n(4,2)$
1	6	$=$	6
2	8	$=$	8
3	16	$=$	16
4	65,536	$>$	256
5	$H_4(2, 65536)$	$\gg$	$4^{4^{256}}$
6	$H_5(2, 65536)$	$\gg$	$H_4(4, H_4(4, (H_4(4,4))))$
...	...	$\gg$	...

In this pairing, the only anti-anticommutative exception is observed, $H_3(2,4) = H_3(4,2)$. Other than that exception, the opposite inequalities are noted compared to Table 3, as when hyperoperations have a base value of 2 and a power value of 4, the heightened power value allows for repeated iterations of those operations and leads to explosive growth far beyond the inverse value inputs, which is limited to only a single iteration of that highest-order hyperoperation. Even with a weaker base value of 2, the final output explosively eclipses of the inverse. This is seen with the shape of general shape of Figure 2, as the $H_n(4,2)$ values fall below the line of $f(2)$.

Once either pairing value reaches 5, the ultimate pattern of hyperoperation inputs is demonstrated.

Table 5: Values for $H_n(2,5) = H_n(5,2)$			
n	$f(n) = H_n(2,5)$	$\stackrel{?}{=}$	$f(n) = H_n(5,2)$
1	7	$=$	7
2	10	$=$	10
3	32	$>$	25
4	$2^{65,536}$	$\gg$	3125
5	$H_4(H_4(H_4(H_4(H_4(2,2)))))$	$\gg$	$5^{5^{3125}}$
6	$H_5(H_5(H_5(H_5(H_5(2,2)))))$	$\gg$	$H_4(5,5)$
...	...	$\gg$	...

From this point, all higher-order hyperoperations feature the same inequality relationships regardless of the size of the pairing value with 2. Again, the growth of the function of $f(n) = H_n(2,y)$ is explosively larger given that many more iterations of the hyperoperation are applied compared to the inverse. Table 6 highlights the inequality relationships.

Table 6: Inequalities for $H_n(x,y) \stackrel{?}{=} H_n(y,x)$			
Tetration, H_4	Pentation, H_5	Hexation, H_6	All higher-order H_n
$H_4(2,3) < H_4(3,2)$	$H_5(2,3) < H_5(3,2)$	$H_6(2,3) \ll H_6(3,2)$	$H_n(2,3) \ll H_n(3,2)$
$H_4(2,4) > H_4(4,2)$	$H_5(2,4) \gg H_5(4,2)$	$H_6(2,4) \gg H_6(4,2)$	$H_n(2,4) \gg H_n(4,2)$
$H_4(2,5) \gg H_4(5,2)$	$H_5(2,5) \gg H_5(5,2)$	$H_6(2,5) \gg H_6(5,2)$	$H_n(2,5) \gg H_n(5,2)$
$H_4(2,6) \gg H_4(6,2)$	$H_5(2,6) \gg H_5(6,2)$	$H_6(2,6) \gg H_6(6,2)$	$H_n(2,6) \gg H_n(6,2)$
$H_4(2,y) \gg H_4(x,2)$	$H_5(2,y) \gg H_5(x,2)$	$H_6(2,y) \gg H_6(x,2)$	$H_n(2,y) \gg H_n(x,2)$

This relationship can be generalized and summarized with the following statements:

$$\text{If } n \geq 3\text{, then } H_n(2,3) < H_n(3,2).$$

$$\text{If } n \geq 4 \text{ and } x \geq 4\text{, then } H_n(2,x) > H_n(x,2).$$

The gap of these inequalities exclusively increases as n and/or y increases. As no whole number exists between 3 and 4, a hyperoperation beyond H_3 cannot have an equality using whole number inputs. Therefore, the following is the sole anti-anticommutative exception in the hyperoperation sequence.

$$H_3(2,4) = H_3(4,2) \rightarrow 2^4 = 4^2 \rightarrow 16 = 16$$

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