# The missing rotation angle of Rodrigues' Rotation Formula

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## Abstract

Rodrigues' Rotation Formula is used to rotate a vector based on the Axis-Angle parameterization of attitude transformation. Given an axis and angle about which an initial vector is rotated, Rodrigues' Formula yields the final orientation of the vector. However, Rodrigues' Formula does not include the associated rotation about the vector axis, which depends on the specific trajectory used in slewing the axis from its initial to final orientation. A theorem in attitude kinematics that was not available when Rodrigues developed his formula, contains attitude components that should be included with Rodrigues' Formula. These attitude components were previously applied to the related problem of the Foucault Pendulum, which rotates with the Earth about the Earth's spin axis. The methods to compute the rotation angle about the vector axis are the same as those used to compute the rotation of the Foucault Pendulum's mounting fixture, since the trajectories are similar. This work presents the derivation of the formula for the missing rotation angle of Rodrigues's Formula. Several numerical examples are presented to illustrate the use of the formula.

Keywords: Rodrigues' Rotation Formula, Axis-Angle, rotational transformation, slewing transformation, solid angle

## Nomenclature

| | |
|---|---|
| **A** | initial axis orientation |
| **a** | pivot vector |
| **B** | final axis orientation |
| **b** | pivot vector |
| c | cos(λ)-1 |
| **d** | unit vector normal to geocentric arc |
| **e** | axis of rotation of Rodrigues' Formula |
| **f, h** | auxiliary unit vectors |
| MF | mounting fixture |
| s | sin(λ) |
| **U** | total transformation |
| $\mathbf{U_S}$ | slewing transformation |
| $\mathbf{U_R}$ | rotational transformation |
| **V** | original vector before rotation |
| $\mathbf{V_M}$ | mid orientation between **V** and $\mathbf{V_R}$ |
| $\mathbf{V_R}$ | final vector after rotation |

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|---|---|
| $\beta$ | auxiliary angle |
| $\delta_i$ | external rotation angles |
| $\Gamma$ | auxiliary angle |
| $\lambda$ | rotation angle about Rodrigues' axis |
| $\psi$ | desired rotation angle |
| $\Omega$ | solid angle |
| $\Theta$ | auxiliary angle |

## 1. Introduction

Euler's Rotation Theorem (Goldstein, 1950) states that a rigid body with a single point held fixed and free to rotate in any manner can be brought from orientation P to orientation Q by a rotation about a fixed axis that passes through the fixed point. Euler's Rotation Theorem is the basis for the Axis-Angle parameterization of attitude transformation, which is the starting point for many other attitude parameterizations. Recent work (Patera, 2020) showed that an attitude transformation can be achieved using the slewing and rotational transformations of a single axis. Therefore, the axis defining an attitude transformation can be slewing instead of being fixed, as in Euler's Rotation Theorem. However, in addition to the slewing motion, a rotation about the axis is required to define the attitude transformation.

Rodrigues' Rotation Formula uses an axis of fixed orientation and rotation angle about the axis to rotate a vector from its initial to final orientation. The associated Direction Cosine Matrix, DCM, is obtained by using Rodrigues' Formula on all three axes of a coordinate system. The DCM contains the complete attitude transformation information. However, the rotation of a single vector using Rodrigues' Formula or the DCM does not contain the complete attitude information, which requires knowledge of the vector axis trajectory. There are an infinite number of trajectories that can be used to take the vector from its initial to final orientation. Rodrigues' Formula itself doesn't contain any information about the vector's trajectory. Each trajectory acquires its own rotation angle about the vector axis during the slewing motion. Nevertheless, Rodrigues' Formula implies a specific trajectory that maintains a constant separation angle between the rotation axis and the vector axis. This trajectory is used to compute the resulting rotation angle about the vector axis. The desired rotation angle about the vector axis depends on the rotation angle about Rodrigues' axis and the separation angle between Rodrigues' axis and the vector axis.

Section 2 provides two methods used to derive the missing rotation angle about the vector axis. Section 3 contains some numerical results that validate the findings based on similar results from the Foucault Pendulum's mounting fixture. Section 4 contains the conclusion of the work.

## 2. Derivation of the missing rotation angle in Rodrigues' Rotation Formula

When a vector with one point fixed slews from one orientation to another, there are three components to the attitude transformation that should be considered based on a recent theorem in kinematics (Patera, 2020). The first component is the transformation associated with the geocentric arc linking the initial and final vector orientations. The second component is the rotational component associated with the solid angle enclosed by the actual trajectory and the geocentric arc trajectory linking the initial and final vector axis orientations. The third component is the integral of the projection of the angular rate along the slewing vector axis throughout its slewing motion.

Rodrigues' Rotation Formula, shown in eq. (1), provides the final vector, $\mathbf{V_R}$ , when given an initial vector, **V,** the unit rotation axis, **e**, and the rotation angle, λ. Note that the magnitudes of **V** and $\mathbf{V_R}$ are the same.

$$\mathbf{V_R} = \mathbf{V}\cos(\lambda) + (\mathbf{e}\times\mathbf{V})\sin(\lambda) + \mathbf{e}\,(\mathbf{e}\cdot\mathbf{V})\,[1-\cos(\lambda)] \tag{1}$$

Rodrigues' Formula, shown in eq. (1), neglects the rotational transformation components about the rotated vector's axis that should be applied to the vector at its final orientation. Many derivations of Rodrigues' Rotation Formula assume that the component of the rotated vector parallel to the rotation axis is invariant and does not rotate (Kuipers, 1999). Nevertheless, the vector axis does rotate according to a recent kinematic theorem (Patera, 2020). For the component of the rotated vector parallel to the rotation axis, **e**, the geocentric arc linking the initial to final orientation is zero, since the vector doesn't change its orientation. There is no rotation about the vector axis due to the associated enclosed solid angle, which is zero. However, the integral of the projection of the angular rate along the slewing vector axis throughout its slewing motion is λ, since the vector axis aligns with the rotation axis. Therefore, the component of the rotated vector parallel to the rotation axis rotates about its axis by angle λ.

Two methods are used to obtain the required rotation angle. The first method is to combine the total transformation, which is a rotation of λ about the axis, **e**, with the inverse of the geocentric arc transformation. The resulting transformation must be the rotational transformation about the vector axis. Therefore, the Euler Rotation Vector of the combined transformation aligns with the final orientation of the rotated vector and the Euler Rotation Vector magnitude is the desired rotation angle.

The second method to determine the rotation angle uses an analytical equation to determine the solid angle enclosed by the actual trajectory of the vector axis and the geocentric arc trajectory. The solid angle is computed based on the method of perimeter points (Patera, 2023). The rotation angle associated with the solid angle is combined with the component of the integral of the angular rate along the vector axis throughout its slewing motion to obtain the desired angle.

## 2a. Attitude Transformation Method to find the missing angle

The total attitude transformation is the product of the geocentric arc linking the initial and final orientations of the vector, the solid angle of the region bounded by the actual trajectory and the geocentric arc trajectory, and the integral of the component of the angular rate along the vector axis throughout the transformation (Patera, 2020). The first component is the geocentric arc slewing transformation, $\mathbf{U_S}$, and the last two components combine into a rotational transformation about the vector axis, $\mathbf{U_R}$, as shown in eq. (2). Note that Rodrigues' Formula represents the $\mathbf{U_S}$ transformation and does not include the $\mathbf{U_R}$ transformation in eq. (2).

$$\mathbf{U}(\mathbf{e},\lambda) = \mathbf{U_R}\,\mathbf{U_S} \tag{2}$$

The total transformation is the DCM provided in eq. (3), where C = cos(λ) – 1, S = sin(λ).

$$\mathbf{U}(\mathbf{e},\boldsymbol{\lambda}) = \begin{pmatrix} 1+(\mathbf{e_z}^2+\mathbf{e_y}^2)\mathrm{C} & -(\mathbf{e_x}\mathbf{e_y}\mathrm{C}+\mathbf{e_z}\mathrm{S}) & \mathbf{e_y}\mathrm{S}-\mathbf{e_x}\mathbf{e_z}\mathrm{C} \\ \mathbf{e_z}\mathrm{S}-\mathbf{e_x}\mathbf{e_y}\mathrm{C} & 1+(\mathbf{e_x}^2+\mathbf{e_z}^2)\mathrm{C} & -(\mathbf{e_z}\mathbf{e_y}\mathrm{C}+\mathbf{e_x}\mathrm{S}) \\ -(\mathbf{e_z}\mathbf{e_x}\mathrm{C}+\mathbf{e_y}\mathrm{S}) & \mathbf{e_x}\mathrm{S}-\mathbf{e_y}\mathbf{e_z}\mathrm{C} & 1+(\mathbf{e_x}^2+\mathbf{e_y}^2)\mathrm{C} \end{pmatrix} \tag{3}$$

The rotational transformation is obtained by multiplying eq. (2) with the inverse of the slewing transformation, as shown in eq. (4).

$$\mathbf{U_R} = \mathbf{U}(\mathbf{e},\lambda)\,\mathbf{U_S^{-1}} \tag{4}$$

The slewing transformation is obtained using the Pivot Vector Method, PVM (Patera, 2017), as shown in eq. (5), where $\mathbf{V_M}$ is given by eq. (6). The transformation in eq. (5) is obtained by rotation angles of π radians about the axis of each Pivot Vector (Patera, 2017). The rotational transformation is shown in eq. (7), where the inverse of $\mathbf{U_S}$ is found by simply reversing the order of the Pivot Vectors. Therefore, the rotation angle, ψ, is obtained as the magnitude of the Euler Rotation Vector associated with rotation matrix, $\mathbf{U_R}$, as shown in eq. (8), where trace is the sum of the diagonal elements of $\mathbf{U_R}$.

$$\mathbf{U_S} = \mathbf{U}(\mathbf{V_M}, \pi)\, \mathbf{U}(\mathbf{V}, \pi) \tag{5}$$

$$\mathbf{V_M} = \frac{(\mathbf{V}+\mathbf{V_R})}{[\mathbf{V}+\mathbf{V_R}]} \tag{6}$$

$$\mathbf{U_R} = \mathbf{U}(\mathbf{e}, \lambda)\, \mathbf{U}(\mathbf{V}, \pi)\, \mathbf{U}(\mathbf{V_M}, \pi) \tag{7}$$

$$\psi = \cos^{-1}\left(\frac{\text{trace}-1}{2}\right) \tag{8}$$

## 2b. Analytic Method to find the missing angle

The analytical method to find the rotation angle about the axis of a rotated vector is based on previous work related to the Foucault Pendulum rotation and its mounting fixture rotation (Patera, 2022). It was shown (Patera, 2020) that the solid angle of any regular spherical polygon of n sides is given by eq. (9), where $\delta_i$ are the exterior rotation angles about the vertices made by adjoining adjacent planes. Each of the n sides is a segment of a great circle arc.

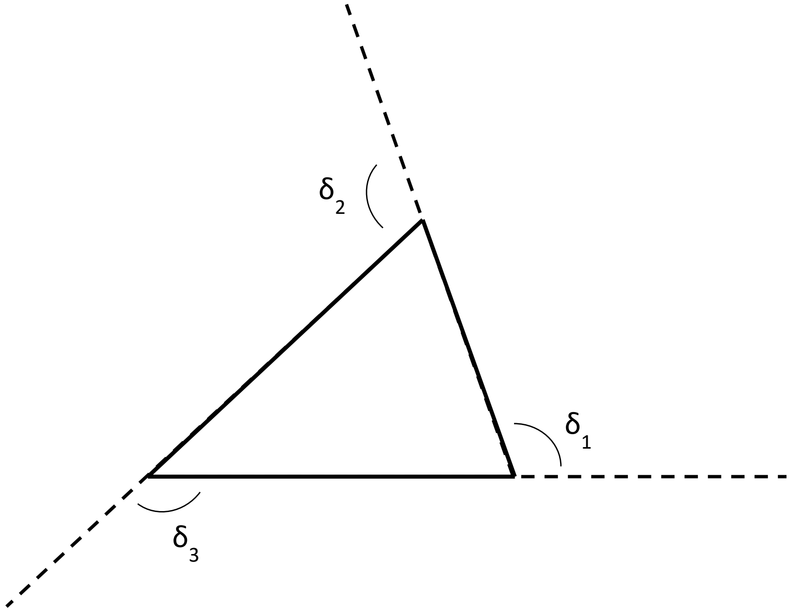

Fig. 1. Exterior angles of a spherical triangle as illustrated on a plane.

Fig. 1 illustrates the case for a spherical triangle as represented on a two dimensional plane. The sum of the exterior angles can be viewed as the rotation of an axis, as the axis moves about the spherical triangle.

$$\Omega = 2\pi - \sum_{i=1}^{n} \delta_i \tag{9}$$

Eq. (9) can be generalized to also include a region of continuous rotation of the axis as it slews about a section of the solid angle region, as shown in eq. (10), where **A** and **B** define the region of continuous rotation.

$$\Omega = 2\pi - \sum_{i=1}^{n} \delta_i - \int_A^B d\delta \tag{10}$$

Fig. 2 illustrates the slewing trajectory of the axis in a plane normal to the rotation axis, as well as, the geocentric arc linking **V** and **V<sub>R</sub>**. The enclosed area on the spherical surface defines the solid angle component of rotation about the slewed axis. The solid circular line defines the plane normal to the rotation axis. The trajectory of the rotated axis starts at point **A** and moves to point **B** along the circular arc indicated by the dashed line. The other dashed line from **B** back to **A** is the geocentric arc trajectory, which is used to define the solid angle region.

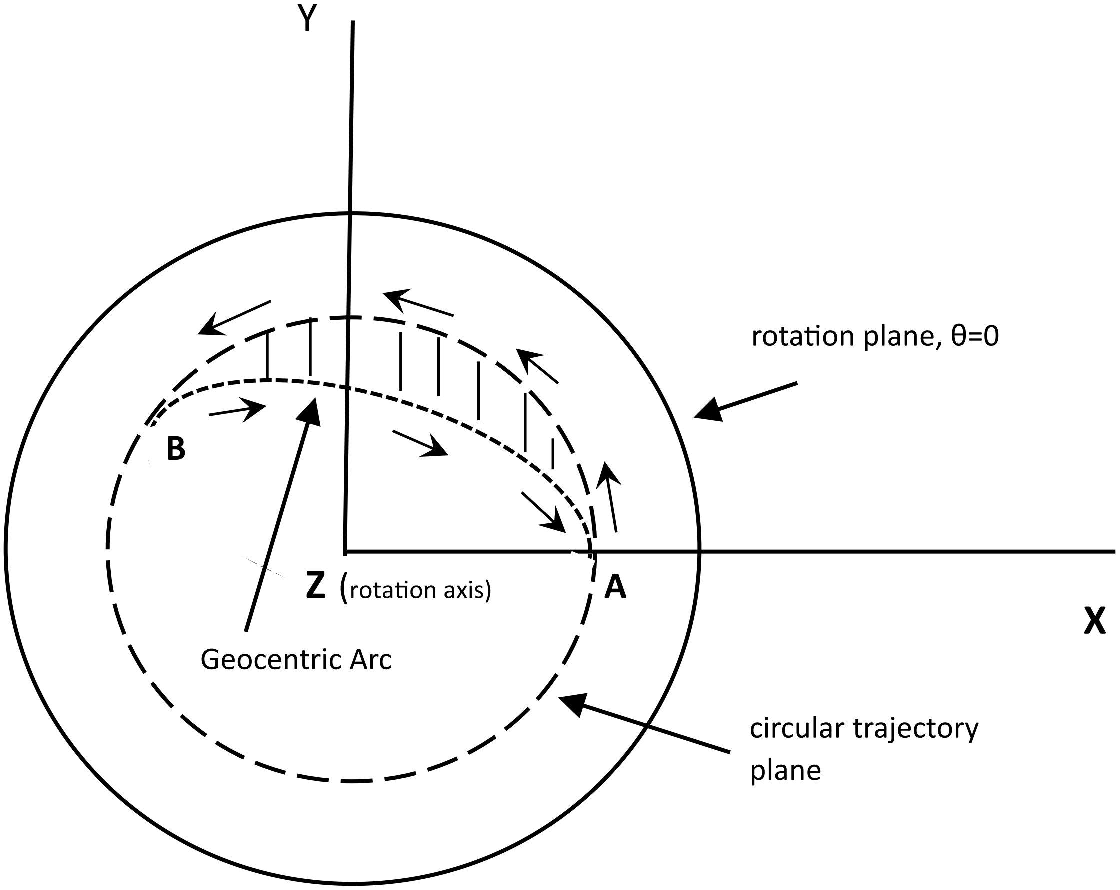

Fig. 2. The solid angle component of rotation about the slewing axis is equal to the solid angle bounded by the circular and geocentric arc trajectories linking points **A** and **B**.

Fig. 3 defines the relevant geometry that is used in eq. (10), where the true spherical geometry is represented on a flat plane for clarity. The unit vectors **A** and **B** are aligned with the vectors **V** and **V<sub>R</sub>**, respectively, and are defined in eq. (11), where Θ is the compliment of the angle between **V** and the axis of rotation and λ is the rotation of **V** about the rotation axis.

$$\mathbf{V} = \begin{pmatrix} \cos(\Theta) \\ 0 \\ \sin(\Theta) \end{pmatrix} \qquad \mathbf{V_R} = \begin{pmatrix} \cos(\Theta)\cos(\lambda) \\ \cos(\Theta)\sin(\lambda) \\ \sin(\Theta) \end{pmatrix} \tag{11}$$

The unit vector **h** is normal to axis **B** and is defined by eq. (12). The unit vector **f** is also normal to axis **B** and is defined in eq. (13).

$$\mathbf{h} = \mathbf{z} \times \mathbf{B} = \begin{bmatrix} -\sin(\lambda) \\ \cos(\lambda) \\ 0 \end{bmatrix} \tag{12}$$

$$\mathbf{f} = \mathbf{h} \times \mathbf{B} = \begin{bmatrix} \sin(\Theta)\cos(\lambda) \\ \sin(\Theta)\sin(\lambda) \\ -\cos(\Theta) \end{bmatrix} \tag{13}$$

The unit vector **d** is normal to the plane that defines the great circle arc connecting **A** and **B**, and is defined by eq. (14). The magnitude of **d** is given by eq. (15).

$$\mathbf{d} = \frac{\mathbf{B}\times\mathbf{A}}{|\mathbf{B}\times\mathbf{A}|} = \left[\frac{\cos(\Theta)}{|\mathbf{d}|}\right] \begin{bmatrix} \sin(\Theta)\sin(\lambda) \\ \sin(\Theta) - \sin(\Theta)\cos(\lambda) \\ -\cos(\Theta)\sin(\lambda) \end{bmatrix} \tag{14}$$

$$|\mathbf{d}| = \cos(\Theta)\sqrt{\sin^2(\lambda) + \sin^2(\Theta)[\cos(\lambda) - 1]^2} \tag{15}$$

The rotation of the axis as it moves around the solid angle region is the sum of the rotations at orientation **B** and at orientation **A** plus the continuous rotation as the axis slews from **A** to **B** along the circular trajectory. The rotation of the axis at orientation **B** is found by first computing the angle between **f** and **d**, which is given by eq. (16), where G is given by eq. (17). The rotation angle at orientation **B** is Γ, which is related to β via eq. (18), as illustrated in Fig. 3.

$$\beta = \cos^{-1}(\mathbf{f}\cdot\mathbf{d}) = \cos^{-1}\left[\frac{\sin(\lambda)}{G}\right] \tag{16}$$

$$G = \sqrt{\sin^2(\lambda) + \sin^2(\Theta)[\cos(\lambda) - 1]^2} \tag{17}$$

$$\Gamma = \pi - \beta \tag{18}$$

Due to symmetry, the rotation of the axis at orientation **A** is the same as the rotation at orientation **B**. The rotation due to the continuous rotation as the axis moves from orientation **A** to orientation **B** is given by λ sin(Θ). Therefore using eq. (10) and eq. (18), the solid angle of the enclosed region is given by eq. (19).

$$\Omega = 2(\pi - \Gamma) - \lambda\sin(\Theta) = 2\beta - \lambda\sin(\theta) \tag{19}$$

The desired rotation angle is given by the solid angle plus the integral of the projection of the angular rate along the slewing axis throughout the slewing motion, as given in eq. (20).

$$\psi = \Omega + \lambda\sin(\theta) = 2\beta - \lambda\sin(\theta) + \lambda\sin(\theta) = 2\beta \tag{20}$$

The sin(λ)/G term in eq. (16) can be simplified using eqs. (21) and eq. (22) which were obtained by using the double angle trigonometric formulas.

$$\sin(\lambda) = 2\cos\left(\frac{\lambda}{2}\right)\sqrt{1 - cos^2\left(\frac{\lambda}{2}\right)} \tag{21}$$

$$\cos(\lambda) = 2\cos^2\left(\frac{\lambda}{2}\right) - 1 \tag{22}$$

Eq. (23) is found by computing the dot product of the axis unit vector, **e**, and unitized rotation vector, **V**.

$$\sin(\theta) = \left(\frac{\mathbf{e}\cdot\mathbf{V}}{[\mathbf{V}]}\right) \tag{23}$$

Using eqs. (21-23) in eqs. (16) and (17), one finds eq. (24) after algebraic simplification.

$$\cos(\beta) = \frac{\cos\left(\frac{\lambda}{2}\right)}{\sqrt{\cos^2\left(\frac{\lambda}{2}\right)+\left(\frac{\mathbf{e}\cdot\mathbf{V}}{[\mathbf{V}]}\right)\left(1-\cos^2\left(\frac{\lambda}{2}\right)\right)}} \tag{24}$$

Therefore, the desired rotation angle about $\mathbf{V_R}$ is found by using eq.(20), eq. (23) and eq. (24) to obtain eq. (25). Eq. (25) is the desired missing rotation angle of Rodrigues' Formula.

$$\psi = 2\cos^{-1}\left[\frac{\cos\left(\frac{\lambda}{2}\right)}{\sqrt{\cos^2\left(\frac{\lambda}{2}\right)+\left(\frac{\mathbf{e}\cdot\mathbf{V}}{[\mathbf{V}]}\right)\left(1-\cos^2\left(\frac{\lambda}{2}\right)\right)}}\right] = 2\cos^{-1}\left[\frac{\cos\left(\frac{\lambda}{2}\right)}{\sqrt{\cos^2\left(\frac{\lambda}{2}\right)+\sin(\theta)\left(1-\cos^2\left(\frac{\lambda}{2}\right)\right)}}\right] \tag{25}$$

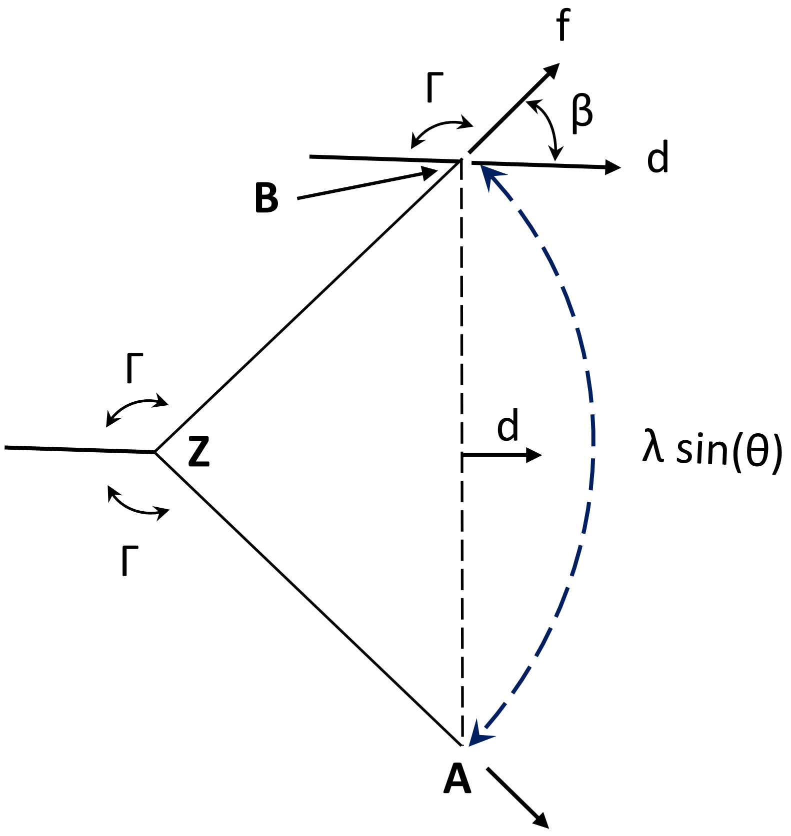

Fig. 3. Geometry related to eqs. (10 - 19) where the solid angle is bounded by the dashed lines.

## 3. Numerical Results

Various values of θ and λ were used in eq. (25) to obtain the associated rotation angles, as shown in Table 1. Results were validated using the attitude transformation method from eq. (8) to ensure that the analytical solution was correct. For a fixed value of θ, the rotation of the vector along its vector axis is a nonlinear function of λ. When λ equals 180 and 360 degrees, the corresponding rotation angles are also 180 and 360 degrees, respectively.

The values in Table 1 are identical of those for a Foucault Pendulum's mounting fixture located at latitude, θ, for an Earth rotation angle of λ, (Patera, 2022). This is because the kinematics of Rodrigues' slewing vector axis is analogous to the slewing of the axis of the mounting fixture. This agreement provides further validation of the results.

Table 1

Rotation Angle Examples for Algorithm Validation

| Rotation angle, λ, deg. | Vector angle, θ, 30 deg. | Vector angle, θ, 45 deg. | Vector angle, θ, 60 deg. |
|---|---|---|---|
| 45 | 23.4 | 32.65 | 39.47 |
| 90 | 53.13 | 70.53 | 81.79 |
| 135 | 100.72 | 119.28 | 128.88 |
| 180 | 180 | 180 | 180 |
| 225 | 259.28 | 240.72 | 231.12 |
| 270 | 306.87 | 289.47 | 278.21 |
| 315 | 336.60 | 327.35 | 320.53 |
| 345 | 352.47 | 349.36 | 346.99 |
| 360 | 360 | 360 | 360 |

Eq. (25) can be used find the rotation angle about the vector's axis when it is aligned with Rodrigues' axis. In this case, θ = 90 degrees and ψ = λ, as expected. If the vector is normal to the axis, θ = 0 and λ is less than 180 degrees, ψ = 0. If θ = 0 and λ is greater than 180 degrees, ψ = 360 degrees. Fig. 4 illustrates values of ψ for various values of θ and λ. Each curve in Fig. 4 represents a unique value of λ, which is shown to the right of the respective curve. Notice that ψ = λ when θ = 90 degrees for all values of λ in Fig. 4. Curves with λ greater than 180 deg. acquire large rotations due to the large solid angle rotation components.

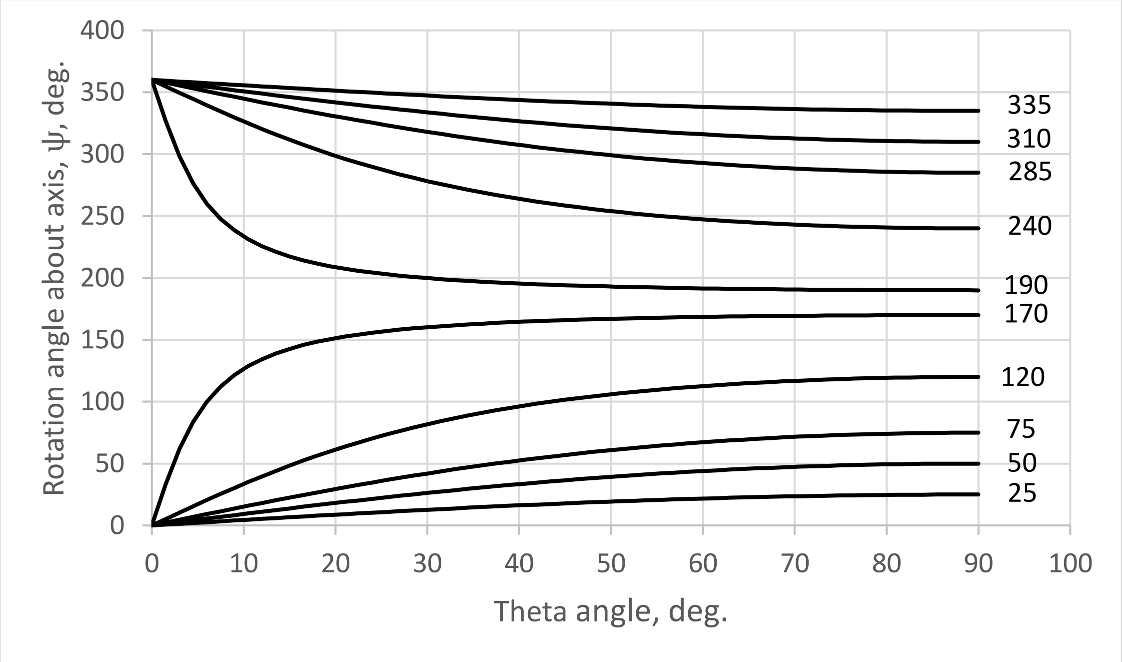

Fig. 4. Rotation angle ψ as a function of θ for curves having various values of λ labeled to the right of each curve.

Numerous cases in Fig. 4 were checked using the derived ψ from eq. (25) and the slewing transformation obtained from **V** and $\mathbf{V_R}$ to derive the transformation $\mathbf{U_R}$ $\mathbf{U_S}$**,** found in eq. (26). The Euler Rotation Vector related to $\mathbf{U_R}$ $\mathbf{U_S}$ yields both the unitized eigenvector, **e**, and the associated eigenvalue, λ, respectively. The derived **e** and λ were found to agree with the original **e** and λ in **U**(**e**, λ), in eq. (26). This agreement proves that the product of the slewing and rotational transformations of the vector axis is equal to the transformation of the original Rodrigues' axis-angle transformation, thereby providing further validation of results presented in Fig. 4.

$$\mathbf{U}(\mathbf{e},\lambda) = \mathbf{U_R}\ \mathbf{U_S} \tag{26}$$

## 4. Conclusion

Rodrigues' Rotation Formula accounts for the change in orientation of a vector when it is rotated about a fixed axis by a given angle. However, the associated rotation about the vector's axis is not provided by Rodrigues's Formula. This work derived the rotation about the slewing vector's axis based on its trajectory implied by Rodrigues' Formula and two rotational contributions given by a recent kinematical theorem. Two methods were used to obtained the desired rotation angle to ensure its correctness. The attitude transformation method yielded the same rotation angle as the analytical method. The results were also in agreement with the analogous rotation of a Foucault Pendulum's mounting fixture, which is driven by the Earth's rotation about the polar axis.

The change in vector orientation given by Rodrigues' Formula combined with the rotation about the vector's axis, provide the complete attitude transformation, which is the same as the original axis-angle attitude transformation of Rodrigues' Rotation Formula.

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