A Proof of Three Integral Representation of Riemann ξ-function Using Divergent Series without using partial integration

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Abstract: In this paper,we proved that for classic integral representation of Riemann ξ-function $\xi(s)=\frac{1}{2}+\frac{s(s-1)}{2}\int_1^\infty\psi(x)(x^{s/2-1}+x^{(1-s)/2-1})dx=-4\psi'(1)+\int_1^\infty\psi'(x)((1-s)x^{s/2}+sx^{(1-s)/2})dx=2\int_1^\infty(\frac{3}{2}\psi'(x)+x\psi''(x))(x^{s/2}+x^{(1-s)/2})dx$, the common lower limitation 1 of the three divergent series equals each other (including $\frac{-1}{2}$ for the first and $(-4)\psi'(1)$ for the second).This provides a new approach to prove the three integral representation without using partial integration.

Key words: Divergent series,Riemann ξ-function.

1. Introduction

We start from the equality of Titchmarsh [10.1.1] and dividing the nominator -1 to $(\frac{-1}{8})\times 8$ for the right side to obtain

$$\begin{aligned}\frac{2\xi(s)}{s(s-1)}&=\frac{1}{s(s-1)}+\int_1^\infty\sum_{n=1}^\infty e^{-n^2\pi x}(x^{s/2-1}+x^{(1-s)/2-1})dx\\&=\frac{\frac{-1}{8}8}{s(1-s)}+\int_1^\infty\sum_{n=1}^\infty e^{-n^2\pi x}(x^{s/2-1}+x^{(1-s)/2-1})dx\end{aligned}\tag{1}$$

Now differentiation of

$$2\sum_{n=1}^\infty e^{-n^2\pi/x}+1=\sqrt{x}(2\sum_{n=1}^\infty e^{-n^2\pi x}+1)\tag{2}$$

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easily gives

$$\frac{-1}{8} = \sum_{n=1}^{\infty} e^{-n^2\pi}(-n^2\pi + \frac{1}{4}) \tag{3}$$

Using the Taylor series-expansion $e^{kx} = \sum_{m=0}^{\infty} \frac{(kx)^m}{m!}$ to obtain

$$\frac{-1}{8} = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{(-n^2\pi)^m(-n^2\pi + \frac{1}{4})}{m!} \tag{4}$$

Making a reduction to obtain

$$\frac{-1}{8} + (-1)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{(-n^2\pi)^m\frac{1}{4}}{m!} = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{(-n^2\pi)^m(-n^2\pi)}{m!} \tag{5}$$

Taking $m = t - 1$ for the right side to obtain

$$\frac{-1}{8} + (-1)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{(-n^2\pi)^m\frac{1}{4}}{m!} = \sum_{n=1}^{\infty}\sum_{t=1}^{\infty} \frac{(-n^2\pi)^{t-1}(-n^2\pi)}{(t-1)!} \tag{6}$$

Putting the index together and write m for the right side for convenience to obtain

$$\frac{-1}{8} + (-1)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{(-n^2\pi)^m\frac{1}{4}}{m!} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(-n^2\pi)^m}{(m-1)!} \tag{7}$$

Multiplying both the nominator and the denominator by m because of $m \neq 0$ (the basic property of two equal terms) for the right side to obtain

$$\frac{-1}{8} + (-1)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{(-n^2\pi)^m\frac{1}{4}}{m!} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(-n^2\pi)^m m}{m!} \tag{8}$$

Transforming $m \geq 1$ to $m \geq 0$ because of $m = 0$ ensures the nominator is zero for the right side to obtain

$$\frac{-1}{8} + (-1)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{(-n^2\pi)^m\frac{1}{4}}{m!} = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{(-n^2\pi)^m m}{m!} \tag{9}$$

Making a reduction to obtain

$$\frac{-1}{8} = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{(-n^2\pi)^m(m + \frac{1}{4})}{m!} \tag{10}$$

Using Equation (10) to replace $\frac{-1}{8}$ of Equation (1) to obtain

$$\frac{2\xi(s)}{s(s-1)} = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{(-n^2\pi)^m(m + \frac{1}{4})}{m!}\frac{8}{s(1-s)} + \int_1^{\infty}\sum_{n=1}^{\infty} e^{-n^2\pi x}(x^{s/2-1} + x^{(1-s)/2-1})dx \tag{11}$$

Using the Taylor series-expansion $e^{kx} = \sum_{m=0}^{\infty} \frac{(kx)^m}{m!}$ to obtain

$$\frac{2\xi(s)}{s(s-1)} = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{(-n^2\pi)^m(m+\frac{1}{4})}{m!}\frac{8}{s(1-s)} + \int_1^{\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^m}{m!}(x^{s/2-1+m} + x^{(1-s)/2-1+m})dx \tag{12}$$

Using the power functional integral formula $\int_1^{\infty} x^k dx = \lim_{x\to+\infty} \frac{x^{k+1}}{k+1} + (-1)\frac{1}{k+1}$ and making a reduction to obtain

$$\begin{aligned}&\frac{2\xi(s)}{s(s-1)} + (-1)\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^m}{m!}(\frac{x^{s/2+m}}{\frac{s}{2}+m} + \frac{x^{(1-s)/2+m}}{\frac{1-s}{2}+m})\\ =&\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^m(m+\frac{1}{4})}{m!}\frac{8}{s(1-s)} + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}(\frac{1}{\frac{s}{2}+m} + \frac{1}{\frac{1-s}{2}+m})\end{aligned} \tag{13}$$

Taking the common denominator for the right side to obtain

$$\begin{aligned}&\frac{2\xi(s)}{s(s-1)} + (-1)\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^m}{m!}(\frac{x^{s/2+m}}{\frac{s}{2}+m} + \frac{x^{(1-s)/2+m}}{\frac{1-s}{2}+m})\\ =&\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^m(m+\frac{1}{4})}{m!}\frac{8}{s(1-s)} + (-1)\frac{8}{(s+2m)(1-s+2m)})\end{aligned} \tag{14}$$

Taking the common denominator for the right side to obtain

$$\begin{aligned}&\frac{2\xi(s)}{s(s-1)} + (-1)\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^m}{m!}(\frac{x^{s/2+m}}{\frac{s}{2}+m} + \frac{x^{(1-s)/2+m}}{\frac{1-s}{2}+m})\\ =&\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^m(m+\frac{1}{4})}{m!}\frac{32(m^2+\frac{1}{2}m)}{s(1-s)(s+2m)(1-s+2m)}\end{aligned} \tag{15}$$

Multiplying both sides by $\frac{s(s-1)}{2}$ to obtain

$$\begin{aligned}&\xi(s) + (-1)\frac{s(s-1)}{2}\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^m}{m!}(\frac{x^{s/2+m}}{\frac{s}{2}+m} + \frac{x^{(1-s)/2+m}}{\frac{1-s}{2}+m})\\ =&(-16)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^m}{m!}\frac{m(m+\frac{1}{4})(m+\frac{1}{2})}{(s+2m)(1-s+2m)}\end{aligned} \tag{16}$$

Transforming $m \geq 0$ to $m \geq 1$ because of the nominator is zero when $m = 0$ for the right side

$$\begin{aligned}&\xi(s) + (-1)\frac{s(s-1)}{2}\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^m}{m!}(\frac{x^{s/2+m}}{\frac{s}{2}+m} + \frac{x^{(1-s)/2+m}}{\frac{1-s}{2}+m})\\ =&(-16)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-n^2\pi)^m}{m!}\frac{m(m+\frac{1}{4})(m+\frac{1}{2})}{(s+2m)(1-s+2m)}\end{aligned} \tag{17}$$

Dividing by m for both the nominator and the denominator because of $m \neq 0$ (basic property of two equal terms) and splitting the index for the right side to obtain

$$\begin{aligned}
&\xi(s)+(-1)\frac{s(s-1)}{2}\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^m}{m!}\Big(\frac{x^{s/2+m}}{\frac{s}{2}+m}+\frac{x^{(1-s)/2+m}}{\frac{1-s}{2}+m}\Big)\\
=&(-16)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-n^2\pi)^{m-1}(-n^2\pi)}{(m-1)!}\frac{(m+\frac{1}{4})(m+\frac{1}{2})}{(s+2m)(1-s+2m)}
\end{aligned}\tag{18}$$

Taking $m-1=t$ for the right side to obtain

$$\begin{aligned}
&\xi(s)+(-1)\frac{s(s-1)}{2}\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^m}{m!}\Big(\frac{x^{s/2+m}}{\frac{s}{2}+m}+\frac{x^{(1-s)/2+m}}{\frac{1-s}{2}+m}\Big)\\
=&(-16)\sum_{n=1}^{\infty}\sum_{t=0}^{\infty}\frac{(-n^2\pi)^{t+1}}{t!}\frac{((t+1)+\frac{1}{4})((t+1)+\frac{1}{2})}{(s+2(t+1))(1-s+2(m+1))}
\end{aligned}\tag{19}$$

Dividing $((t+1)+\frac{1}{2})$ into t and $\frac{3}{2}$ and using m to replace t for convenience for the right side to obtain

$$\begin{aligned}
&\xi(s)+(-1)\frac{s(s-1)}{2}\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^m}{m!}\Big(\frac{x^{s/2+m}}{\frac{s}{2}+m}+\frac{x^{(1-s)/2+m}}{\frac{1-s}{2}+m}\Big)\\
=&(-16)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}\frac{(m+\frac{5}{4})m}{(s+2m+2)(1-s+2m+2)}\\
&+(-16)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}\frac{(m+\frac{5}{4})\frac{3}{2}}{(s+2m+2)(1-s+2m+2)}
\end{aligned}\tag{20}$$

Making a reduction to obtain

$$\begin{aligned}
&\xi(s)+(-1)\frac{s(s-1)}{2}\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^m}{m!}\Big(\frac{x^{s/2+m}}{\frac{s}{2}+m}+\frac{x^{(1-s)/2+m}}{\frac{1-s}{2}+m}\Big)\\
&+(-1)(-16)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}\frac{(m+\frac{5}{4})\frac{3}{2}}{(s+2m+2)(1-s+2m+2)}\\
=&(-16)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}\frac{(m+\frac{5}{4})m}{(s+2m+2)(1-s+2m+2)}
\end{aligned}\tag{21}$$

Transforming $m \geq 0$ to $m \geq 1$ because of the nominator is zero when $m = 0$ for the right side

$$\begin{aligned}
&\xi(s) + (-1)\frac{s(s-1)}{2}\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^m}{m!}(\frac{x^{s/2+m}}{\frac{s}{2}+m}+\frac{x^{(1-s)/2+m}}{\frac{1-s}{2}+m}) \\
&+(-1)(-16)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}\frac{(m+\frac{5}{4})\frac{3}{2}}{(s+2m+2)(1-s+2m+2)} \\
=&(-16)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}\frac{(m+\frac{5}{4})m}{(s+2m+2)(1-s+2m+2)}
\end{aligned} \tag{22}$$

Dividing both the nominator and the denominator by m because of $m \neq 0$ (basic property of two equal terms) and splitting the index for the right side to obtain

$$\begin{aligned}
&\xi(s) + (-1)\frac{s(s-1)}{2}\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^m}{m!}(\frac{x^{s/2+m}}{\frac{s}{2}+m}+\frac{x^{(1-s)/2+m}}{\frac{1-s}{2}+m}) \\
&+(-1)(-16)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}\frac{(m+\frac{5}{4})\frac{3}{2}}{(s+2m+2)(1-s+2m+2)} \\
=&(-16)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-n^2\pi)^{m-1}(-n^2\pi)^2}{(m-1)!}\frac{m+\frac{5}{4}}{(s+2m+2)(1-s+2m+2)}
\end{aligned} \tag{23}$$

Taking $m - 1 = t$ for the right side to obtain

$$\begin{aligned}
&\xi(s) + (-1)\frac{s(s-1)}{2}\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^m}{m!}(\frac{x^{s/2+m}}{\frac{s}{2}+m}+\frac{x^{(1-s)/2+m}}{\frac{1-s}{2}+m}) \\
&+(-1)(-16)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}\frac{(m+\frac{5}{4})\frac{3}{2}}{(s+2m+2)(1-s+2m+2)} \\
=&(-16)\sum_{n=1}^{\infty}\sum_{t=0}^{\infty}\frac{(-n^2\pi)^{t+2}}{t!}\frac{(t+1)+\frac{5}{4}}{(s+2(t+1)+2)(1-s+2(t+1)+2)}
\end{aligned} \tag{24}$$

Using m to replace t for convenience for the right side to obtain

$$\begin{aligned}
&\xi(s) + (-1)\frac{s(s-1)}{2}\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^m}{m!}(\frac{x^{s/2+m}}{\frac{s}{2}+m}+\frac{x^{(1-s)/2+m}}{\frac{1-s}{2}+m}) \\
&+(-1)(-16)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}\frac{(m+\frac{5}{4})\frac{3}{2}}{(s+2m+2)(1-s+2m+2)} \\
=&(-16)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+2}}{m!}\frac{m+\frac{9}{4}}{(s+2m+4)(1-s+2m+4)}
\end{aligned} \tag{25}$$

Making a reduction and taking the common factor $\frac{(-n^2\pi)^{m+1}}{m!}$ for the right side to obtain

$$\begin{aligned}
&\xi(s)+(-1)\frac{s(s-1)}{2}\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^m}{m!}(\frac{x^{s/2+m}}{\frac{s}{2}+m}+\frac{x^{(1-s)/2+m}}{\frac{1-s}{2}+m})\\
=&(-16)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}[\frac{(m+\frac{5}{4})\frac{3}{2}}{(s+2m+2)(1-s+2m+2)}\\
&+(-n^2\pi)\frac{m+\frac{9}{4}}{(s+2m+4)(1-s+2m+4)}]
\end{aligned} \tag{26}$$

Observing that it is the result of combining two denominators that

$$\begin{aligned}
&\xi(s)+(-1)\frac{s(s-1)}{2}\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^m}{m!}(\frac{x^{s/2+m}}{\frac{s}{2}+m}+\frac{x^{(1-s)/2+m}}{\frac{1-s}{2}+m})\\
=&(-2)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}[\frac{3}{2}(\frac{1}{\frac{s}{2}+m+1}+\frac{1}{\frac{1-s}{2}+m+1})\\
&+(-n^2\pi)(\frac{1}{\frac{s}{2}+m+2}+\frac{1}{\frac{1-s}{2}+m+2})]
\end{aligned} \tag{27}$$

Observing that the right side is the result of the power functional integral formula $\int_1^\infty x^k dx = \lim_{x\to+\infty}\frac{x^{k+1}}{k+1}+(-1)\frac{1}{k+1}$ that

$$\begin{aligned}
&\xi(s)+(-1)\frac{s(s-1)}{2}\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^m}{m!}(\frac{x^{s/2+m}}{\frac{s}{2}+m}+\frac{x^{(1-s)/2+m}}{\frac{1-s}{2}+m})\\
=&2\int_1^\infty(\frac{3}{2}\psi'(x)+x\psi''(x))(x^{s/2}+x^{(1-s)/2})dx\\
&+(-2)\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}[\frac{3}{2}[\frac{x^{s/2+m+1}}{\frac{s}{2}+m+1}+\frac{x^{(1-s)/2+m+1}}{\frac{1-s}{2}+m+1}]\\
&+(-n^2\pi)[\frac{x^{s/2+m+2}}{\frac{s}{2}+m+2}+\frac{x^{(1-s)/2+m+2}}{\frac{1-s}{2}+m+2}]]
\end{aligned} \tag{28}$$

where $\psi(x)=\sum_{n=1}^{\infty}e^{-n^2\pi x}$

Edwards [1.8.2] is equivalent to $\xi(s)=2\int_1^\infty(\frac{3}{2}\psi'(x)+x\psi''(x))(x^{s/2}+x^{(1-s)/2})dx$ and it provides a vanishing between the left and the right side that

$$\begin{aligned}
&\frac{s(s-1)}{2}\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^m}{m!}(\frac{x^{s/2+m}}{\frac{s}{2}+m}+\frac{x^{(1-s)/2+m}}{\frac{1-s}{2}+m})\\
=&2\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}[\frac{3}{2}[\frac{x^{s/2+m+1}}{\frac{s}{2}+m+1}+\frac{x^{(1-s)/2+m+1}}{\frac{1-s}{2}+m+1}]\\
&+(-n^2\pi)[\frac{x^{s/2+m+2}}{\frac{s}{2}+m+2}+\frac{x^{(1-s)/2+m+2}}{\frac{1-s}{2}+m+2}]]
\end{aligned} \tag{29}$$

Rewriting both the left and the right side to the integrand from 0 to $+\infty$ because any index of a power function is zero after substituting $x = 0$ ($\lim_{x\to 0^+} x^{\mathrm{const}} = 0$) that

$$\begin{aligned} &\frac{s(s-1)}{2}\int_0^\infty \psi(x)(x^{s/2-1} + x^{(1-s)/2-1})dx \\ =&2\int_0^\infty (\frac{3}{2}\psi'(x) + x\psi''(x))(x^{s/2} + x^{(1-s)/2})dx \end{aligned} \tag{30}$$

Titchmarsh [10.1.3] is equivalent to $\xi(s) = 2\int_0^1(\frac{3}{2}\psi'(x) + x\psi''(x))(x^{s/2} + x^{(1-s)/2})dx = 2\int_1^\infty(\frac{3}{2}\psi'(x) + x\psi''(x))(x^{s/2} + x^{(1-s)/2})dx$ and combining the two integrands provides $\xi(s) = \int_0^\infty(\frac{3}{2}\psi'(x) + x\psi''(x))(x^{s/2} + x^{(1-s)/2})dx$ that

$$2\xi(s) = \frac{s(s-1)}{2}\int_0^\infty \psi(x)(x^{s/2-1} + x^{(1-s)/2-1})dx \tag{31}$$

Using Edwards [1.8.1] $\xi(s) = \frac{s(s-1)}{2}\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ and dividing the right side to two parts that

$$\begin{aligned} 2\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) =& \int_0^\infty \psi(x)(x^{s/2-1} + x^{(1-s)/2-1})dx \\ =&\frac{1}{s(s-1)} + \int_1^\infty \psi(x)(x^{s/2-1} + x^{(1-s)/2-1})dx \\ &+ \frac{1}{s(1-s)} + \int_0^1 \psi(x)(x^{s/2-1} + x^{(1-s)/2-1})dx \end{aligned} \tag{32}$$

Edwards [1.7.3] is $\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \psi(x)(x^{s/2-1} + x^{(1-s)/2-1})dx$ provides a vanishing between the left and the right side that

$$\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) = \frac{1}{s(1-s)} + \int_0^1 \psi(x)(x^{s/2-1} + x^{(1-s)/2-1})dx \tag{33}$$

where it converges only when $Re(s) < 0$ and $Re(s) > 1$

1 Divergent series proof of another integration

Combining Equation (27) with Equation (28) to obtain

$$\begin{aligned}&\xi(s)+(-2)\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}[\frac{3}{2}[\frac{x^{s/2+m+1}}{\frac{s}{2}+m+1}+\frac{x^{(1-s)/2+m+1}}{\frac{1-s}{2}+m+1}]\\&+(-n^2\pi)[\frac{x^{s/2+m+2}}{\frac{s}{2}+m+2}+\frac{x^{(1-s)/2+m+2}}{\frac{1-s}{2}+m+2}]]\\=&2\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}[\frac{3}{2}[\frac{1}{\frac{s}{2}+m+1}+\frac{1}{\frac{1-s}{2}+m+1}]\\&+(-n^2\pi)[\frac{1}{\frac{s}{2}+m+2}+\frac{1}{\frac{1-s}{2}+m+2}]]\end{aligned}\tag{34}$$

where $\psi(x)=\sum_{n=1}^{\infty}e^{-n^2\pi x}$

Using Equation (26) (Taking the common denominator) to obtain

$$\begin{aligned}&\xi(s)+(-2)\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}[\frac{3}{2}[\frac{x^{s/2+m+1}}{\frac{s}{2}+m+1}+\frac{x^{(1-s)/2+m+1}}{\frac{1-s}{2}+m+1}]\\&+(-n^2\pi)[\frac{x^{s/2+m+2}}{\frac{s}{2}+m+2}+\frac{x^{(1-s)/2+m+2}}{\frac{1-s}{2}+m+2}]]\\=&(-16)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}[\frac{(m+\frac{5}{4})\frac{3}{2}}{(s+2m+2)(1-s+2m+2)}\\&+(-n^2\pi)\frac{(m+\frac{9}{4})}{(s+2m+4)(1-s+2m+4)}]\end{aligned}\tag{35}$$

Making a reduction to obtain

$$\begin{aligned}&\xi(s)+(-2)\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}[\frac{3}{2}[\frac{x^{s/2+m+1}}{\frac{s}{2}+m+1}+\frac{x^{(1-s)/2+m+1}}{\frac{1-s}{2}+m+1}]\\&+(-n^2\pi)[\frac{x^{s/2+m+2}}{\frac{s}{2}+m+2}+\frac{x^{(1-s)/2+m+2}}{\frac{1-s}{2}+m+2}]]\\&+16\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}\frac{(m+\frac{5}{4})\frac{3}{2}}{(s+2m+2)(1-s+2m+2)}\\=&(-16)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}\frac{(m+\frac{9}{4})(-n^2\pi)}{(s+2m+4)(1-s+2m+4)}\end{aligned}\tag{36}$$

Taking $m = t - 1$ to obtain

$$\begin{aligned}
&\xi(s) + (-2)\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}[\frac{3}{2}[\frac{x^{s/2+m+1}}{\frac{s}{2}+m+1} + \frac{x^{(1-s)/2+m+1}}{\frac{1-s}{2}+m+1}] \\
&+ (-n^2\pi)[\frac{x^{s/2+m+2}}{\frac{s}{2}+m+2} + \frac{x^{(1-s)/2+m+2}}{\frac{1-s}{2}+m+2}]] \\
&+ 16\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}\frac{(m+\frac{5}{4})\frac{3}{2}}{(s+2m+2)(1-s+2m+2)} \\
=&(-16)\sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\frac{(-n^2\pi)^{t}}{(t-1)!}\frac{((t-1)+\frac{9}{4})(-n^2\pi)}{(s+2(t-1)+4)(1-s+2(t-1)+4)}
\end{aligned} \tag{37}$$

Multiplying both the nominator and the denominator by t because $t \neq 0$ (basic property of two equal terms) and putting the index together for the right side to obtain

$$\begin{aligned}
&\xi(s) + (-2)\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}[\frac{3}{2}[\frac{x^{s/2+m+1}}{\frac{s}{2}+m+1} + \frac{x^{(1-s)/2+m+1}}{\frac{1-s}{2}+m+1}] \\
&+ (-n^2\pi)[\frac{x^{s/2+m+2}}{\frac{s}{2}+m+2} + \frac{x^{(1-s)/2+m+2}}{\frac{1-s}{2}+m+2}]] \\
&+ 16\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}\frac{(m+\frac{5}{4})\frac{3}{2}}{(s+2m+2)(1-s+2m+2)} \\
=&(-16)\sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\frac{(-n^2\pi)^{t+1}}{t!}\frac{t(t+\frac{5}{4})}{(s+2t+2)(1-s+2t+2)}
\end{aligned} \tag{38}$$

Replacing t with m for convenience and expansing $m(m+\frac{5}{4})$ for the right side to obtain

$$\begin{aligned}
&\xi(s) + (-2)\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}[\frac{3}{2}[\frac{x^{s/2+m+1}}{\frac{s}{2}+m+1} + \frac{x^{(1-s)/2+m+1}}{\frac{1-s}{2}+m+1}] \\
&+ (-n^2\pi)[\frac{x^{s/2+m+2}}{\frac{s}{2}+m+2} + \frac{x^{(1-s)/2+m+2}}{\frac{1-s}{2}+m+2}]] \\
&+ 16\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}\frac{(m+\frac{5}{4})\frac{3}{2}}{(s+2m+2)(1-s+2m+2)} \\
=&(-16)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}\frac{m^2+\frac{5}{4}m}{(s+2m+2)(1-s+2m+2)}
\end{aligned} \tag{39}$$

Making a reduction and taking the common factor $\frac{(-n^2\pi)^{m+1}}{m!}$ to obtain

$$
\begin{aligned}
&\xi(s)+(-2)\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}[\frac{3}{2}[\frac{x^{s/2+m+1}}{\frac{s}{2}+m+1}+\frac{x^{(1-s)/2+m+1}}{\frac{1-s}{2}+m+1}] \\
&+(-n^2\pi)[\frac{x^{s/2+m+2}}{\frac{s}{2}+m+2}+\frac{x^{(1-s)/2+m+2}}{\frac{1-s}{2}+m+2}]] \\
=&(-16)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}\frac{m^2+\frac{5}{4}m+(m+\frac{5}{4})\frac{3}{2}}{(s+2m+2)(1-s+2m+2)}
\end{aligned}
\tag{40}
$$

Expansing $m^2+\frac{5}{4}m+(m+\frac{5}{4})\frac{3}{2}$ to obtain

$$
\begin{aligned}
&\xi(s)+(-2)\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}[\frac{3}{2}[\frac{x^{s/2+m+1}}{\frac{s}{2}+m+1}+\frac{x^{(1-s)/2+m+1}}{\frac{1-s}{2}+m+1}] \\
&+(-n^2\pi)[\frac{x^{s/2+m+2}}{\frac{s}{2}+m+2}+\frac{x^{(1-s)/2+m+2}}{\frac{1-s}{2}+m+2}]] \\
=&(-1)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}\frac{4m^2+11m+\frac{15}{2}}{(\frac{s}{2}+m+1)(\frac{1-s}{2}+m+1)}
\end{aligned}
\tag{41}
$$

Observing that it is equivalent to

$$
\begin{aligned}
&\xi(s)+(-2)\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}[\frac{3}{2}[\frac{x^{s/2+m+1}}{\frac{s}{2}+m+1}+\frac{x^{(1-s)/2+m+1}}{\frac{1-s}{2}+m+1}] \\
&+(-n^2\pi)[\frac{x^{s/2+m+2}}{\frac{s}{2}+m+2}+\frac{x^{(1-s)/2+m+2}}{\frac{1-s}{2}+m+2}]] \\
=&(-1)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!} \\
&\times\frac{s(\frac{s}{2}+m+1)+(1-s)(\frac{1-s}{2}+m+1)+4(\frac{s}{2}+m+1)(\frac{1-s}{2}+m+1)}{(\frac{s}{2}+m+1)(\frac{1-s}{2}+m+1)}
\end{aligned}
\tag{42}
$$

Observing that the right side is the result after taking the common denominator that

$$
\begin{aligned}
&\xi(s)+(-2)\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}[\frac{3}{2}[\frac{x^{s/2+m+1}}{\frac{s}{2}+m+1}+\frac{x^{(1-s)/2+m+1}}{\frac{1-s}{2}+m+1}] \\
&+(-n^2\pi)[\frac{x^{s/2+m+2}}{\frac{s}{2}+m+2}+\frac{x^{(1-s)/2+m+2}}{\frac{1-s}{2}+m+2}]] \\
=&(-1)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}(4+\frac{1-s}{\frac{s}{2}+m+1}+\frac{s}{\frac{1-s}{2}+m+1})
\end{aligned}
\tag{43}
$$

Using the Taylor series-expansion $e^{kx} = \sum_{m=0}^{\infty} \frac{(kx)^m}{m!}$ to the right side to obtain

$$\begin{aligned}&\xi(s) + (-2)\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}[\frac{3}{2}[\frac{x^{s/2+m+1}}{\frac{s}{2}+m+1} + \frac{x^{(1-s)/2+m+1}}{\frac{1-s}{2}+m+1}] \\ &+ (-n^2\pi)[\frac{x^{s/2+m+2}}{\frac{s}{2}+m+2} + \frac{x^{(1-s)/2+m+2}}{\frac{1-s}{2}+m+2}]] \\ =&(-4)\psi'(1) + (-1)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}(\frac{1-s}{\frac{s}{2}+m+1} + \frac{s}{\frac{1-s}{2}+m+1})\end{aligned} \tag{44}$$

where $\psi(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}$

Observing that the right side is the result of the power functional integral formula $\int_1^{\infty} x^k dx = \lim_{x\to+\infty}\frac{x^{k+1}}{k+1} + (-1)\frac{1}{k+1}$ that

$$\begin{aligned}&\xi(s) + (-2)\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}[\frac{3}{2}[\frac{x^{s/2+m+1}}{\frac{s}{2}+m+1} + \frac{x^{(1-s)/2+m+1}}{\frac{1-s}{2}+m+1}] \\ &+ (-n^2\pi)[\frac{x^{s/2+m+2}}{\frac{s}{2}+m+2} + \frac{x^{(1-s)/2+m+2}}{\frac{1-s}{2}+m+2}]] \\ =&(-4)\psi'(1) + \int_1^{\infty}\psi'(x)((1-s)x^{s/2} + sx^{(1-s)/2})dx \\ &+ (-1)\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}(\frac{(1-s)x^{s/2+m+1}}{\frac{s}{2}+m+1} + \frac{sx^{(1-s)/2+m+1}}{\frac{1-s}{2}+m+1})\end{aligned} \tag{45}$$

Edwards [1.8.1-1.8.2] (the proving process of Equation (1.8.2]) is equivalent to $\xi(s) = (-4)\psi'(1) + \int_1^{\infty}\psi'(x)((1-s)x^{s/2} + sx^{(1-s)/2})dx$ and it provides a vanishing between the left and the right side that

$$\begin{aligned}&(-2)\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}[\frac{3}{2}[\frac{x^{s/2+m+1}}{\frac{s}{2}+m+1} + \frac{x^{(1-s)/2+m+1}}{\frac{1-s}{2}+m+1}] \\ &+ (-n^2\pi)[\frac{x^{s/2+m+2}}{\frac{s}{2}+m+2} + \frac{x^{(1-s)/2+m+2}}{\frac{1-s}{2}+m+2}]] \\ =&(-1)\lim_{x\to+\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-n^2\pi)^{m+1}}{m!}(\frac{(1-s)x^{s/2+m+1}}{\frac{s}{2}+m+1} + \frac{sx^{(1-s)/2+m+1}}{\frac{1-s}{2}+m+1})\end{aligned} \tag{46}$$

Rewriting both the left and the right side to the integrand from 0 to $+\infty$ because any index of a power function is zero after substituting $x = 0$ ($\lim_{x\to0^+} x^{\text{const}} = 0$) and multiplying both the left and the right side by (-1) to obtain

$$\begin{aligned}&2\int_0^\infty (\frac{3}{2}\psi'(x)+x\psi''(x))(x^{s/2}+x^{(1-s)/2})dx\\&=\int_0^\infty \psi'(x)((1-s)x^{s/2}+sx^{(1-s)/2})dx\end{aligned}\tag{47}$$

Titchmarsh [10.1.3] is equivalent to $\xi(s)=2\int_0^1(\frac{3}{2}\psi'(x)+x\psi''(x))(x^{s/2}+x^{(1-s)/2})dx=2\int_1^\infty(\frac{3}{2}\psi'(x)+x\psi''(x))(x^{s/2}+x^{(1-s)/2})dx$ and combining the two integrands provides

$$2\xi(s)=\int_0^\infty \psi'(x)((1-s)x^{s/2}+sx^{(1-s)/2})dx\tag{48}$$

Dividing the right side to two parts that

$$\begin{aligned}2\xi(s)=&(-4)\psi'(1)+\int_1^\infty \psi'(x)((1-s)x^{s/2}+sx^{(1-s)/2})dx\\&+4\psi'(1)+\int_0^1 \psi'(x)((1-s)x^{s/2}+sx^{(1-s)/2})dx\end{aligned}\tag{49}$$

Edwards [1.8.1-1.8.2] (the proving process of Equation (1.8.2]) is equivalent to $\xi(s)=(-4)\psi'(1)+\int_1^\infty \psi'(x)((1-s)x^{s/2}+sx^{(1-s)/2})dx$ and it provides a vanishing between the left and the right side that

$$\xi(s)=4\psi'(1)+\int_0^1 \psi'(x)((1-s)x^{s/2}+sx^{(1-s)/2})dx\tag{50}$$

where it converges only when $Re(s)<0$ and $Re(s)>1$

2 Summary

We proved that for $\xi(s)=\frac{1}{2}+\frac{s(s-1)}{2}\int_1^\infty \psi(x)(x^{s/2-1}+x^{(1-s)/2-1})dx=-4\psi'(1)+\int_1^\infty \psi'(x)((1-s)x^{s/2}+sx^{(1-s)/2})dx=2\int_1^\infty(\frac{3}{2}\psi'(x)+x\psi''(x))(x^{s/2}+x^{(1-s)/2})dx$,these three divergent series equals each other (consider $\frac{1}{2}$ and $(-4)\psi'(1)$ but don't consider $x\to+\infty$) when substituting the lower limitation 1.

References

[1] Edwards H.M.:Riemann's Zeta Function.Academic Press Inc.,(1974)193-195.

[2] Titchmarsh.E.c¿:The Theory of the Riemann Zeta Function.Cambridge University Press,Cambridge,(1930).