# Hodge classes

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**Abstract**

Let $X$ be a compact Kähler manifold. We show that if $X$ satisfies the extraction condition over $\mathbb{Q}$ (Definition 1.1), then an $\mathbb{R}$-coefficient Hodge class, which is defined as a class in $H^{p,p}(X;\mathbb{Z})\otimes\mathbb{R}$, is represented by a current's weak limit of $\mathbb{R}$-coefficient complex analytic cycles.

## 1 Introduction

**Definition 1.1.** *Let $X$ be a compact Kähler manifold. We say that $X$ satisfies the extraction condition over $\mathbb{Q}$ if for any nonzero rational homology class*

$$\tau \in H_{k,k}(X;\mathbb{Z})\otimes\mathbb{Q}$$

*represented by a closed weakly positive current ([1]) of bidimension $(k,k)$, there is a $k$ dimensional complex analytic cycle $V$ with positive rational coefficients such that*

$$\tau - [T_V] \tag{1.1}$$

*is represented by a closed weakly positive current with the mass*

$$\mathbf{M}(T_V) \geq \boldsymbol{\delta}(\tau) \tag{1.2}$$

*where $\boldsymbol{\delta}(\bullet)$ is a continuous positive function in a neighborhood of $\tau$ in $H_{2k}(X;\mathbb{R})$, $T_\bullet$ denotes the integration current over $\bullet$, and $[\bullet]$ denotes the homology class of $\bullet$.*

**Main theorem 1.2.** *If a compact Kähler manifold $X$ satisfies the extraction condition over $\mathbb{Q}$, then any $\mathbb{R}$-coefficient Hodge class is represented by a current's weak limit of $\mathbb{R}$-coefficient complex analytic cycles.*

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## 2 Proof

*Proof of Main theorem 1.2.* **Step 1**: By the Poincaré duality, we work with the homology where $k = dim(X) - p$ is the dimension. For $\mathbb{G} = \mathbb{R}$ or $\mathbb{Q}$, $H^+_{k,k,\mathbb{G}}(X)$ denotes the closed subset of classes in $H_{k,k}(X;\mathbb{Z}) \otimes \mathbb{G}$ represented by the closed weakly positive currents. Let

$$C^+_k(X) \subset H^+_{k,k,\mathbb{R}}(X) \tag{2.1}$$

be the closure of the set of the complex analytic cycle classes with positive real coefficients. By the lower bound (1.2) and the fact that the subset

$$H_{k,k}(X;\mathbb{Z}) \otimes \mathbb{Q} \subset H_{k,k}(X;\mathbb{Z}) \otimes \mathbb{R} \tag{2.2}$$

is dense, the extraction over $\mathbb{Q}$ can be continuously extended to that over $\mathbb{R}$. For a real homology class $\tau \in H^+_{k,k,\mathbb{R}}(X)$, the extension means that the formulas (1.1) and (1.2) are expressed as

$$\begin{cases} \tau - [\mathscr{V}] = \tau_1 \\ \mathbf{M}(\tau_1) < \mathbf{M}(\tau) \end{cases} \tag{2.3}$$

where $\tau_1 \in H^+_{k,k,\mathbb{R}}(X)$ and $[\mathscr{V}] \in C^+_k(X)$, and the mass $\mathbf{M}(\bullet)$ of a class is well defined if the class is represented by a weakly positive current (see [2]). Let $\lambda \geq 0$ be the infimum of the masses of $\mathbf{M}(\tau_1)$ for all $\tau_1$ and $\mathscr{V}$ satisfying (2.3). Notice that the infimum $\lambda$ can be realized by the current's weak limit as follows. Since all such $\tau_1$ in (2.3) have masses bounded by $\mathbf{M}(\tau)$, then there is a sequence $\tau^N \in H^+_{k,k,\mathbb{R}}(X)$ satisfying (2.3) with the closed current $[\mathscr{V}^\infty]$ and the limit

$$\lim_{N\to\infty} \tau^N = \tau^\infty$$

satisfying $\mathbf{M}(\tau^\infty) = \lambda$. So, if $\lambda > 0$, by the extraction over $\mathbb{R}$, there is another class

$$\tau^b \in C^+_k(X)$$

with strictly smaller mass satisfying (2.3). This is not possible. Therefore $\lambda = 0$. Then $\tau^\infty = 0$. So,

$$\tau - [\mathscr{V}^\infty] = 0. \tag{2.4}$$

Expressing $\tau$ in cohomology, (2.4) proves the main theorem for the cohomology class represented by a closed weakly positive current.

**Step 2**: For an arbitrary real Hodge class $u \in H^{p,p}(X;\mathbb{Z}) \otimes \mathbb{R}$, we write

$$u = \sum_{finite\ j} b_j u_j \tag{2.5}$$

where $b_j$ are real numbers, and $u_j$ are integeral non-torsion classes of $(p,p)$ type. For each $j$, we write

$$u_j = a\omega^p + u_j - a\omega^p \tag{2.6}$$

where $a$ is a real number. Since $u_j$ is of $(p,p)$ type, Demailly in [1] shows that for a sufficiently large number $a$, the cohomology class

$$a\omega^p + u_j$$

is represented by a weakly positive current. Applying the step 1, we obtain the cohomology class $a\omega^p + u_j$ is represented by a current's weak limit of $\mathbb{R}$-coefficient complex analytic cycles. Since the cohomology of $a\omega^p$ lies in the dual of $C_k^+(X)$, $u_j$ is represented by a current's weak limit of $\mathbb{R}$-coefficient complex analytic cycles. We complete the proof.

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# References

[1] J.-P. Demailly, *Courants positifs extrêmaux et conjecture de Hodge*, Inventiones mathematicae (1982), p. 347-374

[2] B. Lawson, *The stable homology of a flat torus* , Math. Scand. 36 (1975), P. 49-73.

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