Dimensions and Colorings of a Graph

VOLKER W. THÜREY
Bremen, Germany *

October 30, 2023

Abstract

In the first part, we introduce several notions of graph dimensions. These concepts are inspired by the classical idea of a 'dimension' which was introduced around 60 years ago. We provide simple examples to illustrate them.
In the second part, we define new types of graph colorings, derived from the Chromatic Number and the Chromatic Index. We apply coloring to both the vertices and the the edges.
Finally, we generalize the concept of the 'Chromatic Number of the Plane'.

Keywords and phrases: Dimension of a Graph; Chromatic Number; Chromatic Index; Chromatic Number of the Plane

MSC: 05C99

1 Introduction

We assume that the reader is familiar with graphs. They should be acquainted with the concept of the 'dimension' of a Graph; otherwise, refer to [1].
Furthermore, the reader should understand the terms Chromatic Index and Chromatic Number. If not, see the chapter Colorings of a Graph.

Two edges are said to be *neighboring* if and only if they are distinct and share a common vertex.
An edge e and a vertex $\vec{a}$ are *incident* if and only if $\vec{a}$ is one of the two endpoints of e.
Two distinct vertices $\vec{v}$ and $\vec{w}$ are *adjacent* if and only if they are endpoints of the same edge.

A graph G is said to be *connected* if and only if, for every pair of vertices $\vec{x}$ and $\vec{y}$ in G, there exists a sequence of edges S_1, S_2 S_3, ... S_{k-1}, S_k such that:
Each consecutive pair S_i and S_{i+1} (for $1 \leq i \leq k-1$) shares a common vertex, i.e., they are neighboring, and both pairs $\vec{x}$ and S_1 and $\vec{y}$ and S_k are incident.

In the paper [2], the mathematicians Erdös, Harary, and Tutte introduced in 1965 a constant called 'dimension' for every graph.

*49 (0)421 591777, volker@thuerey.de

We believe that it is useful to repeat this definition.

First, we define a *display* of a graph G as an isomorphic graph H in $\mathbb{R}^{\mathsf{n}}$ for some n, such that H has only a finite number of intersection points.

The *dimension* of a graph G, abbreviated as dim, is the smallest number k such that G can be represented by an isomorphic graph $H \subset \mathbb{R}^{\mathsf{k}}$ where all the edges of H are straight lines of equal length. See [1].
The concept of *Euclidean dimension*, abbreviated as Edim, is closely related to dim. In an isomorphic graph H, adjacent vertices have equal Euclidean distances, while non adjacent vertices have different distances. See also [1].

2 Dimensions and Distances

Let i and j be natural numbers or zero, and let G be an arbitrary graph.

We define the (i, j) *dimension* of G as the smallest natural number n such that there exists a display of an isomorphic graph H, in which there are at least i pairs of adjacent vertices with equal distances and at least j pairs of non adjacent vertices with equal distances. (These two distances may differ from each other.)

If no such H exits, we write (i, j) dimension $(G) = \infty$.

Now, we assume that G is a connected graph. The following definitions are applicable to both abstract and concrete graphs, where a *concrete* graph is a graph H such that H is a subset of $\mathbb{R}^n$ for any n.

From the vertices, the edges, or both, we construct metric spaces.

For different vertices $\vec{p}$ and $\vec{q}$ of G we define $d_{vertices}(\vec{p}, \vec{q}) := d_{vertices,\ edges}(\vec{p}, \vec{q}) := K$, if $S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_{K-1} \cup S_K$ is a way from $\vec{p}$ to $\vec{q}$, where S_i is an edge for $1 \leq i \leq K$, and there is no shorter way. ($K = 1$ is possible.) It follows that $\vec{p}$ and S_1 and also $\vec{q}$ and S_K are incident pairs.
For different edges e and f of G we define $d_{edges}(e, f) := d_{vertices,\ edges}(e, f) := L - 1$, if $S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_{L-1} \cup S_L$ connect e and f, where S_i is an edge for $1 \leq i \leq L$, and $S_1 = e$ and $S_L = f$, and there is no shorter way. ($L = 1$ is not possible.)
For every edge g, we define $d_{edges}(g, g) := d_{vertices,\ edges}(g, g) := 0$.
For a vertex $\vec{x}$ and an edge e of G we define $d_{vertices,\ edges}(\vec{x}, e) := M$, if $S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_{M-1} \cup S_M$ connect $\vec{x}$ and e , where S_1 is confined by $\vec{x}$ and another vertex, i.e. $\vec{x}$ and S_1 are incident, $S_M = e$, and S_i is an edge for $1 \leq i \leq M$, and there is no shorter way from $\vec{x}$ to e. ($M = 1$ is possible.)

Remark 2.1. It holds that $d_{vertices}(\vec{p}, \vec{q}) = 1$ if and only if $\vec{p}$ and $\vec{q}$ are adjacent. It holds that $d_{edges}(e, f) = 1$ if and only if e and f are neighboring. It holds that $d_{vertices,\ edges}(\vec{x}, e) = 1$ if and only if $\vec{x}$ and e are incident.

To complete the metrics we define that $d_{vertices,\ edges}(\vec{v}, \vec{v})$ and $d_{vertices}(\vec{v}, \vec{v})$ are 0, where $\vec{v}$ is a vertex. Note that the triangle equalities are fulfilled.

In addition, we determine that $d_{vertices}$, d_{edges}, and $d_{vertices,\ edges}$ are symmetric.

We get three metric spaces (The vertices of G, $d_{vertices}$), (The edges of G, d_{edges}), and (G, $d_{vertices,\ edges}$).

Remark 2.2. We have $d_{vertices} = d_{vertices,\ edges}|vertices$ and $d_{edges} = d_{vertices,\ edges}|edges$.

We call '$d_{vertices,\ edges}$' the *Graph distance*.

Proposal
In the usual triangle inequality in the definition of metric spaces we can replace the '$\leq$' sign by the '$<$' sign in the case of three pairwise different elements.
We mean that in a metric space (S, d_S) for all pairwise different elements $x, y, z \in S$ it holds $d_S(x, y) < d_S(x, z) + d_S(y, z)$.

An edge-vertex is defined as a vertex that is part of an edge, i.e. it is incident to this edge.

We introduce three additional types of 'dimensions' for finite connected graphs:

The *metric dimension*, abbreviated as mdim, is defined as the minimum cardinality of a resolving set R. This is a set of vertices with the following property: For any two distinct vertices $\vec{a}$ and $\vec{b}$ of G there exits a vertex $\vec{r} \in \mathsf{R}$ such that the Graph distance between $\vec{a}$ and $\vec{r}$ is different to the Graph distance between $\vec{b}$ and $\vec{r}$.
The *edge metric dimension* (edim) is defined similarly, but we now consider Graph distances between vertices and edges. The constant edim(G) is the minimum cardinality of a resolving set R, which is again a set of vertices. For every pair of distinct edges (e, f), there exits a vertex $\vec{r} \in \mathsf{R}$ such that the Graph distance between e and $\vec{r}$ differs from the Graph distance between f and $\vec{r}$. See [3].
The *edge edge metric dimension* (e - e dim) is defined by considering Graph distances between edges only. It is the minimum cardinality of a resolving set E, where E is a set of edges. For every pair of distinct edges in G there exits an edge $e \in \mathsf{E}$ such that e distinguishes the pair by having different Graph distances to each.

In all cases where no resolving set exits, we denote the corresponding dimension by the symbol '∞'.

We now introduce additional types of 'dimensions'. They are applicable to to finite graphs with at least one edge (except the first one, where no edge is necessary).

The *vertex number dimension* (vert numb dim) is defined as the number of vertices of a graph.
The *edge-vertex number dimension* (edge-vert numb dim) is defined as the number of edge-vertices of the graph.
The *edge number dimension* (edge numb dim) is defined as the number of edges in the graph.

Note that in a connected graph, the vertex number dimension is equal to the edge-vertex number dimension.

For more information about graphs, see, for instance, [1] or [4].

See Figure 1 and Figure 2. Vertices are represented by the symbol '$\times$'.

The graph I has 5 vertices and 6 edges.
It holds that *vertex number dimension*(I) = *edge* $-$ *vertex number dimension*$(I) = 5$, *edge number dimension*$(I) = 6$, $dim(I) = Edim(I) = 3$, and further $mdim(I) = \ edim(I) = 3$, $e - e\ dim(I) = 2$.

The graph J has 3 vertices and 3 edges.
It holds that $dim(J) = Edim(J) = mdim(J) = edim(J) = 2$, *edge number dimension*(J) = *vertex number dimension*(J) = *edge* $-$ *vertex number dimension*$(J) = 3$, and last not least $e - e\ dim(J) = 2$.

Hints:
For $Edim(I) = 3$ use a rhombus that is not a square. For $dim(J) = \ Edim(J) = 2$ use an equilateral triangle.

Figure 1:

The graph I is a square.

It consists of 5 vertices and 6 edges.

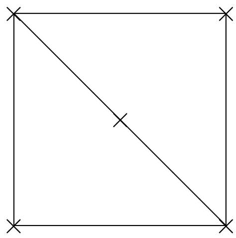
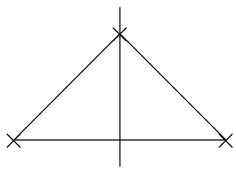


Figure 2:

The graph J consists

of 3 vertices and 3 edges.

It is an isosceles triangle.



s

The vertical s is not a part of J. It is a symmetry line.

3 Colorings of a Graph

In the second part, the reader should be familiar with the terms ‘Chromatic Number’ and ‘Chromatic Index’ of a graph. For clarity, we repeat their definitions.

The Chromatic Index of a graph is the minimum number of colors needed to color the interior of all edges such that neighboring edges have different colors.
The Chromatic Number is the minimum number of colors needed to color all the vertices such that adjacent vertices have different colors.
We define the Thuerey Index as the minimum number of colors needed to color both the vertices and the interior of the edges such that adjacent vertices, neighboring edges, and a vertex and the interior of its incident edge receive different colors.

If no natural numbers satisfies these conditions, we denote the Chromatic Index or the Chromatic Number or the Thuerey Index, respectively, by ∞.

The Chromatic Index of I is 3, its Chromatic Number is 2, and the Thuerey Index of I is 4. The Chromatic Index and the Chromatic Number and the Thuerey Index of J are 3.

Let T be any subset of the natural numbers, i.e. $T \subset \mathbb{N} := \{1, 2, 3, \ldots\}$.

Let $\vec{x}, \vec{y}$ be vertices of a graph G. We say that $\vec{x}$ *is near by* $\vec{y}$ *with respect to* T if and only if $d_{vertices}(\vec{x}, \vec{y})$ is in T.

Let e and f be edges of G. We say that e *is near* f *with respect to* T if and only if $d_{edges}(e, f)$ is in T.

Let $\vec{x}$ be a vertex of G and e an edge. We say that $\vec{x}$ *is near* e *with respect to* T if and only if $d_{vertices,\ edges}(\vec{x}, e)$ is in T.

For every such T, we define coloring constants for all finite graphs G. Specifically, we ask how many colors are needed as a minimum, to color the vertices, the edges, or both the vertices and the edges of G so that all pairs that are near with respect to T receive different colors. We consider either pairs of vertices, or pairs of edges, or pairs consisting of an vertex and edge.

Remark 3.1. For $T = \{1\}$ this has already been defined.

Note that 'to color an edge' means that we provide the interior of an edge with one color.

We may define three additional constants for finite graphs:

The Chromatic Index of the Edges is the minimum number of colors needed to color both the vertices and the edges such that neighboring edges receive different colors.
The Chromatic Number of the Vertices is the minimum number of colors needed to color both the vertices and the edges such that adjacent vertices receive different colors.
The Incident Chromatic Number is the minimum number of colors needed to color both the vertices and the edges such that every vertex and an incident edge receive different colors.

Remark 3.2. Each of the six constants introduced above may be regarded as a type of 'dimensions'.

In the following, we assume that we have as many colors as the sum of the number of vertices and the number of edges. This ensures that all constants are from $\mathbb{N}$.
The in the following defined six constants can also be interpreted as 'dimensions'.

Let G be a nonempty finite graph with at least one edge.

We define the Vertex dimension (abbreviated *vertdim*) as the number of possible colorings of the vertices of G.

We define the Edge dimension (*edgedim*) as the number of possible colorings of the edges of G.

We define the Graph dimension (*graphdim*) as the number of possible colorings of both the vertices and the edges of G.

We define the Adjacent dimension (*addim*) as the number of possible colorings both of the vertices and the edges such that adjacent vertices receive different colors.

We define the Neighboring dimension (*neighdim*) as the number of possible colorings of both the vertices and the edges such that neighboring edges receive different colors.

We define the Incident dimension (*indim*) as the number of possible colorings of both the vertices and the edges such that pairs of a vertex and its incident edge receive different colors.

4 Colorings of the Plane

Definition 4.1. Assume two metric spaces (X, d_X) and (Y, d_Y), and a map f between them. We call f a *distance preserving function* or *Isometry* if and only if for all elements $a, b \in X$ it holds that $d_X(a, b) = d_Y(\mathsf{f}(a), \mathsf{f}(b))$.

We refer to every finite metric space as a *set of vertices.*

Definition 4.2. Let $(\mathsf{V}, d_\mathsf{V})$ be a set of vertices. Let $\chi((\mathsf{V}, d_\mathsf{V}))$ denote the minimum number of colors needed such that every element of $\mathsf{f}(\mathsf{V})$ receives a different color, where $\mathsf{f} : (\mathsf{V}, d_\mathsf{V}) \to (\mathbb{R}^2, ||)$ ranges over all distance preserving functions, and d_V is a metric on V. If no such numbers exists, or if no distance preserving function exits, we define $\chi((\mathsf{V}, d_\mathsf{V})) = 0$.

Remark 4.3. Note that if V has exactly two elements, this corresponds to the problem of the 'Chromatic Number of the Plane', i.e. $\chi((\mathsf{V}, d_\mathsf{V})) = \chi$.

We now define a subset of the natural numbers. Let n be a natural number.

Definition 4.4. We define $\mathbb{N}_n$ as the subset of the first n natural numbers, i.e. $\mathbb{N}_n := \{1, 2, 3, \ldots n-2, n-1, n\}$.

Remark 4.5. There is a canonical embedding $\mathbb{N}_n \hookrightarrow \mathbb{R}^2,\ k \mapsto (k, 0)$

We now define a monotone increasing sequence of coloring constants.

Let χ_n be the least natural number of colors needed to color the plane such that every element of $\mathsf{f}(\mathbb{N}_n)$ receives a different color, where $\mathsf{f} : (\mathbb{R}, |) \to (\mathbb{R}^2, ||)$ ranges over all distance preserving functions. If no such number exists, we write $\chi_n = \infty$.

Remark 4.6. It holds $\chi_2 = \chi$.

Remark 4.7. If we restrict V to finite subsets of $\mathbb{R}^2$, where $\mathbb{R}^2$ is equipped with the standard Eculidean metric $||$, then the number $\chi((\mathsf{V}, d_\mathsf{V}))$ still generalizes the number χ_n. For instance, V could be the set of vertices of a polygon. See [5] p.181 and p.183.

It is easy to generalize Definition 4.2 as follows.

Definition 4.8. Let $(\mathsf{V}, d_\mathsf{V})$ be a set of vertices, and let k be a natural number. We define $\chi((\mathsf{V}, d_\mathsf{V}), \mathsf{k})$ as the minimum number of colors needed such that the points of $\mathsf{f}(\mathsf{V})$ receive at least k different colors, where $\mathsf{f} : (\mathsf{V}, d_\mathsf{V}) \to (\mathbb{R}^2, ||)$ ranges over all distance preserving functions, and d_V is a metric on V. If this is impossible, or if no distance preserving function exists, we write $\chi((\mathsf{V}, d_\mathsf{V}), \mathsf{k}) = 0$.

If $\mathsf{V} = \{(1,0),(2,0),(3,0)\} \subset \mathbb{R}^2$ and $d_{\mathsf{V}} = ||$ and $\mathsf{k} = 3$ it holds $\chi((\mathsf{V}, d_{\mathsf{V}}), \mathsf{k}) = \chi_3$.

The concept can naturally be extended to higher dimensions.

Acknowledgement We thank Lydia Ramachandran for a careful reading, and Yvonne Lüdtke and Dr. Malte von Arnim for support.

References

[1] Alexander Soifer: *The Mathematical Coloring Book. Mathematics of Coloring and the Colorful Life of its Creators*, New York Springer (2009)

[2] Paul Erdös, Frank Harary, William Thomas Tutte: *ON THE DIMENSION OF A GRAPH*, Mathematica 12, London (1965)

[3] https://arxiv.org/abs/1602.00291

[4] Joan M. Aldous and Robin J. Wilson: *Graphs and Applications*, Springer (2000)

[5] Xingting Wang (Editor): *Research Highlights in Mathematics and Computer Science Vol. 3*, B P International (2022)

[6] Alexander Soifer: *The New Mathematical Coloring Book. Mathematics of Coloring and the Colorful Life of its Creators*, New York Springer (2024)