Why was it so difficult to prove the twin prime conjecture?

Óscar E. Chamizo Sánchez.
Redonda Kingdom University, Faculty of Sciences, Department of Mathematics.

Abstract:

The twin prime conjecture, asserting there are infinitely many pairs of primes differing by 2, was popularized by French mathematician Alphonse de Polignac in 1849 [1] [2] We are pleased to present an astounding and overwhelming proof revealing, by the way, a perhaps not so amazing relationship with the Goldbach conjecture and testing, since the core of reasoning is the same, that both statements are strongly connected [3].

Definitions:

From now on, *m* and *n* are positive integer numbers, *p, q* are prime numbers and p_i (i = 1,2,3..., k) is the prime number sequence begining with $p_1=5$.

Twin prime conjecture states that there are infinitely many pairs of primes that differ by 2: 11-13, 17-19, 29-31, and so on.

Our goal is to prove that it cannot happen that inside the interval $p_k < m < p_k^2$ there is not at least a pair of twin primes.

If q and q+2 are twin prime numbers greater than 3 they are of the form 6n ± 1 so let´s see the conditions that $6n \pm 1$ ($p_k < 6n < p_k^2$) must fullfill to become twin primes: Obviously $6n \pm 1$ must not be multiple of any prime number less than or equal to p_k

Twin prime conditions
for 6n

$6n \pm 1 \not\equiv 0 \bmod 5$ or $6n \not\equiv \pm 1 \bmod 5$
$6n \pm 1 \not\equiv 0 \bmod 7$ or $6n \not\equiv \pm 1 \bmod 7$
$6n \pm 1 \not\equiv 0 \bmod 11$ or $6n \not\equiv \pm 1 \bmod 11$
$6n \pm 1 \not\equiv 0 \bmod 13$ or $6n \not\equiv \pm 1 \bmod 13$
…………..   …………..
$6n \pm 1 \not\equiv 0 \bmod p_k$ or $6n \not\equiv \pm 1 \bmod p_k$

Hence for each p_i there are p_i-2 remainders moduli p_i that fullfill the conditions. That amounts up to $(p_1-2)(p_2-2)(p_3-2)\ldots(p_k-2)$, id est, 3.5.9.11….$(p_k-2)$ different systems of linear congruences with prime moduli. The chinese remainder theorem ensures that each one of them has a different and unique solution moduli 5.7.11.13… p_k.

It´s necessary then to prove that exists at least a multiple of 6 that fulfills the preceding conditions inside the aformentioned interval:

$$A = (p_k^2 - p_k)/6 - 1 = p_k(p_k - 1)/6 - 1$$

Let be M the greatest number of consecutive occurrences of 6m that do not fulfill the conditions. It is not easy to figure out an upper bound for M, given the unpredictable nature of prime number distribution[1].

However, an empirically unquestionable fact is that the number of twin primes **Tn** inside **A** seems to grow and grow forever faster than p_k:

k	**p_k**	**Tn**	**A**
2	7	4	6
3	11	7	17
4	13	9	25
5	17	15	45
6	19	17	56
…	…	…	…
17	67	110	737
18	71	121	828
…	…	…	…
167	1009	8278	169512
168	1013	8332	170859

Given a series of S consecutive 6n´s, each residue class mod p appears about S/p times, so prime p covers roughly 2S/p of these S multiples of 6 and hence the density ***d*** of multiples not covered by any prime is about:

$$d = \prod_{5 \le p \le p_k} \left(1 - \frac{2}{p}\right)$$

Taking adventage of the simplification log (1 – x) ≈ − x for small

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[1] For all those who, like myself, enjoy practical questions that sometimes shed light on some more abstract matter of discussion, the problem to determine an accurate value for **M** is the same as the following: Suppose you may not work on 2 predetermined days in five, 2 predetermined days in seven, 2 days in 11, 2 in 13 and so on until 2 days in p_k days. What is the maximum number, as a function of p_k, of consecutive days off?

values of x [4]:

$$\prod_{5\leq p\leq p_k}\left(1-\frac{2}{p}\right)\approx \exp\left(-2\sum_{5\leq p\leq p_k}\frac{1}{p}\right)$$

Since the series between brackets is the well known partial summation of the reciprocal of the primes[5]:

$$\sum_{p\leq x}\frac{1}{p}\approx \operatorname{loglog}(x)$$

Then:

$$\prod_{5\leq p\leq p_k}\left(1-\frac{2}{p}\right)\approx \exp(-2\log\log p_k)=\frac{1}{\log^2 p_k}$$

And the tipical gap between uncovered numbers is $\log^2 p_k$ so the longest run of consecutive multiples of 6 that do not generate a twin prime within the aforementioned interval grows at most on the order of magnitude of the square of the logarithm of p_k.

Although this approach to the problem offers an interesting heuristic, it seems hopeless because we remain stuck in the forbidden leap from the general to the specific, because the local density is unknown.

However, given our large interval A, it's possible to focus on finding the opposite: the minimum number of multiples of 6 that fulfill the conditions within that interval. This involves successfully applying a suitable sieve.

The width of the interval is so large that a simple inclusion-exclusion count with immediate truncation can do the job perfectly [6], providing the following lower bound:

$$\mathrm{L}=\frac{p^2}{6\log^2 p_k}$$

There are at most p/6 multiples of 6 in the range $0 < n < p$, hence an explicit lower bound for the number of twin primes in the interval $p < n < p^2$ is:

$$\frac{p^2}{6\log^2 p_k}-\frac{p}{6}$$

But:

$$\mathrm{L} = \frac{p^2}{6 \log^2 \mathrm{p_k}} \gg p$$

So for any $p \geq 7$

$$\mathrm{L} - \frac{p}{6} > 0$$

Hence there exist an integer n:

$$p < 6n < p^2$$

such that

$$6\mathrm{n} \not\equiv \pm 1 \bmod \mathrm{q} \;\; \forall\, \mathrm{q} \in [5, \mathrm{p}]$$

We have rigorously proven:

For every prime p, there exists a multiple of 6 between p and p^2 that is not congruent to ±1 modulo any prime q with $5 \leq q \leq p$.

Now, given prime p_k, there is always at least a pair of twin primes between p_k and p_k^2. Hence it is inmediate to conclude that there are infinitely many twin primes.

That completes the demonstration.

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roundkuniversity@gmail.com
PA³.

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