Structural Analysis of the Generalized Collatz Tree (k-Tree)

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Abstract

In earlier work [1], we introduced a refined and more structurally representative Collatz tree, within which we identified a singularity. A subsequent preprint [2] established a methodological generalization of the Collatz sequences that preserves this singularity by extending it to a generalized singularity. In the present paper, we investigate the structure of the generalized Collatz tree—referred to as the *k-Tree*—arising from this transformation. Our analysis focuses on the ordering, propagation, and interaction of branch beginnings across ranks, with particular attention to the structural sets $\mathcal{B}_k$ and $\mathcal{A}_k$. This study aims to elucidate the internal architecture of the generalized tree and to clarify the extent to which the geometric and dynamical features of the classical Collatz tree persist under the generalization.

1 Introduction

The Collatz conjecture is traditionally explored through the inverse Collatz tree, whose root is the natural number 1 and whose branches encode all trajectories eventually reaching this fixed point. Despite its apparent simplicity, the structural organization of this tree has long remained insufficiently understood.

In a previous preprint [1], we proposed a new formulation of the Collatz tree that more faithfully reflects the internal dynamics of the Collatz map. This construction revealed a recurring structural phenomenon—referred to as a *singularity*—that is represented by rank 1, branch beginnings. Building upon this discovery, a second preprint [2] introduced a methodological generalization of the Collatz sequences. The purpose of this generalization was to preserve the intrinsic binary singularity of the classical system while extending it to a parameterized framework governed by a transformation of the form

$$(1+2^k)n + S_k(n) \qquad \text{or} \qquad \frac{n}{2^k}$$

depending on the congruence class of n mod 2^k.The present work continues this program by examining the structure of the generalized Collatz tree denoted as the **k-Tree**. This tree is generated by parameter-dependent branch generators. Our objective is to analyze the behavior, ordering, and propagation of branch beginnings across the ranks of the k-Tree, and we are not aiming to provide any proof of Collatz conjecture.

2 Generalized Collatz sequences $(1+2^k)n+S_k(n), n/2^k$ Statment

$$\text{Let } \mathbb{N} = \{0, 1, 2, \dots\}, \text{ and } \qquad \mathbb{N}^* = \mathbb{N} \setminus \{0\}$$

In the previous work [1] We first defined the two auxiliary functions:

$$g_k : \mathbb{E} \to \mathbb{N}^*$$

$$g_k(n) = \frac{n}{2^k} \quad \text{where } \mathbb{E} \text{ is the set of } n \in \mathbb{N}^* \text{ that are}$$

congruent to zero modulo 2^k

$$f_k : \mathbb{L} \to \mathbb{E}$$

$$f_k(n) = (1+2^k)n + S_k(n) \quad \text{where } S_k(n) = 2^k - \big(n \bmod 2^k\big)$$

and $\mathbb{L}$ is the set of $n \in \mathbb{N}^*$ that are not

congruent to zero modulo 2^k

k is called the **Exponent parameter**.

We have:

$$\mathbb{L} \cup \mathbb{E} = \mathbb{N}^*$$

and

$$\mathbb{N}^* \cup \mathbb{E} = \mathbb{N}^*$$

so:

The generalized Collatz sequence is then defined as follows:

$$C_k : \mathbb{N}^* \to \mathbb{N}^*$$

$$C_{n+1} = \begin{cases} f_k(C_n) & \text{if } C_n \not\equiv 0 \pmod{2^k} \\ g_k(C_n) & \text{if } C_n \equiv 0 \pmod{2^k} \end{cases}$$

3 Properties of the generalized Collatz Sequences

3.1 Trajectory in the generalized Collatz Sequences

A *trajectory* of a $C_0 \in \mathbb{N}^*$, refers to the ordered sequence of terms obtained by successively applying the recurrence rule of the generalized Collatz sequence, until the value **1** is reached for the first time, if it is ever reached.
Thus, the trajectory includes the initial term C_0, all the intermediate terms, and stops as soon as **1** is reached for the first time.

3.2 The Trivial Cycle

For all $k \in \mathbb{N}^* \setminus \{2, 4\}$

We have:

$$\forall k \in \mathbb{N}^*,\ \forall n \in \mathbb{N}^*,\ \text{if } n < 2^k$$

then the term following **n** is obtained by an iteration via the auxiliary function f_k, and it is equal to:

$$\text{Since } n \in [1, 2^k - 1] \text{ then } n \bmod 2^k = n, \text{ so } S_k(n) = 2^k - n$$

$$f_k(n) = (1 + 2^k) \cdot n + s_k(n) = n + n \cdot 2^k + 2^k - n = (n + 1) \cdot 2^k$$

We also have:

$$\forall k \in \mathbb{N}^*,\ \forall n \in \mathbb{N}^*$$

if $(n + 1) \bmod 2^k = 0$, then the term following $(n + 1) \cdot 2^k$ is obtained via the second auxiliary function, and it is equal to:

$$\frac{(n+1)\cdot 2^k}{2^k} = (n+1)$$

So, from these two rules, and taking **1** as the first term of the generalized Collatz sequence, we get the following sequence:

$$\{1, 2\cdot 2^k, 2, 3\cdot 2^k, 3, \ldots, 2^k\cdot 2^k, 2^k\}$$

And since the term that succeeds the term 2^k is:

$$\frac{2^k}{2^k} = 1$$

We conclude that there is a trivial generalized cycle of the form:

$$\{1, 2\cdot 2^k, 2, 3\cdot 2^k, 3, \ldots, 2^k\cdot 2^k, 2^k\}$$

Since **1** belongs to the cycle, then the set $\{1, 2\cdot 2^k, 2, 3\cdot 2^k, 3, \ldots, 2^k\cdot 2^k, 2^k\}$ satisfies the generalized Collatz conjecture.

Moreover, Any $a \in [1, 2^k - 1]$ came from $g_k(a\cdot 2^k)$, and we know that $a\cdot 2^k$ such that $a \in [1, 2^k]$ is an element of the trivial cycle. So all elements $a \in [1, 2^k - 1]$ cannot be reached directly from the outside of the trivial cycle.

For $a = 2^k$, we know that it cams from $g_k(2^k\cdot 2^k)$, so all elements $a \in [1, 2^k]$ of the trivial cycle cannot be reached directly from the outside of the trivial cycle.

For the elements of the trivial cycle of the form $a\cdot 2^k$ with $a \in [2, 2^k]$, we know that they came from $f_k(a-1)$, which is an element of the trivial cycle, but it can also come from $g_k(a\cdot 2^k\cdot 2^k)$ with $a\cdot 2^k\cdot 2^k$ not an element of the trivial cycle.

So the trivial cycle has $2^k - 1$ gateways of the form $a\cdot 2^k$ such that $a \in [2, 2^k]$. These gateways are the only elements of the trivial cycle that can be reached directly from the outside of the cycle.

3.3 Property

It can be shown that every $n \in \mathbb{N}^*$ converges to a gateway $2^k\cdot 2^k$ of the trivial cycle before reaching **1**. To demonstrate this, we define the inverse generalized Collatz sequence C_k^{-1}.

3.4.1 Inverse Functions

- $g_k^{-1} : \mathbb{N}^* \to \mathbb{E}$
$$g_k^{-1}(n) = n \cdot 2^k$$

- Let $k \in \mathbb{N}^* \setminus \{2, 4\}$ be the exponent parameter, so:
$$f_k(m) = (1 + 2^k) \cdot m + S_k(m)$$
where $S_k(m) = 2^k - \left(m \bmod 2^k\right)$ is the smallest non-negative additive inverse of $m \mod 2^k$, satisfying:
$$0 < S_k(m) < 2^k$$
We define the inverse function f_k^{-1} as follows:
$$f_k^{-1}(n) = \frac{n - t}{1 + 2^k}$$
where $t = S_k(f_k^{-1}(n))$ is the smallest number in$\mathbb{N}^*$ such that:
$$(1 + 2^k) f_k^{-1}(n) + t \equiv 0 \pmod{2^k}$$
Domain of the Inverse. The inverse function f_k^{-1} is defined for all $n \in \mathbb{N}^*$ such that:
$$\begin{cases} n \equiv 0 \pmod{2^k} \\ \exists\ t \in \mathbb{N}^*,\ 0 < t < 2^k,\ \text{such that } n - t \equiv 0 \pmod{1 + 2^k} \end{cases}$$

3.4.2 Inverse generalized Collatz Sequence

$$C_k^{-1} : \mathbb{N}^* \to \mathbb{N}^*$$

$$\begin{aligned} &C_{n+1}^{-1} = g_k^{-1}(C_n^{-1}), \\ &C_{n+1}^{-1} = f_k^{-1}(C_n^{-1}) \quad \text{if} \quad f_k^{-1}(C_n^{-1}) \in \mathbb{L} \end{aligned}$$

$\exists j \in \mathbb{N}^* \ : \ C_j = 1$

So $C_{j-1} = g_k^{-1}(C_j) = g_k^{-1}(1) = 2^k \cdot 1 = 2^k$

We have $C_{j-1} = 2^k$ is not a gateway of the trivial cycle, then can only be an element of the trivial cycle.

$C_{j-2} = g_k^{-1}(C_{j-1}) = g_k^{-1}(2^k) = 2^k \cdot 2^k$

We conclude that:

$\forall C_0 \in \mathbb{N}^* \setminus \{1, 2 \cdot 2^k, 2, 3 \cdot 2^k, 3, \ldots, 2^k \cdot 2^k, 2^k\} :$

$\exists j \in \mathbb{N}^* \ : \ C_j = 1 \ \Rightarrow \ \exists m < j \ : \ C_m = 2^k \cdot 2^k$

3.4.3 Theorem

Every $C_0 \in \mathbb{N}^* \setminus \{1, 2 \cdot 2^k, 2, 3 \cdot 2^k, 3, \ldots, 2^k \cdot 2^k, 2^k\}$, if it converges to 1, then it first converges to $2^k \cdot 2^k$

Formal statement:

$\forall k \in \mathbb{N}^* \setminus \{2, 4\} \quad$ then:

$\forall C_0 \in \mathbb{N}^* \setminus \{1, 2 \cdot 2^k, 2, 3 \cdot 2^k, 3, \ldots, 2^k \cdot 2^k, 2^k\} :$
$\exists j \in \mathbb{N}^*$ such that $C_j = 1$

$\Rightarrow \quad \exists m \in \mathbb{N}^*$ such that $C_m = 2^k \cdot 2^k \quad$ with $m = j - 2$

3.5 Property

- $\forall \ k \in \mathbb{N}^*, \forall \ C_0 \in \mathbb{N}^*, \forall \ j \in \mathbb{N}$ we have

  $C_j \not\equiv 0 \pmod{2^k} \quad \Rightarrow \quad C_{j+1} \equiv 0 \pmod{2^k}$ with $C_{j+1} = f_k(C_j)$

  That is to say, every non-congruent to zero modulo 2^k term is followed by a term that is congruent to zero modulo 2^k.

- $\forall \ k \in \mathbb{N}^*, \forall \ C_0^{-1} \in \mathbb{N}^*, \forall \ j \in \mathbb{N}$ we have:

  $C_j^{-1} \not\equiv 0 \pmod{2^k} \quad \Rightarrow \quad C_{j+1}^{-1} \equiv 0 \pmod{2^k}$ with $C_{j+1}^{-1} = g_k(C_j^{-1})$

That is to say, every non-congruent to zero modulo 2^k term in the inverse generalized Collatz sequence is followed by a term that is congruent to zero modulo 2^k.

- $\forall\ k \in \mathbb{N}^*, \forall\ C_0^{-1} \in \mathbb{N}^*, \forall\ j \in \mathbb{N}$ we have:

$$C_j^{-1} \equiv 0 \pmod{2^k} \quad \Rightarrow$$

$$\begin{aligned} &C_{j+1}^{-1} = g_k^{-1}(C_j^{-1}),\\ &C_{j+1}^{-1} = f_k^{-1}(C_j^{-1}) \quad \text{if} \quad f_k^{-1}(C_j^{-1}) \in \mathbb{L} \end{aligned}$$

That is to say, every congruent to zero modulo 2^k term in the inverse generalized Collatz sequence is followed by a term that is congruent to zero modulo 2^k, but it can also be followed by a non-congruent to zero modulo 2^k term if C_j^{-1} is a branch generator.

- $\forall\ k \in \mathbb{N}^*,\ \forall\ C_0 \in \mathbb{N}^*, \forall\ j \in \mathbb{N}$ we have :

$$C_j \equiv 0 \pmod{2^k} \Rightarrow \begin{cases} C_{j+1} = g_k(C_j) \equiv 0 \pmod{2^k} \text{ if } C_j \equiv 0 \pmod{2^{2k}} \\ C_{j+1} = g_k(C_j) \not\equiv 0 \pmod{2^k} \end{cases}$$

Every congruent to zero modulo 2^k term in the generalized Collatz sequence is followed by a congruent to zero modulo 2^k if the term is congruent to zero modulo 2^{2k}, else it is followed by a non-congruent to zero modulo 2^k term.

4 Construction of the generalized Collatz Tree

4.1 Division of $\mathbb{N}^*$ into Subsets

For a given $k \in \mathbb{N}^* \setminus \{2, 4\}$ of generalized Collatz sequence,

We define a partition of $\mathbb{N}^*$ into subsets.

- If $n \in \mathbb{N}^*$ is a power of 2^k, then n belongs to the subset $\mathcal{A}_k$:

$$\mathcal{A}_k = \left\{ n \in \mathbb{N}^* \,\middle|\, \exists j \in \mathbb{N},\ n = (2^k)^j \right\}$$

That is to say $\mathcal{A}_k$ is the set of $n \in \mathbb{N}^*$ that are a power of 2^k.
Here, **1** is included in $\mathcal{A}_k$, because $1 = (2^k)^0$.

- If $n \in \mathbb{N}^*$, is not congruent to zero modulo 2^k, then n belongs to the subset:

$$\mathcal{B}_k = \left\{ n \in \mathbb{N}^* \,\middle|\, n \not\equiv 0 \pmod{2^k} \right\}$$

$$\text{we have} \quad \mathcal{A}_k \cap \mathcal{B}_k = \{1\}$$

- If $n \in \mathbb{N}^*$, is congruent to zero modulo 2^k, but not a power of 2^k, then **n** belongs to the subset:

$$\mathcal{M}_k = \left\{ n \in \mathbb{N}^* \,\middle|\, n = p \cdot (2^k)^j,\ p \not\equiv 0 \pmod{2^k},\ j \geq 1 \right\}$$

  That is to say $\mathcal{M}_k$ is the set of $n \in \mathbb{N}^*$ that are divisible by 2^k, but not a pure power of 2^k.

$$\text{we have} \quad \mathcal{M}_k \cap \mathcal{A}_k = \emptyset$$

  Thus, we have the following partition:

$$\mathbb{N}^* = \mathcal{A}_k \cup \mathcal{B}_k \cup \mathcal{M}_k$$

4.2 Description of the Generalized Collatz k-*Tree* to Construct

We define the k-*Tree* as the tree structure generated by the generalized Collatz auxiliary functions $(1+2^k)n + S_k(n), n/2^k,\ k \in \mathbb{N}^* \setminus \{2,4\}$ is the exponent parameter.

The Collatz k-*Tree* we are about to construct represents the set of numbers in $\mathbb{N}^*$ that converge to **1** with $k \in \mathbb{N}^* \setminus \{2,4\}$, the representation of these numbers will be based on the **rank** of the natural number.

The **rank** of a $n \in \mathbb{N}^*$ **n**, denoted **rank(n)**, is defined as the number of f_k^{-1} iterations required to reach a value equal to **n**, using the inverse Collatz sequence C_k^{-1}, and starting from $C_0^{-1} = 1$ as the initial term.

To determine the **rank** of $n \in \mathbb{N}^*$ and its representation in the Collatz k-*Tree*, it is not possible to deduce it directly from **n** using C_k^{-1}, Instead, it is necessary to:

- Verify whether **n** converges to **1** (otherwise, it cannot be represented in the *k-Tree*);
- Determine the **rank** of **n**, **rank(n)**, as well as the number of successive g_k iterations required to go from one non-congruent to zero modulo 2^k term to the next non-congruent to zero modulo 2^k term.
- We agree that when referring to the notions of *k-Tree*, **rank**, or generalized Collatz sequence, we are considering a **fixed** exponent parameter $k \in \mathbb{N}^* \setminus \{2, 4\}$.

According to the generalized Collatz sequence, for each term, there is only one possible next term. This implies that every number in $\mathbb{N}^*$ has a unique trajectory in the generalized Collatz sequence C_k. Therefore, every number in $\mathbb{N}^*$ that converges to **1** for a given C_k can only have one unique representation in the Collatz *k-Tree*.

4.2.1 Properties of the Elements in Subset $\mathcal{A}_k$

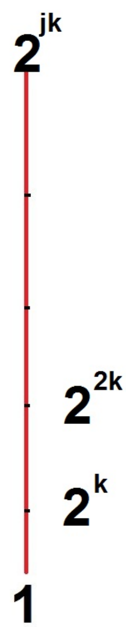

Figure1: Trunk of the Collatz k-Tree

$$k \in \mathbb{N}^* \setminus \{2, 4\}, \forall\ C_0 \in \mathcal{A}_k :\ C_0 > 1 \quad \text{then} \quad C_0 = (2^k)^j : \quad j \in \mathbb{N}^*$$

$$\Rightarrow \quad C_1 = g_k(C_0) = \frac{C_0}{2^k} = \frac{(2^k)^j}{2^k} = (2^k)^{j-1}$$

$$\Rightarrow \quad C_j = \frac{2^j}{2^j} = 1$$

Therefore, for any $C_0 \in \mathcal{A}_k : C_0 > 1$, C_0 converges to 1.

For $C_0 = 1$, we have previously shown that 1 converges to 1 since it is an element of the generalized trivial cycle.

Thus:

$$\forall\, C_0 \in \mathcal{A}_k, \quad C_0 \text{ converges to } 1.$$

The elements of subset $\mathcal{A}_k$ will be represented by a vertical line, which we will call the *Trunk* of the Collatz k-*Tree* (Figure 1).

4.2.2 Remark:

The elements of subset $\mathcal{A}_k$ have rank zero.

Let $n \in \mathcal{M}_k$

$$\mathcal{M}_k = \left\{ n \in \mathbb{N}^* \,\middle|\, n = p \cdot (2^k)^j,\ \ p \not\equiv 0 \pmod{2^k},\ j \geq 1 \right\}$$

Thus,

$$g_k^j(n) = g_k^j(p \cdot (2^k)^j) = p$$

where g_k is the division by 2^k function.

$p \in \mathcal{B}_k$, so any $n \in \mathcal{M}_k$ will be transformed into an element of $\mathcal{B}_k$ in the Collatz sequence by applying j times the g_k iteration.

Therefore, if all the elements of $\mathcal{B}_k$ converges to **1**, then all the elements of $\mathcal{M}_k$ also converges to **1**.

This is expressed as:

$$\forall\, n \in \mathcal{B}_k,\ \exists\, j \in \mathbb{N}^*,\ C_j = 1 \quad \Rightarrow \quad \forall\, m \in \mathcal{M}_k,\ \exists\, i \in \mathbb{N}^*,\ C_i = 1$$

4.2.3 Conclusion

We have shown that forall $k \in \mathbb{N}^* \setminus \{2, 4\}$:

- All elements of the subset $\mathcal{A}_k$ converge to **1**.

- If all elements of the subset $\mathcal{B}_k$ converge to **1**, then all elements of the subset $\mathcal{M}_k$ also converges to **1**.

And since:

$$\mathcal{A}_k \cup \mathcal{B}_k \cup \mathcal{M}_k = \mathbb{N}^*$$

we derive the following Theorem:

4.2.4 Theorem

The convergence to **1** for all $n \in \mathbb{N}^*$ is equivalent to the convergence to **1** for all element of subset $\mathcal{B}_k$ of non-congruent to zero modulo 2^k.

4.2.5 Representation of Elements from $\mathcal{B}_k$ and $\mathcal{M}_k$

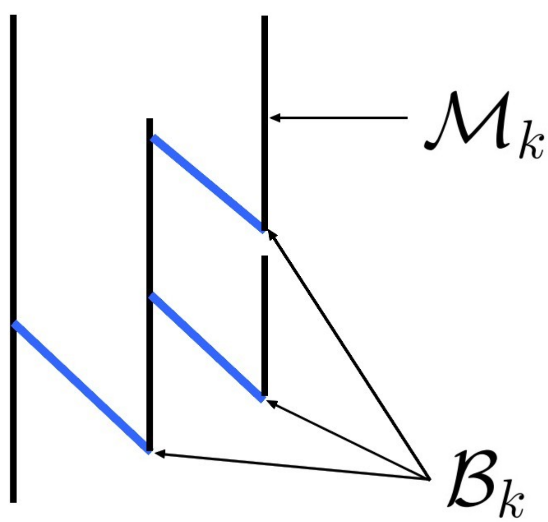

Figure2: Diagram of a Branch

Each element $n \in \mathcal{B}_k$ that converge to 1 will be represented by a distinct point on the Collatz k-**tree**.
All elements $m \in \mathcal{M}_k$ Reducible to an $n \in \mathcal{B}_k$ which means $m = n{\cdot}(2^k)^j$ with $n > 1,\ n \not\equiv 0 \pmod{2^k},\ j \geq 1$, will be represented on a half-line starting from it's original point in **n**.

This half-line is called a ***Branch*** (Figure 2) and the origin point **n** is called the **Beginning of a branch**.

A branch that is emerging directly from the trunk is called a **Daughter Branch**. If a branch have a Daughter Branch emerging from one of its $\mathcal{M}$ elements, we call it **Mother Branch**.

4.2.6 Exception

For the number **1**, it will be represented on the trunk, and represented as a branch beginning, without representing the $\mathcal{M}_k$ elements otherwise it will reproduce a second Trunk; from the trivial cycle $\{1, 2 \cdot 2^k, 2, 3 \cdot 2^k, 3, \ldots, 2^k \cdot 2^k, 2^k\}$ we can deduce that $\text{rank}(1) = 2^k - 1$.

4.2.7 Branch Generator

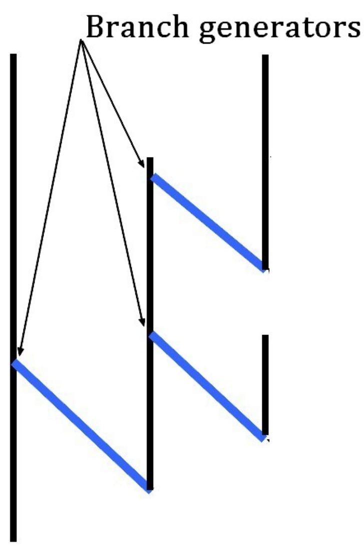

Figure3: Branch generator

A **branch generator** (Figure 3) is a number in $\mathbb{N}^*$ that belongs to a mother branch or the trunk, that is to say, it is an element of $\mathcal{A}_k$ or $\mathcal{M}_k$.

A $g \in \mathbb{N}^*$ is a branch generator if and only if $f_k^{-1}(g) = n$, with n being a branch beginning.

$$f_k^{-1}(g) = n \quad \Rightarrow \quad n = \frac{g - S_k(n)}{1 + 2^k}$$

$$\Rightarrow \quad g \equiv 0 \pmod{2^k} \quad \text{and} \quad g - S_k(n) \equiv 0 \pmod{1 + 2^k}$$

Therefore, g is a branch generator if and only if:

$$\exists\ n \in \mathcal{B}_k,\ g = (1 + 2^k)n + S_k(n)$$

We represent the connection between a branch and its mother branch (or the trunk) by an inclined **line segment**, running from the top (the branch generator) down to the bottom (the branch beginning).

4.2.9 Sister Branches

Two branches are **sister branches** (Figure 6) if and only if they have the same mother branch or they are daughter branches of the trunk.

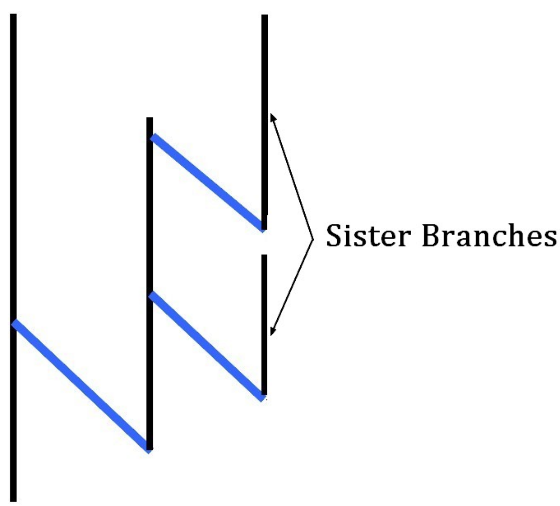

Figure4: Sister Branches

5 Recognition of Structural Patterns in the Dynamics of generalized Collatz

5.1 The Form of a Branch Generator

Let $g \in \mathcal{M}_k$, g is a branch generator if and only if:

$$\exists\, n \in \mathcal{B}_k : g = f_k(n) \quad \text{and this is equivalent to} \quad \exists\, n \in \mathcal{B}_k : n = f_k^{-1}(g)$$

We have:

$$\exists\, n \in \mathcal{B}_k : g = f_k(n) \Rightarrow g \equiv 0 \pmod{2^k}$$

The smallest branch generator must be superior than any gateway, witch means:

$$g = 2^k \cdot 2^k$$

And we have:

$$\exists\, n \in \mathcal{B}_k : n = f_k^{-1}(g) \Rightarrow \exists\, 0 < S_k(n) < 2^k \quad \text{such that} \quad \frac{g - S_k(n)}{1 + 2^k} = n$$

with

$$S_k(n) = 2^k - (n \bmod 2^k)$$

is always a number in $\mathbb{N}^*$ that is not congruent to zero modulo 2^k

5.1.1 Theorem

A $g \in \mathbb{N}^*$ is a branch generator if and only if:

- $g \in \mathcal{M}_k$
- $g \geq 2^k \cdot 2^k$
- $\exists\, n \in \mathcal{B}_k : \frac{g - S_k(n)}{1 + 2^k} = n$

5.2 The First Branch Generator

Now we ask the question: how many times must we multiply by 2^k a branch beginning or the beginning of the trunk to obtain the first branch generator?

We must first prove that if we have two branch beginnings d_1, d_2 that come respectively from the branch generators $g_1, g_2,$ then

$$d_1 < d_2 \implies g_1 < g_2$$

To prove this, let $d_1 < d_2$ be two branch beginnings generated respectively from the branch generators g_1, g_2. Then we have:

$$d_1(1+2^k) < d_2(1+2^k)$$

Since the smallest value of d_2 is (d_1+1), we get:

$$d_2(1+2^k) \geq d_1(1+2^k) + (1+2^k)$$

Thus,

$$d_2(1+2^k) > d_1(1+2^k) + 2^k$$

We also have:

$$|S_k(d_1) - S_k(d_2)| < 2^k$$

Therefore,

$$d_2(1+2^k) + S_k(d_2) > d_1(1+2^k) + 2^k + S_k(d_1) - 2^k$$

which simplifies to:

$$d_2(1+2^k) + S_k(d_2) > d_1(1+2^k) + S_k(d_1)$$

Hence:

$$d_1 < d_2 \implies g_1 < g_2$$

5.2.1 Representation of a Branch Beginning

Let **n** be a branch beginning, meaning $n \in \mathcal{B}_k$, or the beginning of the trunk, in case $n = 1$. Then:

$$n \in \mathcal{B}_k \cup \{1\}.$$

We have, for all $\mathbf{n} \in \mathcal{B}_k \cup \{1\}$,

$$n = \begin{cases} (1+2^k)j, \\ \textit{or} \\ (1+2^k)j+1, \\ \textit{or} \\ . \\ . \\ \textit{or} \\ (1+2^k)j+(2^k-1), \\ \textit{or} \\ (1+2^k)j+2^k, \end{cases} \qquad j \in \mathbb{N}.$$

Hence, for all $\mathbf{n} \in \mathcal{B}_k \cup \{1\}$,

$$n = (1+2^k)j + \varphi, \qquad \varphi \in [0, 2^k], \qquad (j = 0 \Rightarrow \varphi > 0).$$

If we multiply $\mathbf{n}$ by 2^{ik}, we obtain:

$$g = (1+2^k)j \cdot 2^{ik} + \varphi \cdot 2^{ik}.$$

If g is a branch generator, then for some $d \in \mathcal{B}_k$:

$$d = \frac{g - S_k(d)}{1+2^k} = \frac{(1+2^k)j \cdot 2^{ik} + \varphi \cdot 2^{ik} - S_k(d)}{1+2^k}.$$

Thus:

$$d = j\, 2^{ik} + \frac{\varphi\, 2^{ik} - S_k(d)}{1+2^k}.$$

The smallest number of times we must multiply $\mathbf{n}$ by 2^k to become a branch generator is the smallest natural number $\mathbf{i} > \mathbf{0}$ such that:

$$\begin{gathered} j \in \mathbb{N}, \\ \varphi \in [0, 2^k], \\ j = 0 \Rightarrow \varphi > 0, \\ d \in \mathcal{B}_k, \\ S_k(d) \in [1, 2^k - 1]. \end{gathered}$$

5.3 The smallest successor of a branch generator

Let d_1, d_2 be two branch beginnings that are generated respectively from g_1, g_2 such that: $g_1 < g_2$

$$\text{We have} \quad g_1 = (1 + 2^k)\, d_1 + S_k(d_1)$$
$$g_2 = (1 + 2^k)\, d_2 + S_k(d_2)$$

$$\text{We also have} \quad g_2 = g_1 \cdot 2^{ik} = (1 + 2^k)\, d_1 \cdot 2^{ik} + S_k(d_1) \cdot 2^{ik}$$

What is the smallest $i \in \mathbb{N}^*$ such that

$$\begin{cases} d_1 \cdot 2^{ik} \neq d_2, & \text{because } d_2 \not\equiv 0 \pmod{2^k}, \\ S_k(d_1) \cdot 2^{ik} \neq S_k(d_2), & \text{because } S_k(d_2) \in [1, 2^k - 1] \end{cases}$$

Let $\varphi \in \mathbb{N}^*$.

$$g_2 = (1 + 2^k)\, d_1 \cdot 2^{ik} + S_k(d_1) \cdot 2^{ik} - (1 + 2^k)\, \varphi + (1 + 2^k)\, \varphi$$

$$= (1 + 2^k)\, (d_1 \cdot 2^{ik} + \varphi) + \left[S_k(d_1) \cdot 2^{ik} - (1 + 2^k) \cdot \varphi\right]$$

$$\begin{cases} d_1 \cdot 2^{ik} + \varphi = d_2, \\ \left[S_k(d_1) \cdot 2^{ik} - (1 + 2^k) \cdot \varphi\right] = S_k(d_2) \end{cases}$$

So we must find the smallest $i \in \mathbb{N}^*$ such that we can find $\varphi \in \mathbb{N}^*$ satisfying

$$\begin{cases} S_k(d_1) \cdot 2^{ik} - \varphi(1 + 2^k) \in [1, 2^k - 1], \\ d_1 \cdot 2^{ik} + \varphi \not\equiv 0 \pmod{2^k} \end{cases}$$

$$\text{The second condition is equivalent to} \quad d_1 \cdot 2^{ik} + \varphi \not\equiv 0 \pmod{2^k}$$

6 Conclusion

The structural analysis of the generalized Collatz tree, or k-Tree, has revealed that its organization does not replicate the exact structural formulas observed in the classical Collatz tree. While several analogies persist—particularly in the behavior of branch beginnings and their hierarchical propagation—the generalized framework displays distinctions that require deeper investigation before a complete structural correspondence can be established.

In this preprint, our exploration of the k-Tree remained intentionally limited. A full structural characterization of the generalized tree, including closed-form descriptions of ranks, branch generators, and their asymptotic behavior, has not yet been undertaken.

References

[1] Ammar Hamdous, *Revealing a Singularity in Collatz Sequences*, Zenodo, 2025.

[2] Ammar Hamdous, *Methodological Generalization of the Collatz Sequences*, Zenodo, 2025.