On the Constant Real Part of Non-Trivial Zeros: A Proof of the Riemann Hypothesis

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Abstract

This paper presents a proof of the Riemann Hypothesis by examining the geometric and analytic properties of the Dirichlet eta function *η(s)* along an arbitrary, constant imaginary component *t* of the complex argument *s.* The core logic of this proof stems from the idea of isolating the individual dynamics of the real part of *η(s)* along the horizontal slice *t.*

We track the real part of *η(s)* independently along the slice *t* as the real component *σ* of the argument *s* increases from zero to infinity. As *σ* approaches infinity, except the first term 1, every other term of *η(s)* gradually shrinks in modulus, forcing the real part to converge asymptotically to 1. Through a rigorous analysis, we demonstrate that the real part of *η(s)* can admit at most one conditional change in monotonicity for any constant height *t*. This structural constraint strictly rules out the possibility of the real part vanishing more than once across the slice *t*.

Therefore, for *η(s)*—and consequently for the Riemann zeta function ζ(s)— two distinct non-trivial zeros symmetric to the critical line cannot occur, thereby proving that all non-trivial zeros must possess a real component of exactly $\frac{1}{2}$.

1 Introduction

The Riemann zeta function is defined by

$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$ ($s \in \mathbb{C}$) where Re(s) > 1. (1)

For the domain beyond Re(s) > 1, there is a functional equation that represents the function through analytic continuation except at s = 1.

$\zeta(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2}) \Gamma(1-s) \zeta(1-s).$ (2)

Within the *critical strip* (0 < Re(s) < 1), the zeta function *ζ(s)* is also represented in terms of the *Dirichlet eta function η(s)* as:

$\zeta(s) = (1 - 2^{1-s})^{-1}\eta(s)$. (3)

That is, $\zeta(s) = (1 - 2^{1-s})^{-1}\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}\right)$. (4)

η(s) in (3) is defined by the alternating series in (4).

Equation (3) defines a strict analytic identity between $\zeta(s)$ and *η(s).* Although the alternating series in (4) is only conditionally convergent within the critical strip, this identity holds provided the terms maintain their natural, ascending order (n = 1, 2, 3, …).

It is known that all non-trivial zeros of *η(s)* lie within the critical strip; as a consequence, from the identity, the non-trivial zeros of *ζ(s)* must also lie within that strip.

The Riemann Hypothesis (RH) states that all non-trivial zeros of *ζ(s)* lie exactly on the *critical line* defined by Re(s) = $\frac{1}{2}$.

From Equation (4), we see that *ζ(s)* vanishes whenever the summation representing *η(s)* is zero. Therefore, to validate this constant real part $\frac{1}{2}$ of the non-trivial zeros of *ζ(s)*, it is sufficient to determine whether all these zeros of *η(s)* lie exactly on the critical line.

2 Distribution of terms of *η(s)* and the Domain

2.1 The Distribution of terms

Now consider,

$\eta(s) = \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}\right)$ $= 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} +$ ... where $s = \sigma + it$; $\sigma > 0$,

$t \in \mathbb{R}^*$ being an arbitrary height.

The terms of *η(s)* are radially distributed around the origin of the complex plane. There are an infinite number of terms in the right half-plane just as in the left with reference to the origin. The angles of the terms with the positive real axis are governed by both the alternating factor $(-1)^{n+1}$ and the imaginary part *t.*

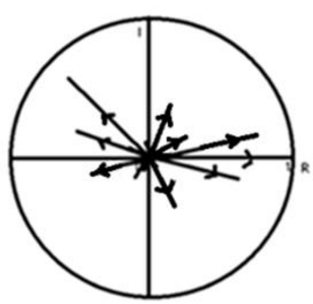

Figure 1: Distribution of Terms

2.2 The Domain of *η(s)* under study

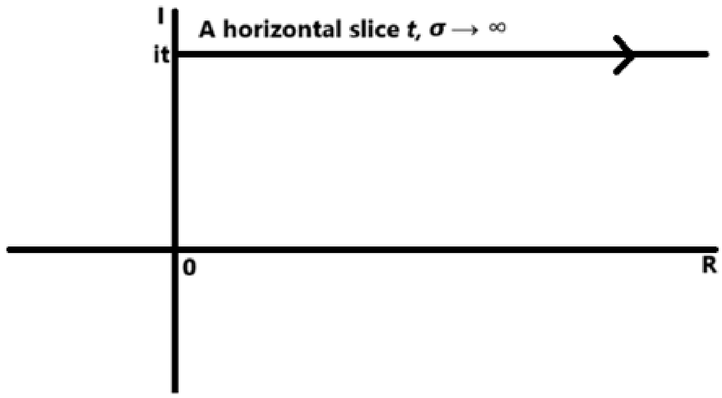

Figure 2: A constant imaginary component *t* of *s* of *η(s)* - Input Complex Plane

By fixing an arbitrary horizontal height *t* across the positive domain, we observe that as *σ* varies continuously, the transformation modifies only the modulus of each term from the second onward in the alternating series *η(s)*, while keeping its phase angle strictly invariant. Consequently, the trajectory of each term is confined to a phase-locked radial ray from the origin, preserving the initial orientation of every term throughout this transformation.

This implies that no term crosses any coordinate axis. Moreover, for any two terms *m* and *n* of *η(s)*, if $|m| > |n|$, *n* possesses a faster decay rate than *m*. These two fundamental properties form the geometric foundation of the proof.

3 Monotonicity Analysis of the Real Part of *η(s)… Along the Slice t*

Therefore, if one side (left/right) of the origin possesses asymptotically a greater number of terms with faster decay rate and consequently, faster decay in net moduli than the other side, this decay dominance of the former side persists. This is due to the fixed angles(phase-lock) of the terms that cause them to strictly lie on one side of the Complex Plane, as *σ* increases continuously from 0^+ *toward* ∞.

The real part of *η(s)* asymptotically approaches 1 as the real component *σ* approaches infinity, for any constant imaginary component *t*.

Case A Consider the real part of *η(s)* is initially negative (at the left of the origin) and also faster decay in net moduli occurs to the left. There are two possible scenarios.

Case A_1 As *σ* increases, the real part is seamlessly monotonic rightward until it reaches its asymptotic limit 1.

Case A_2 As σ increases, the initial real part may continue moving leftward for some time due to the strength of the initial bias to the left. At a particular instance, the faster decaying left-side loses to the right in the net moduli during this transformation. This triggers a unique reversal at a well-defined leftmost point, after which it is monotonic toward right until it reaches its asymptotic limit 1. This is because after losing once to the right, the left cannot become dominant again.

In both cases of A_1 and A_2, the real part vanishes exactly once as illustrated in Figure 3 below.

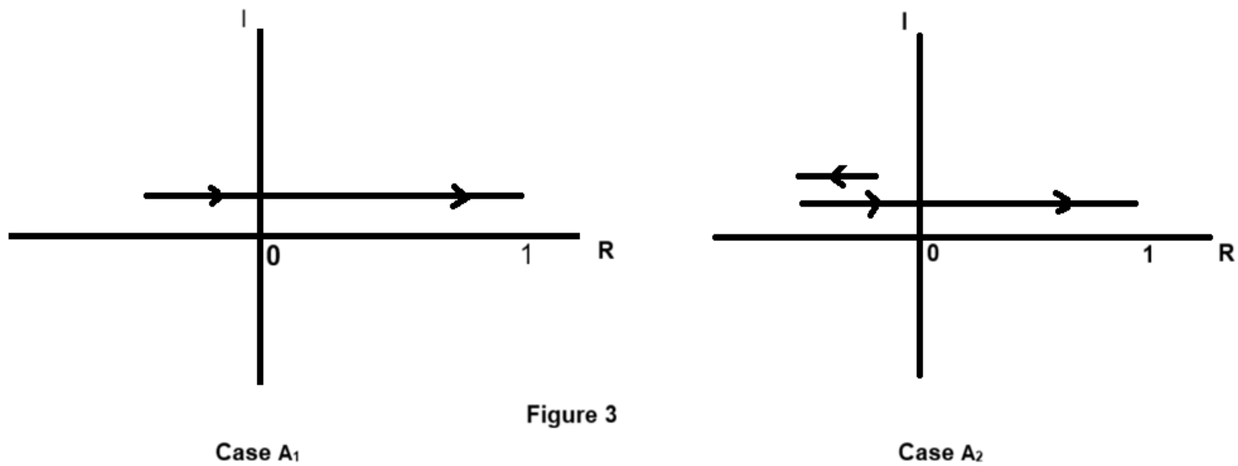

Figure 3

Case B Consider the real part of *η(s)* is initially positive (at the right of the origin) and also the faster decay in net moduli occurs to the right. There are three possible scenarios.

Case B_1 As σ increases, the real part is seamlessly monotonic rightward until it reaches its asymptotic limit 1 due to the strength of the initial bias to the right.

Case B_2 As σ increases, the initial real part may continue moving rightward and reach a well-defined point greater than 1 due to the initial bias to the right. At a particular instance, the faster decaying right-side loses to the left in the net moduli during this transformation. This triggers a unique reversal at that rightmost point, after which it is monotonic toward left until it reaches its asymptotic limit 1. This is because after losing once to the left, the right cannot become dominant again.

Case B_3 The initial real part may be greater than 1. As σ increases, it is seamlessly monotonic leftward until it reaches its asymptotic limit 1.

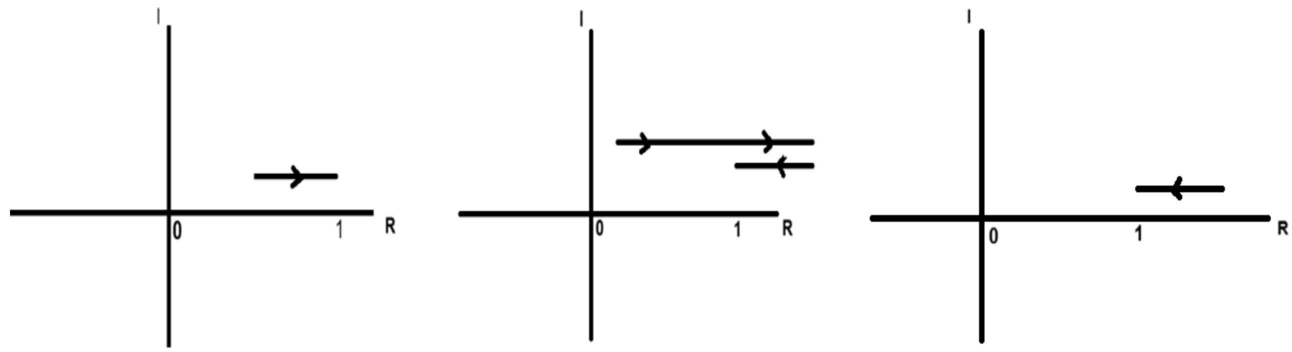

Figure 4

Case B_1 Case B_2 Case B_3

Remark. In Case B_2 illustration, the initial value of the real part is depicted to lie between 0 and 1. It can also be greater than 1.

Case C Consider the real part of *η(s)* is initially positive (between 0 and 1) and the faster decay in net moduli to the left. There is only one possible scenario. As *σ* increases, the real part is seamlessly monotonic rightward until it reaches its asymptotic limit 1.

Case D In case the decay rates of terms are asymptotically balanced on either side, the movement will continue monotonically in the opposite direction to the initial bias. That is, the initial value of the real part must be greater than 1 so that it is seamlessly monotonic leftward until it reaches its asymptotic limit 1.

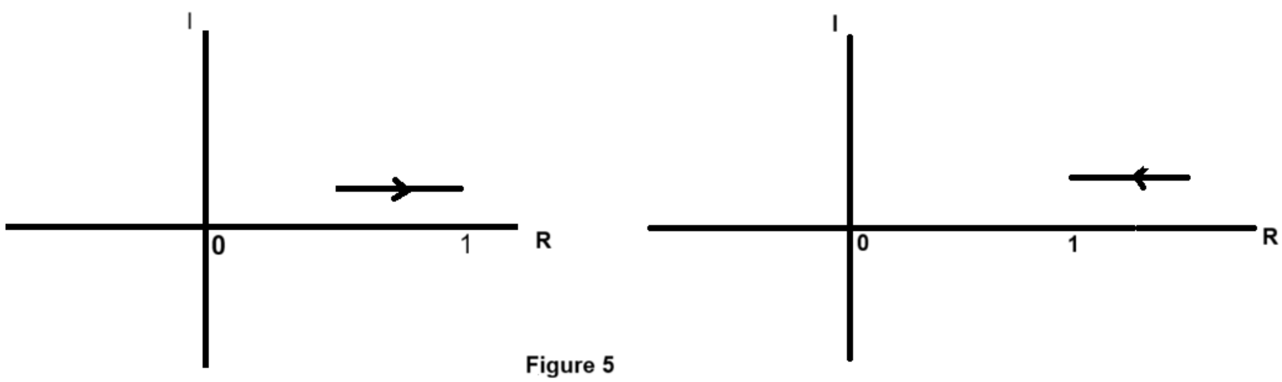

Figure 5

Case C **Case D**

Remark. The cases B_1 and C are similar, and so are the cases B_3 and D. In all cases B_1, B_2, B_3, C and D, the real part does not vanish at all as illustrated in Figure 4 and Figure 5.

Hence, we conclude that the real part of *η(s)* can undergo at most a single change in monotonicity across the entire positive domain of σ through a slice *t.* Consequently, it can at most vanish once along that slice *t* (both Case A_1 and A_2).

Similarly, we can establish that the imaginary part can also vanish only once across the slice *t.*

4 Conclusion

As a result, only one non-trivial zero for the Dirichlet eta function $\eta(s)$ can occur for any given constant imaginary component *t.* Therefore, it must lie exactly on the critical line. Given the identity in Eq. (3), the Riemann zeta function $\zeta(s)$ inherits this same property with regard to its non-trivial zeros.

Theorem 1 (Main Result). *All non-trivial zeros of the Riemann zeta function $\zeta(s)$ lie exactly on the critical line Re(s) =* $\frac{1}{2}$ *.*

Reference

- **Riemann, B. (1859).** *Über die Anzahl der Primzahlen unter einer gegebenen Grösse* (On the Number of Primes Less than a Given Magnitude). Monatsberichte der Berliner Akademie.