

# STRUCTURES OF FINITE AND INFINITE TYPES OF NON-COMPACT HYPERBOLIC SURFACES VIA FUNDAMENTAL GROUPS

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ABSTRACT. The purpose of this article is to introduce the notion of constructing any arbitrary finite and infinite types of non-compact hyperbolic Riemann surfaces via (non-abelian) fundamental groups equipped with various types of classical Schottky structures, with limit sets as uncountable sets (but not necessarily Cantor sets), emphasising the cases in which the surfaces are of infinite hyperbolic areas. In particular, in this paper, the primary goal is to establish the existence of a canonical non-compact infinite area Fuchsian polygon with the help of various classical Schottky structures in the hyperbolic plane. After that, we have initiated a structure of an arbitrary finite type non-compact hyperbolic Riemann surface with genus, conformal holes, cusps, and funnel ends by using the canonical Fuchsian Schottky polygons. Furthermore, in this manuscript, we have proposed the ideas of infinite types conformally compact and semi-conformally compact hyperbolic Riemann surfaces, respectively. Indeed, we have constructed four new and interesting types of infinite type hyperbolic Riemann surfaces (we call generalized flute surfaces) that are constructed from infinite sequences of infinite area hyperbolic pair of pants, each glued to the next along a common geodesic boundary with certain methodologies.

## 1. INTRODUCTION

In [17], Keen initiated the construction of all possible canonical polygons for finitely generated Fuchsian groups that give rise to mainly compact hyperbolic Riemann surfaces. Furthermore, Keen [18] introduced the notion of constructing any compact (hyperbolic) Riemann surface of type  $(g, 0, 0, 0, 0)$  via fundamental (Fuchsian) groups, where  $g$  indicates the genus. So, it is natural to raise the following questions:

**Question 1.** Is it possible to extend Keen's works ([17] and [18]) from finite area (not necessarily compact) hyperbolic Riemann surfaces to infinite area non-compact ones?

**Question 2.** What happens if someone wishes to generalize Keen's work ([18]) from finitely generated Fuchsian groups to infinitely generated ones? Does it create new surfaces in the infinite type Riemann surface theory?

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In this paper, we have provided affirmative answers to the above questions by establishing the following Theorems 1.1, 1.2, and 1.3. In particular, Theorems 1.1 and 1.2 indicate the answer to Question 1; however, Theorem 1.3 responds to Question 2.

We have adopted the idea, first used by Fenchel-Nielsen ([13] and [14]) and later by Keen [18], of regarding the (surface) groups as free products with amalgamation of (certain) subgroups with smaller signatures, although our procedure is completely different from that in [18] or [14].

More precisely, the aim of this study is threefold.

- (1) To extend Keen's work [17] from compact to non-compact hyperbolic Riemann surfaces, we have fabricated three types of canonical non-compact infinite area Fuchsian polygons equipped with various types of classical Schottky structures within the Poincaré disk model (see Theorem 1.1).
- (2) Secondly, we have formulated a strategy for how to construct any finite type arbitrary non-compact hyperbolic Riemann surfaces via fundamental (Fuchsian) groups. In fact, this investigation generalizes the paper of Keen [18] from compact Riemann surfaces to non-compact ones (see Lemma 4.1, Lemma 5.1, Lemma 6.1, Lemma 7.1, and Theorem 1.2).
- (3) Finally, we have introduced four new types of infinite type hyperbolic surfaces, which we named as generalized flute surfaces (flute surface was first introduced in a paper of Basmajian [6]), linked to various types of infinite dimensional Teichmüller spaces. Of course, the way flute surface was structured by Basmajian ([5], [6]), the flute surfaces and generalized flute surfaces are almost equivalent, but from the point of view of the infinite dimensional Teichmüller theory (in particular, various types of metrics, viz., length spectrum metric, Fenchel-Nielsen metric, bi-Lipschitz metric, and quasiconformal metric, see [2], [3], [1], and [20]), these two kinds of infinite type surfaces (i.e., flute surface and generalized flute surface) are not analogous. In particular, we constructed four types of infinite type generalized non-tight flute surfaces and four types of infinite type generalized tight flute surfaces, with limit sets that are Cantor sets and the whole circle at infinity, respectively. In fact, Basmajian [5] introduced the notion of infinitely generated Fuchsian groups from the perspective of hyperbolic Riemann surface theory. Subsequently, Schwartz [27] initiated the idea of infinitely generated Schottky groups equipped with the notion of limit sets. Although these two works were developed from completely different perspectives. In this paper, the infinitely generated groups that we aim to construct are basically Fuchsian groups with Schottky flavor, that is, Fuchsian groups equipped with classical Schottky structures. More precisely, the infinitely generated groups that Basmajian [5] formulated, although the associated hyperbolic surface is of infinite area, each block is non-compact and

of finite area. On the other hand, Schwartz's [27] constructed infinitely generated group gives rise to a compact hyperbolic Riemann surface. However, in this study, we deal with infinitely generated groups that lead to non-compact Riemann surfaces; additionally, each block may carry an infinite hyperbolic area. Furthermore, it is well known that the limit sets of any non-trivial finitely generated and infinitely generated Schottky groups are usually Cantor sets (see [25]). In this investigation, we have proposed some finitely generated and infinitely generated Schottky groups whose limit sets are uncountable sets, but not necessarily Cantor sets, rather the whole circle at infinity. In essence, the study of infinitely generated Fuchsian groups of signatures  $(0, 0, \infty, 0, 0)$ ,  $(0, 0, 0, \infty, 0)$ , and  $(0, 0, \infty, \infty, 0)$  is somewhat known in literature. In this article, we have introduced the notion of infinitely generated Fuchsian groups of signatures  $(0, 0, 0, 0, \infty)$ ,  $(0, 0, 0, \infty, \infty)$ ,  $(0, 0, \infty, 0, \infty)$ , and  $(0, 0, \infty, \infty, \infty)$  (see Theorem 1.3). For an encounter of the infinite type Riemann surface theory of types  $(0, 0, \infty, 0, 0)$ ,  $(0, 0, 0, \infty, 0)$ , and  $(0, 0, \infty, \infty, 0)$ , we refer the readers to the following expositions, e.g., [6], [7], [4], [31], [20], [2], and [8].

In this paper, we have proved the following theorems in this regard. Indeed, the following Theorem 1.1 supplies the construction of any non-compact Fuchsian polygon with infinite hyperbolic areas. On the other hand, Theorem 1.2 provides the structure of an arbitrary finite type non-compact Fuchsian (surface) group with Schottky settings, whereas Theorem 1.3 indicates the construction of infinite types non-compact Fuchsian groups of signatures  $(0, 0, 0, 0, \infty)$ ,  $(0, 0, 0, \infty, \infty)$ ,  $(0, 0, \infty, 0, \infty)$ , and  $(0, 0, \infty, \infty, \infty)$ . In particular, Theorem 1.1 is a generalization of [[17], Theorems 6 and 7], from finite area (not necessarily compact) hyperbolic Riemann surfaces to infinite area non-compact ones. On the other hand, Theorem 1.2 generalizes the work [[18], Theorem 5] from compact hyperbolic Riemann surfaces to non-compact (hyperbolic) Riemann surfaces. Furthermore, Theorem 1.3 extends the result [[18], Theorem 5] from finitely generated Fuchsian groups to infinitely generated ones. In particular, Theorems 1.3 (a), (b), (c), and (d) provide four types of infinite types conformally compact hyperbolic Riemann surfaces; on the other hand, Theorems 1.3 (e), (f), (g), and (h) supply four types of infinite types semi-conformally compact hyperbolic (Riemann) surfaces.

**Theorem 1.1.** *For any non-compact infinite area hyperbolic Riemann surface,  $\exists$  a canonical Fuchsian polygon with Schottky settings.*

**Theorem 1.2.** *Let  $\Omega$  be an arbitrary finite type Fuchsian surface group of signature  $(2, 0, 1, 2, 8)$ . Then, by utilizing the Klein-Maskit combination theorem as a tool, we have proved that  $\Omega = \langle \Omega_1^*, \Omega_3, \Omega_2^{**}, \Omega_5, \Omega_4^{**}, \Omega_6^*, \Omega_8, \Omega_7^* \rangle$ , where  $\Omega_i$ ,  $i = 1, 2, \dots, 8$  are non-compact Fuchsian groups*

with some specified signatures of finite types and the symbol ‘\*’ indicates the number of operations that act in that group (see Figure 1.1).

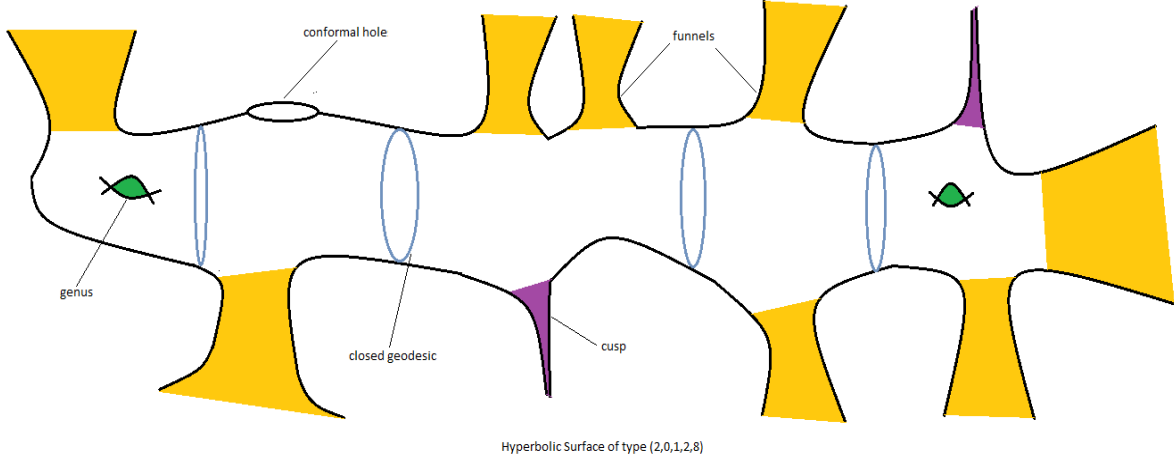


Figure : 1.1

**Theorem 1.3.** Suppose  $\Gamma$ ,  $\Delta$ ,  $\Omega$ ,  $\Phi$ ,  $\Xi$ ,  $\Psi$ ,  $\Sigma$ , and  $\Upsilon$  are Fuchsian groups, where ‘\*’ represents the number of operations that act in that group. Then, by using the Klein-Maskit combination theorem as a tool, we have proved the following results:

(a) Let  $\Gamma_i$  be Fuchsian groups with signatures  $(0, 0, 0, 0, 4)$ ,  $i \in 2\mathbb{N} \cup \{1\}$ . Further, we assume that  $\Gamma_{3+i}$  is a Fuchsian group of signature  $(0, 0, 2, 0, 2)$ , where  $i \in 2\mathbb{N} \cup \{0\}$ . Then, the group  $\Gamma = \langle \Gamma_1^*, \Gamma_3, \Gamma_2^{**}, \Gamma_5, \Gamma_4^{**}, \Gamma_7, \Gamma_6^{**}, \dots \rangle$  is an infinitely generated Fuchsian group of type  $(0, 0, 0, 0, \infty)$  (see Figure 8 (b) in Section 8).

(b) Let  $\Delta_i$  be Fuchsian groups with signatures  $(0, 0, 0, 0, 3)$ ,  $i \in 2\mathbb{N} \cup \{1\}$ . Furthermore, suppose that  $\Delta_{3+i}$  is a Fuchsian group of signature  $(0, 0, 2, 0, 1)$ , where  $i \in 2\mathbb{N} \cup \{0\}$ . Then, the group  $\Delta = \langle \Delta_1^*, \Delta_3, \Delta_2^{**}, \Delta_5, \Delta_4^{**}, \Delta_7, \Delta_6^{**}, \dots \rangle$  is an infinitely generated Fuchsian group of type  $(0, 0, 0, 0, \infty)$  (see Figure 8 (c)).

(c) Let  $\Omega_i$  be Fuchsian groups with signatures  $(0, 0, 0, 0, 3)$ ,  $i \in 2\mathbb{N} \cup \{1\}$ . Additionally, let  $\Omega_{3+i}$  be a Fuchsian group of signature  $(0, 0, 3, 0, 0)$ , where  $i \in 2\mathbb{N} \cup \{0\}$ . Then, the group  $\Omega = \langle \Omega_1^*, \Omega_3, \Omega_2^{**}, \Omega_5, \Omega_4^{**}, \Omega_7, \Omega_6^{**}, \dots \rangle$  is an infinitely generated Fuchsian group of type  $(0, 0, \infty, 0, \infty)$  (see Figure 8 (d)).

(d) Let  $\Phi_i$  be Fuchsian groups with signatures  $(0, 0, 1, 0, 3)$ ,  $i \in 2\mathbb{N} \cup \{1\}$ . Moreover, we assume that  $\Phi_{3+i}$  is a Fuchsian group of signature  $(0, 0, 3, 0, 1)$ , where  $i \in 2\mathbb{N} \cup \{0\}$ . Then, the group  $\Pi = \langle \Phi_1^*, \Phi_3, \Phi_2^{**}, \Phi_5, \Phi_4^{**}, \Phi_7, \Phi_6^{**}, \dots \rangle$  is an infinitely generated Fuchsian group of type  $(0, 0, \infty, 0, \infty)$  (see Figure 8 (e)).

(e) Let  $\Xi_i$  be Fuchsian groups with signatures  $(0, 0, 0, 1, 3)$ ,  $i \in 2\mathbb{N} \cup \{1\}$ . In addition, suppose that  $\Gamma_{3+i}$  is a Fuchsian group of signature  $(0, 0, 2, 1, 1)$ , where  $i \in 2\mathbb{N} \cup \{0\}$ . Then, the group  $\Xi = \langle \Xi_1^*, \Xi_3, \Xi_2^{**}, \Xi_5, \Xi_4^{**}, \Xi_7, \Xi_6^{**}, \dots \rangle$  is an infinitely generated Fuchsian group of type  $(0, 0, 0, \infty, \infty)$  (see Figure 8 (f)).

(f) Let  $\Psi_i$  be Fuchsian groups with signatures  $(0, 0, 0, 0, 3)$ ,  $i \in 2\mathbb{N} \cup \{1\}$ . Furthermore, let  $\Pi_{3+i}$  be a Fuchsian group of signature  $(0, 0, 2, 1, 0)$ , where  $i \in 2\mathbb{N} \cup \{0\}$ . Then, the group  $\Psi = \langle \Psi_1^*, \Pi_3, \Psi_2^{**}, \Pi_5, \Psi_4^{**}, \Pi_7, \Psi_6^{**}, \dots \rangle$  is an infinitely generated Fuchsian group of type  $(0, 0, 0, \infty, \infty)$  (see Figure 8 (g)).

(g) Let  $\Sigma_i$ ,  $\Pi_i$ , and  $\Upsilon_i$  be Fuchsian groups with signatures  $(0, 0, 0, 0, 3)$ ,  $(0, 0, 1, 0, 2)$ , and  $(0, 0, 0, 1, 2)$  respectively,  $i \in \mathbb{N}$ . Then, the group  $\Upsilon = \langle \Sigma_1^*, \Pi_1, \Upsilon_1^{**}, \Sigma_2, \Pi_2^{**}, \Upsilon_2, \Sigma_3^{**}, \Pi_3, \Upsilon_3^{**}, \dots \rangle$  is an infinitely generated Fuchsian group of type  $(0, 0, \infty, \infty, \infty)$  (see Figure 8 (h)).

(h) Let  $\Lambda_i$ ,  $\Omega_i$ , and  $\Theta_i$  be Fuchsian groups with signatures  $(0, 0, 0, 0, 4)$ ,  $(0, 0, 0, 1, 3)$ , and  $(0, 0, 3, 0, 1)$  respectively,  $i \in \mathbb{N}$ . Then, the group  $\Phi = \langle \Lambda_1^*, \Theta_1, \Omega_1^{**}, \Lambda_2, \Theta_2^{**}, \Omega_2, \Lambda_3^{**}, \Theta_3, \Omega_3^{**}, \dots \rangle$  is an infinitely generated Fuchsian group of type  $(0, 0, \infty, \infty, \infty)$  (see Figure 8 (i)).

It is well known that, unlike the case of a hyperbolic surface of finite type,  $\exists$  various types of Teichmüller spaces corresponding to an infinite type hyperbolic surface, and each of these Teichmüller spaces (heavily) depends on the choice of a basepoint for that Teichmüller space. Moreover, even if someone fixes a base point, the Teichmüller space depends set theoretically on a distance function that can be placed in that space. In fact, there are various types of distances that can be used in the Teichmüller space. In this point of view, the infinite type surfaces coming out from the Fuchsian groups of signatures  $(0, 0, 0, 0, \infty)$  (Theorem 1.2 (a) and (b)),  $(0, 0, \infty, 0, \infty)$  (Theorem 1.2 (c) and (d)),  $(0, 0, 0, \infty, \infty)$  (Theorem 1.2 (e) and (f)), and  $(0, 0, \infty, \infty, \infty)$  (Theorem 1.2 (g) and (h)) are not analogous. In particular, both the infinitely generated groups,  $\Gamma$  and  $\Delta$  with signatures  $(0, 0, 0, 0, \infty)$  provide infinite type surfaces, where each block is of infinite hyperbolic area, but each block of  $\Delta$  contains one funnel end (apart from the first one), whereas for the group  $\Gamma$ , two funnels appear in each block. Furthermore, both the infinitely generated groups  $\Omega$  and  $\Pi$  of signature  $(0, 0, \infty, 0, \infty)$  supply infinite type surfaces, but they are different from the point of view of infinite dimensional Teichmüller spaces. In fact, for group  $\Omega$ , the 1st, 3rd, 5th, 7th, ... blocks have infinite hyperbolic areas, but the 2nd, 4th, 6th, 8th, ... blocks have finite areas (i.e.,  $\pi$ ). For the group  $\Pi$ , each block consists of an infinite hyperbolic area. Moreover, the infinitely generated groups  $\Xi$  and  $\Psi$  of signature  $(0, 0, 0, \infty, \infty)$ , the associated hyperbolic surfaces, for the group  $\Xi$ , each block is of infinite hyperbolic area, whereas, for  $\Psi$ , 1st, 3rd, 5th, 7th, ... blocks are of infinite hyperbolic areas,

but 2nd, 4th, 6th, 8th, ... blocks are of finite areas. Finally, the infinitely generated groups  $\Upsilon$  and  $\Phi$  of signature  $(0, 0, \infty, \infty, \infty)$ , the corresponding infinite type surfaces, for the group  $\Upsilon$ , 1st, 4th, 7th, .... blocks are non-compact hyperbolic areas, whereas 2nd, 5th, 8th, ... blocks are compact finite areas, and 3rd, 6th, 9th, ... blocks are non-compact but finite areas. However, for the group  $\Phi$ , each block has an infinite hyperbolic area.

**Outline of this paper.** The rest of this article is arranged as follows: In Section 2, we discuss some preliminaries that will be needed throughout this paper. In Section 3, we have proved Theorem 1.1 by constructing three (completely different) types of non-compact infinite area Fuchsian polygons with various kinds of classical Schottky structures in the Poincaré disk model. To prove Theorems 1.2 and 1.3, we have utilized Sections 4 to 7 as the technical core. In particular, in Section 4, we have initiated the strategy of gluing two hyperbolic surfaces of type  $(0, 0, 0, 0, f)$  via fundamental groups (see Lemma 4.1). After that, we have approached the technique of attaching two hyperbolic surfaces of type  $(0, 0, 0, c, f)$  in Section 5 (see Lemma 5.1). In Section 6, we have investigated the situations when the non-compact infinite area hyperbolic surface contains conformal holes (see Lemma 6.1). Furthermore, in Section 7, we have studied the circumstances when non-compact hyperbolic surfaces carry conformal holes but of finite area (see Lemma 7.1). Finally, in Section 8, we have proved Theorems 1.2 and 1.3. At the end, we proposed a brief notion about the limit sets of our constructed infinitely generated groups.

## 2. PRELIMINARIES

**Definition 2.1.** A Fuchsian group is a discrete subgroup of  $\mathrm{PSL}(2, \mathbb{R})$  (or one can say  $\mathrm{SL}(2, \mathbb{R})$ ). In particular, a Fuchsian group is basically the fundamental group of the hyperbolic surface (see [9], [19], and [16]).

**Definition 2.2.** A Kleinian group is the three-dimensional version of the Fuchsian group. More precisely, a Kleinian group is a discrete subgroup of  $\mathrm{PSL}(2, \mathbb{C})$  (see [23]).

**Definition 2.3.** A Schottky group is basically a special type of Kleinian group equipped with a standard construction within the Riemann sphere. Interestingly, in this article, we use the two-dimensional version of the Schottky group, that is, a Schottky group is a Fuchsian group equipped with a certain classical structure within the hyperbolic plane with a circle at infinity as the boundary. See [32] and [12] (or [29]) for the constructions of classical Schottky structures in three-dimension and two-dimension, respectively.

**Definition 2.4.** A hyperbolic Riemann surface is said to be of finite type if it provides a finitely generated fundamental group. On the other hand, a hyperbolic surface is called of infinite type if it supplies an infinitely generated fundamental group.

**Definition 2.5.** A funnel is basically a hyperbolic surface having one (geodesic) boundary component that is isometric to  $D / \langle z \rightarrow e^l z \rangle$ , where  $D = \{z \in U : \Re(z) \leq 0\}$  has the induced metric as a subspace of the upper-half plane model  $U$  of the hyperbolic plane, where  $\Re(z)$  denotes the real part of  $z = x + iy$ .

**Definition 2.6.** A cusp is also a hyperbolic surface having one horocyclic boundary component that is isometric to the quotient  $\{z : \Im(z) \geq 1\} / \langle z \rightarrow z + 1 \rangle$ , where  $\{z : \Im(z) \geq 1\}$  has the induced metric as a subspace of the upper-half plane model, where  $\Im(z)$  denotes the imaginary part of  $z$ .

**Definition 2.7.** A conformally compact Riemann surface is basically a non-compact hyperbolic surface. A hyperbolic Riemann surface is said to be conformally compact if it is of infinite hyperbolic area endowed with a metric of Gaussian curvature  $-1$ , providing a non-abelian finitely generated fundamental group, and without a cusp end.

**Definition 2.8.** A semi-conformally compact hyperbolic Riemann surface is of type  $(g, p, h, c, f)$ , where  $g$  is the number of genus,  $p$  and  $h$  are the distinguished points and conformal disks (that are removed from its compact core),  $c$  and  $f$  are the numbers of cusp and funnel ends respectively.

We assert that  $g, p, h$ , and  $c \geq 0$ , but  $f > 0$  always. Moreover, if  $p = p_1, p_2, \dots, p_m$ , and  $h = h_1, h_2, \dots, h_n$ ,  $p_i \neq p_j$ , for  $i \neq j$ , and  $h_i \cap h_j = \emptyset$ ,  $i \neq j$ , with  $p_i \notin h_j$ . Now, if the genus  $g = 0$  and  $h + c + f = 3$ , we say that the quotient surface is a pair of pants (see [28] and [30] for more details).

In fact, a conformally compact hyperbolic surface can contain only funnel ends in the boundary of its compact core. On the other hand, a semi-conformally compact hyperbolic Riemann surface is also a non-compact hyperbolic surface, but the notion is slightly more general compared to a conformally compact hyperbolic surface. In particular, a semi-conformally compact hyperbolic surface may also contain cusp ends. For the notion of infinite area hyperbolic surfaces with finitely generated fundamental groups, see [10], [22], and [26].

In this article, we have explored torsion-free Fuchsian groups, so there is no elliptic element in the group. In fact, Keen ([17] and [18]) studied the situations for containing elliptic elements

in Fuchsian groups. So, we have exempted that case here. More precisely, in this manuscript, we are mainly interested in the type  $(g, 0, h, c, f)$ . It is well known that a Fuchsian (surface) group is said to be a pair of pants if its (associated) quotient surface is topologically a sphere with three holes. In particular, a pair of pants is a surface of three types, e.g.,  $(0, 0, 3, 0, 0)$ ,  $(0, 0, 2, 1, 0)$ , and  $(0, 0, 1, 2, 0)$ .

**Definition 2.9.** A flute surface is a special type of infinite type hyperbolic surface that looks like an infinite chain of (finite area) pairs of pants glued end to end.

Topologically, a flute surface is homeomorphic to a sphere, removing a Cantor set of points. Although geometrically, a flute surface carries a hyperbolic metric.

**Definition 2.10.** We define a flute surface to be tight if it contains at least one (hence countably infinite number) cusp end. If an infinite type flute surface doesn't contain any cusp ends, we call it non-tight.

For more on the basics of a hyperbolic manifold, funnel, and cusp, pair of pants, and flute surfaces, the readers may go through the expositions, for instance, [21], [11], [28], [7], [10], [6], and [24].

### 3. PROOF OF THE FIRST MAIN THEOREM

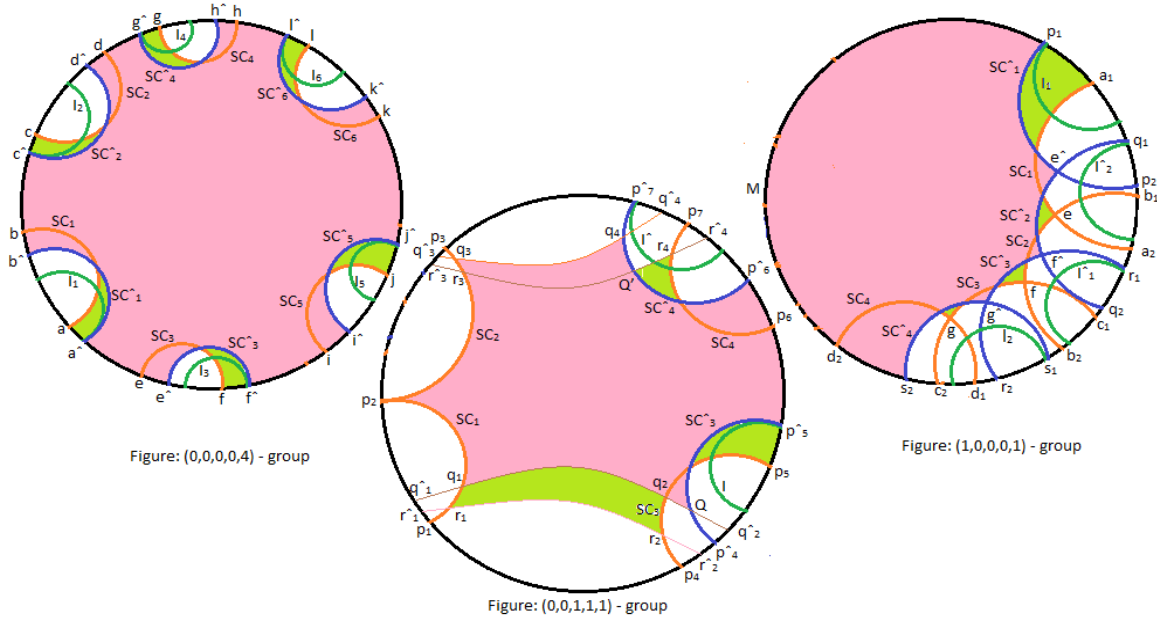


Figure : 3.1

**Proof of Theorem 1.1.** To prove this theorem, we have constructed three different types of non-compact infinite area Fuchsian polygons equipped with various kinds of classical Schottky structures in the Poincaré disk model. In particular, we initiated the canonical polygons of the groups  $(0, 0, 0, 0, 4)$ ,  $(0, 0, 1, 1, 1)$ , and  $(1, 0, 0, 0, 1)$ , which we have presented as follows (see Figure 3.1).

Constructing canonical Fuchsian Schottky polygon 1.

Step 1. Let us consider six semi-circles  $SC_1, SC_2, SC_3, SC_4, SC_5,$  and  $SC_6$  within the Poincaré disk model with end points  $a, b; c, d; e, f; g, h; i, j; k, l$  respectively lying on the boundary of the disk.  $SC_1$  is paired with  $SC_2$ ,  $SC_3$  twins with  $SC_4$ , and  $SC_5$  is linked with  $SC_6$  by three hyperbolic Möbius transformations. Now, the open region (looks like a polygon) bounded by these six semi-circles with arcs lying on the boundary of the disk represents the open non-compact polygon of a  $(0, 0, 0, 0, 4)$  group. In particular, the area surrounded by the non-Euclidean lines joining the points  $a, e; f, i; j, k; l, h; g, d, c; d$ , and the semi-circles  $SC_1, SC_2, SC_3, SC_4, SC_5,$  and  $SC_6$  exhibits the fundamental domain for the group  $(0, 0, 0, 0, 4)$ . Now, for a  $(0, 0, 0, 0, 4)$  group, to make the non-compact polygon canonical, we have utilized the following procedure:

Step 2. Let,  $\Omega^1 = \langle g_1, g_2, g_3 \rangle$  be a group with signature  $(0, 0, 0, 0, 4)$ . Note that all three generators  $g_1, g_2,$  and  $g_3$  are hyperbolic Möbius transformations. Now, we consider the three isometric circles  $I_1, I_2,$  and  $I_3$  corresponding to the right side of the three axes of the hyperbolic elements  $g_1, g_2,$  and  $g_3$  respectively (see [15] for the notion of isometric circles). Suppose the end points of the isometric circles  $I_1, I_2,$  and  $I_3$  are  $\hat{a}$  (between the points  $a$  and  $e$ ),  $\hat{f}$  (between the points  $f$  and  $i$ ), and  $\hat{j}$  (between the points  $j$  and  $k$ ) respectively. Let,  $g_1(\hat{a}) = \hat{d}$ ,  $g_2(\hat{f}) = \hat{h}$ , and  $g_3(\hat{j}) = \hat{k}$ . Now, to fabricate a canonical polygon for the group  $(0, 0, 0, 0, 4)$ , we draw six semi-circles in the following way. We translate six semi-circles  $SC_1$  to  $\hat{S}\hat{C}_1$  with end point at  $\hat{a}$  (by the hyperbolic curve length  $(\hat{a} - a)$ ),  $SC_2$  with end point at  $\hat{d}$  (by the hyperbolic curve length  $(\hat{d} - d)$ ),  $SC_3$  to  $\hat{S}\hat{C}_3$  with end point at  $\hat{f}$  (by the hyperbolic curve length  $(\hat{f} - f)$ ),  $SC_4$  to  $\hat{S}\hat{C}_4$  with end point at  $\hat{h}$  (by the hyperbolic curve length  $(\hat{h} - h)$ ),  $SC_5$  to  $\hat{S}\hat{C}_5$  with end point at  $\hat{j}$  (by the hyperbolic curve length  $(\hat{j} - j)$ ), and  $SC_6$  to  $\hat{S}\hat{C}_6$  with end point at  $\hat{k}$  (by the hyperbolic curve length  $(\hat{k} - k)$ ). Let  $\hat{b}, \hat{c}, \hat{e}, \hat{g}, \hat{i}, \hat{l}$  be the starting points of the semi-circles  $\hat{S}\hat{C}_1, \hat{S}\hat{C}_2, \hat{S}\hat{C}_3, \hat{S}\hat{C}_4, \hat{S}\hat{C}_5, \hat{S}\hat{C}_6,$  respectively. Now, the region bounded by the semi-circles  $\hat{S}\hat{C}_1, \hat{S}\hat{C}_2, \hat{S}\hat{C}_3, \hat{S}\hat{C}_4, \hat{S}\hat{C}_5, \hat{S}\hat{C}_6,$  and the arcs joining by the points  $\hat{a}$  to  $\hat{e}, \hat{f}$  to  $\hat{i}, \hat{j}$  to  $\hat{k}, \hat{l}$  to  $\hat{h}, \hat{g}$  to  $\hat{d}, \hat{c}$  to  $\hat{b}$  lying on the boundary of the disk represents the canonical polygon of the  $(0, 0, 0, 0, 4)$  group.

### Constructing canonical Fuchsian Schottky polygon 2.

Step 1. Let us consider four semi-circles  $SC_1$ ,  $SC_2$ ,  $SC_3$ , and  $SC_4$  within the Poincaré disk model with end points  $p_1, p_2; p_2, p_3; p_4, p_5; p_6, p_7$  respectively lying on the boundary of the disk.  $SC_1$  is paired with  $SC_2$  by a parabolic Möbius transformation  $p$ , and  $SC_3$  is linked with  $SC_4$  by a hyperbolic Möbius transformation  $h$ . Now, the open region (looks like a polygon) bounded by these four semi-circles  $SC_1$ ,  $SC_2$ ,  $SC_3$ , and  $SC_4$  represents an open non-compact polygon of a  $(0, 0, 0, 1, 2)$  group. In particular, the area surrounded by the non-Euclidean lines joining the points  $p_1, p_4; p_5, p_6; p_7, p_3$  (all lying on the boundary of the disk) and the semi-circles  $SC_1$ ,  $SC_2$ ,  $SC_3$ , and  $SC_4$  exhibits the fundamental domain for the group  $(0, 0, 0, 1, 2)$ . Now, we consider two non-Euclidean lines; one intersects at  $r_1$  in  $SC_1$  and  $r_2$  in  $SC_3$ ; the other one intersects at  $r_3$  in  $SC_2$  and  $r_4$  in  $SC_4$ . We produce  $r_2r_1$  to  $\hat{r}_1$ ,  $r_1r_2$  to  $\hat{r}_2$ ,  $r_4r_3$  to  $\hat{r}_3$ , and  $r_3r_4$  to  $\hat{r}_4$  in the boundary of the disk. The region surrounded by the arcs joining the points  $p_2, r_1, r_2, p_5, p_6, r_4, r_3$ , and  $p_7$  represents the fundamental polygon of a  $(0, 0, 1, 1, 1)$  group. Now, to make this polygon canonical, we have used the following procedure:

Step 2. Let  $\Omega^2 = \langle p, h \rangle$  be a group with signature  $(0, 0, 1, 1, 1)$ . Here,  $p$  is parabolic, whereas  $h$  is a hyperbolic Möbius transformation. Now, let us consider the isometric circle  $I$  corresponding to the right side of the axis of the hyperbolic element  $h$ . Assume that the end point of the isometric circle  $I$  is  $\hat{p}_5$  (between the points  $p_5$  and  $p_6$ ) on the boundary of the Poincaré disk. Let,  $h(\hat{p}_5) = \hat{p}_6$ . Now, to construct a canonical polygon for the group  $(0, 0, 1, 1, 1)$ , we draw two semi-circles  $I$ , and  $\hat{I}$ , in the subsequent way. We translate these two semi-circles  $SC_3$  to  $\hat{SC}_3$  with end point at  $\hat{p}_4$  (by the hyperbolic curve length  $(\hat{p}_5 - p_5)$ ),  $SC_4$  with end point at  $\hat{p}_7$  (by the hyperbolic curve length  $(\hat{p}_6 - p_6)$ ). Let  $\hat{p}_5$  and  $\hat{p}_6$  be the starting points of the semi-circles  $\hat{SC}_3$  and  $\hat{SC}_4$ , respectively. Now, on the boundary of the disk, we take four points  $\hat{q}_1$  (between  $p_2$  and  $\hat{r}_1$ ),  $\hat{q}_2$  (between  $\hat{p}_5$  and  $\hat{p}_4$ ),  $\hat{q}_3$  (between  $\hat{r}_3$  and  $p_3$ ),  $\hat{q}_4$  (between  $\hat{p}_7$  and  $\hat{r}_4$ ). We produce the non-Euclidean arcs  $\hat{q}_1\hat{q}_2$  (intersecting  $SC_1$  at  $q_1$ , and  $\hat{SC}_3$  at  $Q$ ), and  $\hat{q}_3\hat{q}_4$  (intersecting  $SC_2$  at  $q_3$ , and  $\hat{SC}_4$  at  $Q'$ ). The region bounded by the non-Euclidean arcs joining by the points  $p_2$  to  $q_1$ ,  $q_1$  to  $q_2$ ,  $q_2$  to  $Q$ ,  $Q$  to  $\hat{p}_5$ ,  $\hat{p}_5$  to  $p_6$ ,  $p_6$  to  $\hat{p}_6$ ,  $\hat{p}_6$  to  $Q'$ ,  $Q'$  to  $q_4$ ,  $q_4$  to  $q_3$ ,  $q_3$  to  $r_3$ , and  $r_3$  to  $p_2$  represents the canonical polygon for the group  $(0, 0, 1, 1, 1)$ .

### Constructing canonical Fuchsian Schottky polygon 3.

Step 1. Let us consider four semi-circles  $SC_1$ ,  $SC_2$ ,  $SC_3$ , and  $SC_4$  within the Poincaré disk model with end points  $a_1, a_2; b_1, b_2; c_1, c_2; d_1, d_2$  respectively that are lying on the boundary of the disk.  $SC_1$  is paired with  $SC_3$ , and  $SC_4$  twins with  $SC_2$  by two hyperbolic Möbius transformations  $h_1$  and  $h_2$ , respectively. The open region (looks like a polygon) encircled by

these four semi-circles equipped with the arc lying on the boundary of the disk represents the non-compact polygon for a  $(1, 0, 0, 0, 1)$  group. Let  $M$  be a point situated on the boundary of the disk at the point  $(-1, 0)$ . Furthermore, we assume that the semi-circles  $SC_1$  and  $SC_2$  intersect at the point  $e$ ,  $SC_2$  and  $SC_3$  meet at  $f$ , and  $SC_3$  and  $SC_4$  intersect at the point  $g$ . Note that all the points  $e, f$ , and  $g$  are lying within the disk. Now, the area surrounded by the non-Euclidean lines joining the points  $d_2, g; g, f; f, e; e, a_1; a_1, M; M, d_2$  exhibits the fundamental domain for the group  $(1, 0, 0, 0, 1)$ . Now, our goal is to produce the canonical polygon of the group  $(1, 0, 0, 0, 1)$ .

Step 2. Let  $\Omega^3 = \langle h_1, h_2 \rangle$  be a group with signature  $(1, 0, 0, 0, 1)$ . Here, both the generators  $h_1$  and  $h_2$  are hyperbolic Möbius transformations. Now, we consider two isometric circles  $I_1$  and  $I_2$ , corresponding to the two axes of the hyperbolic transformations  $h_1$  and  $h_2$ , respectively. Suppose the end points of the isometric circles  $I_1$  and  $I_2$  are  $p_1$  (between the points  $a_1$  and  $M$ ) and  $s_1$  (between the points  $d_1$  and  $b_2$ ), respectively. Let,  $h_1(p_1) = r_1$ , and  $h_2(s_1) = q_1$ . Now, to fabricate a canonical polygon for the group  $(1, 0, 0, 0, 1)$ , we draw four semi-circles in the following way. We translate four semi-circles  $SC_1$  to  $\hat{S}C_1$  with end point at  $p_1$  (by the hyperbolic curve length  $(p_1 - a_1)$ ),  $SC_2$  to  $\hat{S}C_2$  with end point at  $q_1$  (by the hyperbolic curve length  $(q_1 - b_1)$ ),  $SC_3$  to  $\hat{S}C_3$  with end point at  $r_1$  (by the hyperbolic curve length  $(r_1 - c_1)$ ),  $SC_4$  to  $\hat{S}C_4$  with end point at  $s_1$  (by the hyperbolic curve length  $(s_2 - d_1)$ ). Let,  $p_2, q_2, r_2$ , and  $s_2$  are the starting points of the semi-circles  $\hat{S}C_1, \hat{S}C_2, \hat{S}C_3$ , and  $\hat{S}C_4$  respectively. Let the semi-circles  $\hat{S}C_1$  and  $\hat{S}C_2$  intersect at the point  $\hat{e}$ ,  $\hat{S}C_2$  and  $\hat{S}C_3$  meet at  $\hat{f}$ , and finally,  $\hat{S}C_3$  and  $\hat{S}C_4$  intersect at the point  $\hat{g}$ . Now, the region bounded by the arcs joining the points  $s_2$  to  $\hat{g}$ ,  $\hat{g}$  to  $\hat{f}$ ,  $\hat{f}$  to  $\hat{e}$ ,  $\hat{e}$  to  $p_1$ ,  $p_1$  to  $M$ ,  $M$  to  $s_2$  represents the canonical polygon of the  $(1, 0, 0, 0, 1)$  group.

In this way, by utilizing the above three constructions, one can originate any canonical non-compact infinite area Fuchsian polygons equipped with various kinds of classical Schottky structures in the hyperbolic plane. This completes the proof of Theorem 1.1.  $\square$

In the subsequent Sections 4, 5, 6, and 7, we have described a general method for constructing new non-compact finite and infinite area hyperbolic surfaces by gluing along infinite funnel boundaries with certain methodologies. The essential tool we employ here is the (Klein-Maskit) combination theorem (see [18], Theorem 1 for an exposition of the combination theorem) that allows us to amalgamate two non-compact Fuchsian groups to form a larger discrete group. In particular, Sections 4 to 6 deal with infinite area hyperbolic surfaces, whereas Section 7 concerns finite area surfaces.

4. GLUING NON-COMPACT HYPERBOLIC SURFACES OF TYPE  $(0,0,0,0,f)$  VIA  
FUNDAMENTAL GROUPS

**Lemma 4.1.** (Construction 1.) *Let  $\Omega_1$  be a Fuchsian group of signature  $(0,0,0,0,3)$  with neck lengths  $\kappa_1, \kappa_2, 2\kappa_3$ , and  $\Omega_2$  be a Fuchsian group of signature  $(0,0,0,0,3)$  with neck lengths  $\kappa_4, \kappa_5, 2\kappa_6$ . Suppose both the corresponding surfaces possess different neck lengths  $\kappa_i, i = 1, 2, \dots, 6$ , where  $\kappa_i \neq \kappa_j$ , for  $i \neq j$ . Then:*

(1) *For a group  $\Pi_3$  of signature  $(0,0,2,0,1)$  consisting of the lengths of their closed geodesics (of the conformal holes)  $\kappa_2$  and  $\kappa_4$ , with neck length  $2\kappa_7$ , we prove that  $\Omega^1 = \langle \Omega_1^*, \Pi_3, \Omega_2^* \rangle$ , where ‘\*’ indicates the number of operations acting on that group.*

(2) *Gluing two surfaces of types  $(0,0,0,0,3)$  yields a hyperbolic surface of type  $(0,0,0,0,5)$ . In particular,  $\Omega^1$  is a hyperbolic surface of five funnel ends with neck lengths  $\kappa_1, 2\kappa_3, 2\kappa_7, 2\kappa_6$ , and  $\kappa_5$ .*

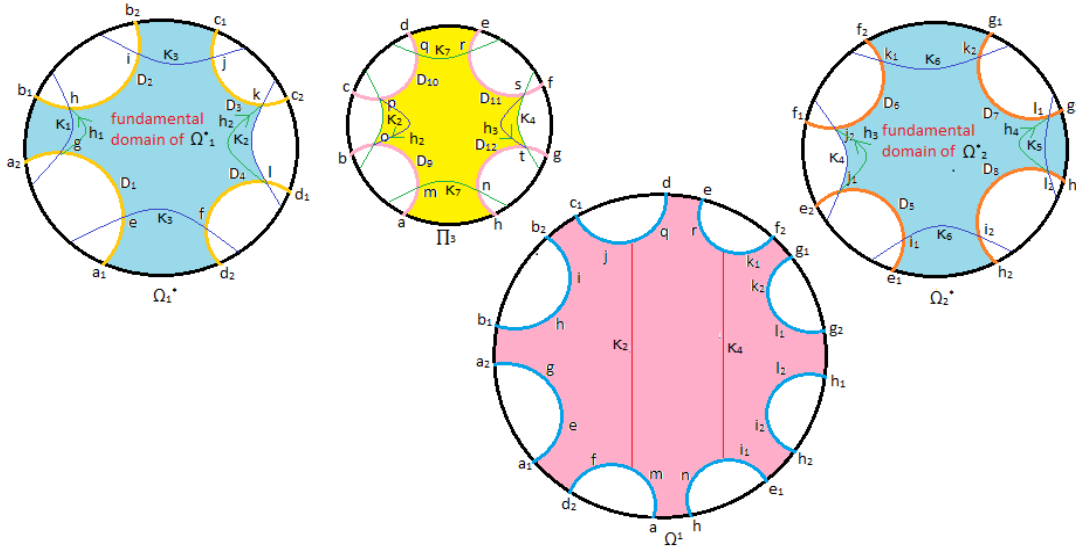


Figure : 4.1

*Proof.* Let  $D_1, D_2, D_3$ , and  $D_4$  be four semi-circles within the Poincaré disk model. In particular,  $D_1$  is paired with  $D_2$  by the hyperbolic Möbius transformation  $h_1$ , whereas  $D_4$  twins with  $D_3$  by the hyperbolic Möbius transformation  $h_2$ . Suppose these four semi-circles meet the boundary of the disk at the points  $a_1, a_2; b_1, b_2; c_1, c_2; d_1, d_2$  respectively. Now, on the boundary of the disk, we take eight points, two between  $a_1$  and  $a_2$ , viz.,  $\tilde{e}$  (near the point  $a_1$ ) and  $\tilde{g}$  (near

the point  $a_2$ ), two between  $b_1$  and  $b_2$ , viz.,  $\tilde{h}$  (near the point  $b_1$ ) and  $\tilde{i}$  (near the point  $b_2$ ), two between  $c_1$  and  $c_2$ , viz.,  $\tilde{j}$  (near the point  $c_1$ ) and  $\tilde{k}$  (near the point  $c_2$ ), and finally two, between the points  $d_1$  and  $d_2$ , viz.,  $\tilde{l}$  (near the point  $d_1$ ) and  $\tilde{f}$  (near the point  $d_2$ ). We draw the non-Euclidean curves lines by joining the points  $\tilde{e}$  to  $\tilde{f}$ ,  $\tilde{g}$  to  $\tilde{h}$ ,  $\tilde{i}$  to  $\tilde{j}$ , and  $\tilde{k}$  to  $\tilde{l}$ . Note that these four curves intersect in the four semi-circles  $D_1$ ,  $D_2$ ,  $D_3$ , and  $D_4$  at the points  $e, g, h, i, j, k, l$ , and  $f$ . Assume that the lengths of the curve line  $\overline{gh}$  is  $\kappa_1$ ,  $\overline{lk}$  is  $\kappa_2$ , and  $\overline{ef}$  (or  $\overline{ij}$ ) is  $\kappa_3$ . Now, the area (denoted by  $R_0$ ) joining the points  $a_1, d_2, f, l, d_1, c_2, k, j, c_1, b_2, i, h, b_1, a_2, g, e$ , and  $a_1$  represents the fundamental domain of a Fuchsian group (denoted by  $\Omega_1 = \langle h_1, h_2 \rangle$ ) with signature  $(0, 0, 0, 0, 3)$ . Therefore,  $\Omega_1$  is a hyperbolic Riemann surface with three funnel ends consisting of neck lengths  $\kappa_1$ ,  $\kappa_2$ , and  $2\kappa_3$ . However, the region (denoted by  $R_1$ ) joining the points  $a_1, d_2, f, l, k, j, c_1, b_2, i, h, b_1, a_2, g, e$ , and  $a_1$  represents the fundamental domain of a Fuchsian group (denoted by  $\Omega_1^*$ ) with signature  $(0, 0, 1, 0, 2)$  with neck lengths  $\kappa_1$ , and  $2\kappa_3$  consisting a closed geodesic boundary with length  $\kappa_2$ . Now, we consider another Fuchsian group  $\Omega_2$  with signature  $(0, 0, 0, 0, 3)$  with neck lengths  $\kappa_4$ ,  $\kappa_5$ , and  $2\kappa_6$ . We proceed in a situation similar to that described above. Now, the area (denoted by  $R_2$ ) joining the points  $e_1, h_2, i_2, l_2, h_1, g_2, l_1, k_2, g_1, f_2, k_1, j_2, f_1, e_2, j_1, i_1$ , and  $e_1$  represents a fundamental domain of a Fuchsian group (denoted by  $\Omega_2 = \langle h_3, h_4 \rangle$ ) with signature  $(0, 0, 0, 0, 3)$ . On the other hand, the region (denoted by  $R_3$ ) joining the points  $e_1, h_2, i_2, l_2, h_1, g_2, l_1, k_2, g_1, f_2, k_1, j_2, j_1, i_1$ , and  $e_1$  represents a fundamental domain of a Fuchsian group (denoted by  $\Omega_2^*$ ) with signature  $(0, 0, 1, 0, 2)$  consisting one closed geodesic boundary of length  $\kappa_4$ , and two funnels with neck lengths  $\kappa_5$ , and  $2\kappa_6$ . Let us consider a group  $\Pi_3 = \langle h_2, h_3 \rangle$  with signature  $(0, 0, 2, 0, 1)$  with one funnel end of neck length  $2\kappa_7$ . Then, from Figure 4.1 one can easily see that, the region (denoted by  $R_4$ ) joining the points  $o, m, a, h, n, t, s, r, e, d, q, p$ , and  $o$  represents the fundamental domain of a group  $\Pi_3$  with signature  $(0, 0, 2, 0, 1)$  with neck length  $2\kappa_7$  and two boundary geodesics with lengths  $\kappa_2$  and  $\kappa_4$ .

To glue two groups  $\Omega_1$  and  $\Omega_2$  with signatures  $(0, 0, 0, 0, 3)$ , we cut the two corresponding surfaces, one funnel end each, along the neck of their funnels, where neck lengths are  $\kappa_2$  and  $\kappa_4$ , respectively. After this operation, two groups are formed: one is  $\Omega_1^*$  and the other is  $\Omega_2^*$ . Then, a group  $\Pi_3$  is added in such a way that the neck length can match with the groups  $\Omega_1^*$  and  $\Omega_2^*$ . Note that, when we paste with  $\Omega_1^*$  to  $\Pi_3$  the curve lines (i.e.,  $\overline{kl}$  in  $\Omega_1^*$  and  $\overline{op}$  in  $\Pi_3$ ) can match, that is, the lengths are the same and their directions are opposite to each other. Now, using the combination theorem, one can obtain a new group  $\Omega_{13} = \langle \Omega_1^*, \Pi_3 \rangle = \langle h_1, h_3 \rangle$  with signature  $(0, 0, 1, 0, 3)$ . Subsequently, we will attach the groups  $\Omega_{13}$  and  $\Omega_2^*$  on the analogy of the technique

that we have mentioned just before (i.e., matching the curve lines  $\overline{st}$  in  $\Omega_{13}$  and  $\overline{j_1 j_2}$  in  $\Omega_2^*$ ). Consequently, one can again utilize the combination theorem to obtain a new group  $\Omega_{132} = \langle \Omega_{13}, \Omega_2^* \rangle = \langle h_1, h_4 \rangle (= \Omega^1, \text{ say})$  with signature  $(0, 0, 0, 0, 5)$ . Therefore, the fundamental domain of the arising group  $\Omega_{132}$  is the area bounded by the non-Euclidean lines joining the points  $a_1, d_2, f, m, a, h, n, i_1, e_1, h_2, i_2, l_2, h_1, g_2, l_1, k_2, g_1, f_2, k_1, r, e, d, q, j, c_1, b_2, i, h, b_1, a_2, g, e$ , and  $a_1$ . Therefore, the proof of Lemma 4.1 is complete.  $\square$

**Remark 1.** Note that the polygon of a Fuchsian group with signature  $(0, 0, 3, 0, 0)$  that Keen [[18], see construction 2, Figure 3] introduced for compact Riemann surface, is completely different from our constructed polygon for Fuchsian group equipped with classical Schottky structures of signature  $(0, 0, 3, 0, 0)$  in non-compact settings.

**Remark 2.** For the group  $\Omega_1$  with signature  $(0, 0, 0, 0, 3)$  consists of three funnel components with three necks, whose two (out of three) neck lengths (that is,  $\kappa_1$  and  $\kappa_2$ ) are coming from the axes of the two hyperbolic elements  $h_1, h_2$ , respectively, and the third one (that is,  $\kappa_3$ ) arises from the product of these two hyperbolic elements (that is,  $h_1 h_2$ ). Moreover, for the other two groups,  $\Omega_2$  and  $\Pi_3$ , we apply the analogous technique as we have done for the group  $\Omega_1$ .

## 5. ATTACHING SURFACES OF TYPES $(0, 0, 0, c, f)$ VIA FUNDAMENTAL GROUPS

**Lemma 5.1.** (Construction 2.) *Let  $\Omega_3$  be a Fuchsian group of signature  $(0, 0, 0, 1, 2)$  with neck lengths  $2\kappa_1, \kappa_2$ , and  $\Omega_4$  be a Fuchsian group of signature  $(0, 0, 0, 1, 2)$  with neck lengths  $2\kappa_3, \kappa_4$ . Assume that both the corresponding surfaces possess different neck lengths  $\kappa_i, i = 1, 2, \dots, 4$ , where  $\kappa_i \neq \kappa_j$ , for  $i \neq j$ . Then:*

(1) *For a group  $\Pi_5$  of signature  $(0, 0, 2, 0, 1)$  consisting of the lengths of their closed geodesics (of the conformal holes)  $\kappa_2$  and  $\kappa_4$ , with neck length  $2\kappa_5$ , we prove that  $\Omega^2 = \langle \Omega_3^*, \Pi_5, \Omega_4^* \rangle$ , where ‘\*’ represents the number of operations acting on that group.*

(2) *Gluing two surfaces of types  $(0, 0, 0, 1, 2)$  yields a hyperbolic surface of type  $(0, 0, 0, 2, 3)$ . In particular,  $\Omega^2$  represents a hyperbolic surface of two cusps and three funnel ends with neck lengths  $2\kappa_1, 2\kappa_5$ , and  $2\kappa_3$ .*

*Proof.* Let  $D_1, D_2, D_3$ , and  $D_4$  be four semi-circles within the Poincaré disk model. In fact,  $D_1$  is paired with  $D_2$  by the parabolic Möbius transformation  $p_1$ , whereas  $D_4$  is linked with  $D_3$  by the hyperbolic Möbius transformation  $h_1$ . Assume that these four semi-circles meet the boundary of the disk at the points  $a_1, a_2; a_2, a_3; a_4, a_5; a_6, a_7$  respectively. Now, on the boundary of the disk, we take six points, two between  $a_7$  and  $a_6$ , viz.,  $\tilde{c}$  (near the point  $a_7$ ) and  $\tilde{d}$  (near the point  $a_6$ ), two between  $a_5$  and  $a_4$ , viz.,  $\tilde{e}$  (near the point  $a_5$ ) and  $\tilde{f}$  (near

the point  $a_4$ ), one between  $a_1$  and  $a_2$ , viz.,  $\tilde{b}$  (near the point  $a_1$ ), and finally one, between the points  $a_2$  and  $a_3$ , viz.,  $\tilde{g}$  (near the point  $a_3$ ). We draw the non-Euclidean curves lines by joining the points  $\tilde{b}$  to  $\tilde{c}$ ,  $\tilde{d}$  to  $\tilde{e}$ , and  $\tilde{f}$  to  $\tilde{g}$ . Notice that these three curves intersect in the four semi-circles  $D_1$ ,  $D_2$ ,  $D_3$ , and  $D_4$  at the points  $b; g; f; e; d$ , and  $c$ . Suppose the lengths of the curve lines  $\overline{bc}$  (or  $\overline{fg}$ ) is  $\kappa_1$ , and  $\overline{de}$  is  $\kappa_2$ . Now, the region (denoted by  $R_5$ ) joining the points  $a_1, a_7, c, d, a_6, a_5, e, f, a_4, a_3, g, a_2, b$ , and  $a_1$  represents the fundamental domain of a Fuchsian group (denoted by  $\Omega_3 = \langle p_1, h_1 \rangle$ ) with signature  $(0, 0, 0, 1, 2)$ . Consequently, the group  $\Omega_3$  is a hyperbolic Riemann surface with one cusp and two funnel ends consisting of neck lengths  $2\kappa_1$ , and  $\kappa_2$ . However, the region (denoted by  $R_6$ ) joining the points  $a_1, a_7, c, d, e, f, a_4, a_3, g, a_2, b$ , and  $a_1$  represents the fundamental domain of a Fuchsian group (denoted by  $\Omega_3^*$ ) with signature  $(0, 0, 1, 1, 1)$  with one funnel of neck length  $2\kappa_1$  consisting a closed geodesic boundary with length  $\kappa_2$ . Now, we consider another group  $\Omega_4$  with signature  $(0, 0, 0, 1, 2)$  with one cusp end and two funnels of neck lengths  $2\kappa_3$ , and  $\kappa_4$ . We proceed in a similar situation to the one we did in the above construction. Now, the domain (denoted by  $R_7$ ) joining the points  $b_1, b_7, j, b_6, k, b_5, b_4, l, m, b_3, b_2, n, i$ , and  $b_1$  represents a fundamental region of a Fuchsian group (denoted by  $\Omega_4 = \langle h_2, p_2 \rangle$ ) with signature  $(0, 0, 0, 1, 2)$ . However, the area (denoted by  $R_7$ ) joining the points  $b_1, b_7, j, b_6, k, b_5, b_4, l, m, n, i$ , and  $b_1$  represents a fundamental domain of a Fuchsian group (denoted by  $\Omega_4^*$ ) with signature  $(0, 0, 1, 1, 1)$  consisting one closed geodesic boundary of length  $\kappa_4$ , and one funnel with neck length  $2\kappa_3$ . Now, we are considering a group  $\Pi_5 = \langle h_1, h_2 \rangle$  with signature  $(0, 0, 2, 0, 1)$  consisting of one funnel end of neck length  $2\kappa_5$ . Then, from the following Figure 5.1, one can easily observe that, the region (denoted by  $R_8$ ) joining the points  $c_1, c_8, p, q, r, s, c_5, c_4, t, u, v, o$ , and  $c_1$  represents the fundamental domain of a group  $\Pi_5$  with signature  $(0, 0, 2, 0, 1)$  with one funnel of neck length  $2\kappa_5$  and two closed geodesic boundary with lengths  $\kappa_2$  and  $\kappa_4$ .

To glue two groups  $\Omega_3$  and  $\Omega_4$  with signatures  $(0, 0, 0, 1, 2)$ , we cut the two corresponding surfaces, one funnel end each, along the neck of their funnels, where neck lengths are  $\kappa_2$  and  $\kappa_4$  respectively. After this operation, two groups are formed: one is  $\Omega_3^*$  and the other is  $\Omega_4^*$ . Then, adding a group  $\Pi_5$  in such a way that the neck length can match the groups  $\Omega_3^*$  and  $\Pi_5$ . Note that, when we paste with  $\Omega_3^*$  to  $\Pi_5$  the curve lines (i.e.,  $\overline{de}$  in  $\Omega_3^*$  and  $\overline{uv}$  in  $\Pi_5$ ) can match, that is, their directions are opposite to each other. Now, by using the combination theorem, one can get a new group  $\Omega_{35} = \langle \Omega_3^*, \Pi_5 \rangle = \langle p_1, h_2 \rangle$  with signature  $(0, 0, 1, 1, 2)$ . After that, we will attach the groups  $\Omega_{35}$  and  $\Omega_4^*$  on the analogy of the technique that we have mentioned just before (i.e., matching the curve lines  $\overline{rq}$  in  $\Omega_{35}$

and  $\overline{nm}$  in  $\Omega_4^*$ ). So, we can utilize the combination theorem to obtain a new group  $\Omega_{354} = \langle \Omega_{35}, \Omega_4^* \rangle = \langle p_1, p_2 \rangle (= \Omega^2, \text{ say})$  with signature  $(0, 0, 0, 2, 3)$ . Therefore, the fundamental domain of the arising group  $\Omega_{354}$  is the area bounded by the non-Euclidean lines joining the points  $a_1, a_7, c, o, c_1, c_8, p, i, b_1, b_7, j, b_6, k, b_5, b_4, l, s, c_5, c_4, t, f, a_4, a_3, g, a_2, b$ , and  $a_1$ . This proves Lemma 5.1.

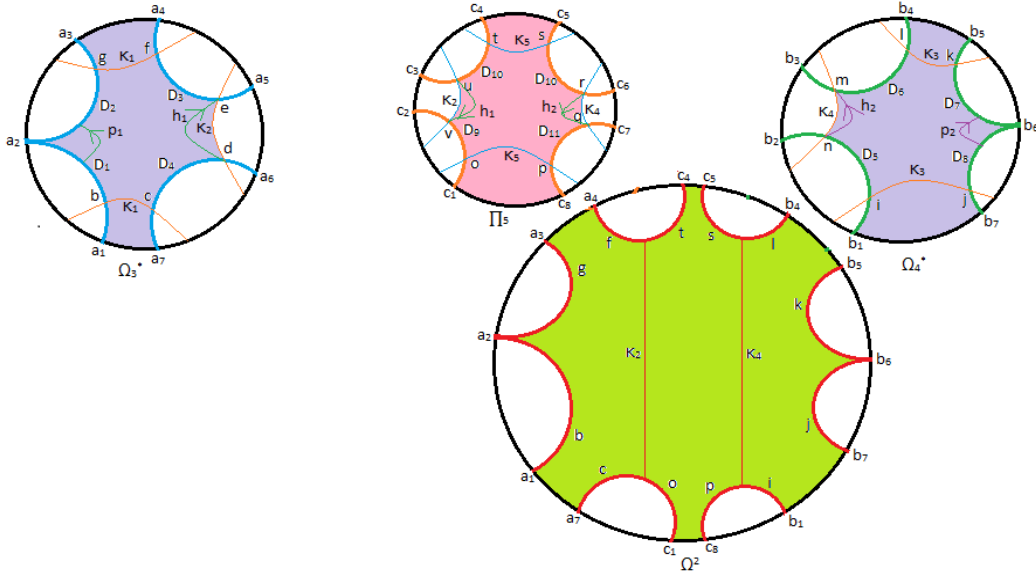


Figure : 5.1

□

**Remark 3.** Note that the neck lengths  $\kappa_2$  and  $\kappa_4$  arise from the axes of the hyperbolic elements  $h_1$  and  $h_2$ , respectively. Furthermore, for the group  $\Omega_3$  (or  $\Omega_3^*$ ), the neck length  $\kappa_1$  is coming from the product of the two elements  $p_1$  and  $h_1$ , whereas for the group  $\Pi_5$ , the neck length  $\kappa_5$  is emerging from the product of the (axes) of the two hyperbolic elements  $h_1$  and  $h_2$ , on the other hand, for the group  $\Omega_4$  (or  $\Omega_4^*$ ), the neck length  $\kappa_3$  is coming from the product of the two elements  $p_2$  and  $h_2$ .

### 6. AMALGAMATION OF NON-COMPACT HYPERBOLIC SURFACES OF TYPES $(0,0,m,0,f)$ AND $(0,0,m,c,f)$ VIA FUNDAMENTAL GROUPS

**Lemma 6.1.** (Construction 3.) Let,  $\Omega_5$  be a Fuchsian group of signature  $(0, 0, 1, 0, 2)$  consisting length of the closed geodesic (of the conformal hole)  $2\kappa_3$ , with neck lengths  $\kappa_1, \kappa_2$  and  $\Omega_6$

be a Fuchsian group of signature  $(0, 0, 1, 1, 1)$  consisting length of the closed geodesic (of the conformal hole)  $2\kappa_5$ , with neck length  $\kappa_4$ . Suppose the lengths of their necks and closed geodesic boundaries in the conformal holes are all distinct. Then:

(1) For a group  $\Pi_7$  of signature  $(0, 0, 2, 0, 1)$  consisting of the lengths of the closed geodesic (of conformal holes)  $\kappa_2, \kappa_4$ , with neck length  $2\kappa_6$ , we prove that  $\Omega^3 = \langle \Omega_5^*, \Pi_7, \Omega_6^* \rangle$ , where “\*” represents the number of operations acting on that group.

(2) Gluing two surfaces of types  $(0, 0, 1, 0, 2)$  and  $(0, 0, 1, 1, 1)$  yields a hyperbolic surface of type  $(0, 0, 2, 1, 2)$ . In particular,  $\Omega^3$  represents a hyperbolic surface with two cusps, two conformal holes with lengths of their closed geodesics  $2\kappa_3, 2\kappa_5$ , and two funnel ends with neck lengths  $\kappa_1$  and  $2\kappa_6$ .

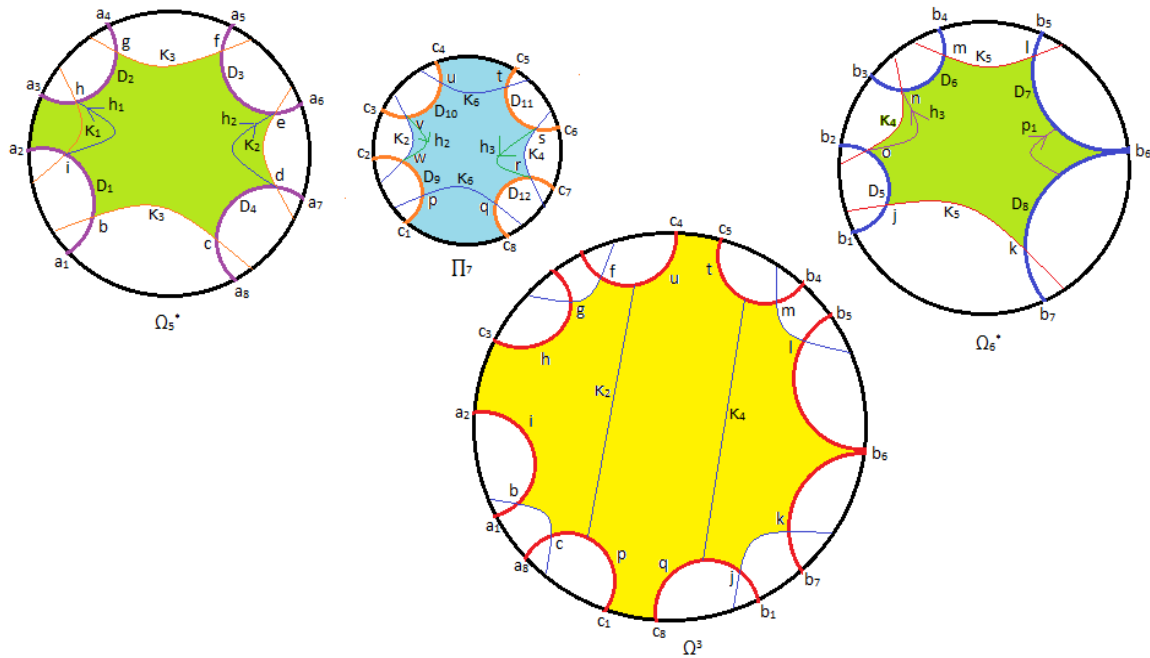


Figure : 6.1

*Proof.* Let  $D_1, D_2, D_3$ , and  $D_4$  be four semi-circles within the Poincaré disk model. More precisely,  $D_1$  twins with  $D_2$  by the hyperbolic Möbius transformation  $h_1$ , on the other hand,  $D_4$  is linked with  $D_3$  by the hyperbolic Möbius transformation  $h_2$ . Suppose these four semi-circles meet the boundary of the disk at the points  $a_1, a_2; a_3, a_4; a_5, a_6; a_7, a_8$  respectively. We consider eight points, on the boundary of the disk, two between  $a_1$  and  $a_2$ , viz.,  $\tilde{b}$  (near the point  $a_1$ ) and  $\tilde{i}$  (near the point  $a_2$ ), two between  $a_3$  and  $a_4$ , viz.,  $\tilde{h}$  (near the point  $a_3$ ) and  $\tilde{g}$  (near the

point  $a_4$ ), two between  $a_5$  and  $a_6$ , viz.,  $\tilde{f}$  (near the point  $a_5$ ), and finally two, between the points  $a_7$  and  $a_8$ , viz.,  $\tilde{d}$  (near the point  $a_7$ ) and  $\tilde{c}$  (near the point  $a_8$ ). Now, we draw the non-Euclidean curves lines by joining the points  $\tilde{b}$  to  $\tilde{c}$ ,  $\tilde{i}$  to  $\tilde{h}$ ,  $\tilde{g}$  to  $\tilde{f}$ , and  $\tilde{e}$  to  $\tilde{d}$ . Observe that these four curves intersect in the four semi-circles  $D_1, D_2, D_3$ , and  $D_4$  at the points  $b, i; h, g; f, e; d$ , and  $c$ . Assume that the lengths of the curve lines  $\overline{i\tilde{h}}$  is  $\kappa_1$   $\overline{bc}$  (or  $\overline{fg}$ ) is  $\kappa_3$ , and  $\overline{de}$  is  $\kappa_2$ . Note that, the region (denoted by  $F_1$ ) joining the points  $i, b, c, d, a_7, a_6, e, f, g, h, a_3, a_2$ , and  $i$  represents the fundamental domain of a Fuchsian group (denoted by  $\Omega_5 = \langle h_1, h_2 \rangle$ ) with signature  $(0, 0, 1, 0, 2)$ . Now, the area (denoted by  $F_2$ ) joining the points  $i, b, c, d, e, f, g, h, a_3, a_2$ , and  $i$  represents the fundamental region of a Fuchsian group (denoted by  $\Omega_5^* = \langle h_1, h_2 \rangle$ ) with signature  $(0, 0, 2, 0, 1)$ . Hence, the group  $\Omega_5^*$  is a hyperbolic Riemann surface consisting of one funnel end of neck length  $\kappa_1$  with two conformal holes where the lengths of their boundary components are  $\kappa_2$  and  $2\kappa_3$ , respectively. Now, we consider another group  $\Omega_6$  with signature  $(0, 0, 1, 1, 1)$  with one cusp end, one funnel end of neck length  $\kappa_4$ , and one conformal hole with a length of the closed geodesic boundary is  $2\kappa_5$ . We proceed in a similar manner to that arranged in the overhead construction. Observe that the region (denoted by  $F_3$ ) joining the points  $o, j, k, b_6, l, m, n, b_3, b_2$ , and  $o$  represents a fundamental domain of a Fuchsian group (denoted by  $\Omega_6 = \langle h_3, p_1 \rangle$ ) with signature  $(0, 0, 1, 1, 1)$ . However, the region (denoted by  $F_4$ ) joining the points  $o, j, k, b_6, l, m, n$ , and  $o$  represents a fundamental domain of a Fuchsian group (denoted by  $\Omega_6^*$ ) with signature  $(0, 0, 2, 1, 0)$  consisting of two closed geodesic boundaries of lengths  $\kappa_4$  and  $2\kappa_5$  and one cusp end.

Now, if we want to glue the groups  $\Omega_5^*$  and  $\Omega_6^*$ , two cases will occur.

(i) Assume that  $\kappa_3 = \kappa_5$ . Then, one can easily paste these two groups,  $\Omega_5^*$  and  $\Omega_6^*$ , in a direct way. This act gives a Fuchsian group with signature  $(0, 0, 0, 1, 3)$ . If possible, this will not happen (say). Then, we will move forward to the second case.

(ii) Suppose  $\kappa_3 \neq \kappa_5$ . So, in this situation, we can not attach  $\Omega_5^*$ , and  $\Omega_6^*$  directly. So, we are going to fabricate a new group in the following way:

Let  $D_9, D_{10}, D_{11}$ , and  $D_{12}$  be four semi-circles within the Poincaré disk model. More precisely,  $D_{10}$  is linked with  $D_9$  by the hyperbolic Möbius transformation  $h_2$ , whereas  $D_{11}$  twins with  $D_{12}$  by the hyperbolic Möbius transformation  $h_3$ . Let these four semi-circles meet the boundary of the disk at the points  $c_1, c_2; c_3, c_4; c_5, c_6; c_7, c_8$  respectively. Now, on the boundary of the disk, we take eight points, two between  $c_1$  and  $c_2$ , viz.,  $\tilde{p}$  (near the point  $c_1$ ) and  $\tilde{w}$  (near the point  $c_2$ ), two between  $c_3$  and  $c_4$ , viz.,  $\tilde{v}$  (near the point  $c_3$ ) and  $\tilde{u}$  (near the point  $c_4$ ), two between  $c_5$  and  $c_6$ , viz.,  $\tilde{t}$  (near the point  $c_5$ ) and  $\tilde{s}$  (near the point  $c_6$ ), and finally two between  $c_7$  and

$c_8$ , viz.,  $\tilde{r}$  (near the point  $c_7$ ) and  $\tilde{q}$  (near the point  $c_8$ ). Now, one can draw four non-Euclidean lines by joining the points  $\tilde{p}$  to  $\tilde{q}$ ,  $\tilde{w}$  to  $\tilde{v}$ ,  $\tilde{u}$  to  $\tilde{t}$ , and  $\tilde{s}$  to  $\tilde{r}$ . Assume that these four curves intersect in the four semi-circles  $D_9$ ,  $D_{10}$ ,  $D_{11}$ , and  $D_{12}$  at the points  $p, w; v, u; t, s; r$ , and  $q$ . Suppose the lengths of the curve lines  $\overline{wv}$ ,  $\overline{rs}$ , and  $\overline{pq}$  (or  $\overline{tu}$ ) are  $\kappa_2$ ,  $\kappa_4$ ,  $2\kappa_6$ , respectively. The region (denoted by  $F_5$ ) joining the points  $w, p, c_1, c_8, q, r, s, t, c_5, c_4, u, v$ , and  $w$  represents the fundamental domain of the Fuchsian group  $(\Pi_7 = \langle h_2, h_3 \rangle)$  with signature  $(0, 0, 2, 0, 1)$ .

Now, to glue the groups  $\Omega_5$  and  $\Omega_6$ , we cut their corresponding hyperbolic surfaces one funnel each along the neck of their suitable funnel ends, consisting of neck lengths  $\kappa_2$  and  $\kappa_4$ , respectively. After this operation, two groups are formed: one is  $\Omega_5^*$ , and the other is  $\Omega_6^*$ . Then, a group  $\Pi_7$  is added in such a way that the neck length can match with the groups  $\Omega_5^*$ , and the other one  $\Omega_6^*$ . Note that, when we paste with  $\Omega_5^*$  to  $\Pi_7$  the curve lines (i.e.,  $\overline{de}$  in  $\Omega_5^*$  and  $\overline{vw}$  in  $\Pi_7$ ) can match, that is, their lengths are the same, but their directions are opposite to each other. Now, by using the combination theorem, one can easily obtain a new group  $\Omega_{57} = \langle \Omega_5^*, \Pi_7 \rangle = \langle h_1, h_3 \rangle$  with signature  $(0, 0, 2, 0, 2)$ . Furthermore, we will attach the groups  $\Omega_{57}$  and  $\Omega_6^*$  on the analogy of the technique that we mentioned just above (i.e., matching the curve lines  $\overline{rs}$  in  $\Omega_{57}$  and  $\overline{no}$  in  $\Omega_6^*$ ). Now, again, by utilizing the combination theorem, we get  $\Omega_{576} = \langle \Omega_{57}, \Omega_6^* \rangle = \langle h_1, p_1 \rangle (= \Omega^3, \text{ say})$  with signature  $(0, 0, 2, 1, 2)$ . Therefore, the fundamental domain of the group  $\Omega_{576}$  ( $= \Omega^3$ , say) is the region bounded by the non-Euclidean lines joining the points  $a_2, i, b, c, p, c_1, c_8, q, j, k, b_6, l, m, t, c_5, c_4, u, f, g, h, a_3$ , and  $a_2$ . So, Lemma 6.1 is proven.  $\square$

**Remark 4.** Observe that the neck lengths  $\kappa_1$ ,  $\kappa_2$ , and  $\kappa_4$  are arising from the axes of the hyperbolic elements  $h_1$ ,  $h_2$ ,  $h_3$ , respectively. Moreover, for the group  $\Omega_5$  (or  $\Omega_5^*$ ), the neck length  $\kappa_3$  is emerging from the product of the (axes) of the two hyperbolic elements  $h_1$  and  $h_2$ , whereas for the group  $\Pi_7$ , the neck length  $\kappa_6$  is coming from the product of the (axes) of the two hyperbolic elements  $h_2$  and  $h_3$ , on the other hand, for the group  $\Omega_6$  (or  $\Omega_6^*$ ), the neck length  $\kappa_5$  is created from the product of the two elements  $p_1$  and  $h_3$ .

## 7. GLUING NON-COMPACT HYPERBOLIC RIEMANN SURFACES OF TYPES $(0,0,m,c,0)$ AND $(g,0,m,c,0)$ VIA FUNDAMENTAL GROUPS

**Lemma 7.1.** (Construction 4.) *Let  $\Omega_7$  be a Fuchsian group of signature  $(0, 0, 2, 1, 0)$  consisting of the lengths of their closed geodesics (of two conformal holes)  $2\kappa_1$ ,  $\kappa_2$ , and  $\Omega_8$  be a Fuchsian group of signature  $(1, 0, 2, 1, 0)$  with lengths of their closed geodesics (of two conformal holes)*

$\kappa_3, 2\kappa_4$ . Assume that both the corresponding surfaces possess different lengths of boundaries in their conformal holes,  $\kappa_i, i = 1, 2, \dots, 4$ , where  $\kappa_i \neq \kappa_j$ , for  $i \neq j$ . Then:

(1) For a group  $\Pi_9$  of signature  $(0, 0, 3, 0, 0)$  with lengths of their closed geodesics (of three conformal holes)  $\kappa_2, 2\kappa_5, \kappa_3$ , we prove that  $\Omega^4 = \langle \Omega_7, \Pi_9, \Omega_8 \rangle$ .

(2) Gluing two surfaces of types  $(0, 0, 2, 1, 0)$  and  $(1, 0, 2, 1, 0)$  yields a hyperbolic surface of type  $(1, 0, 3, 2, 0)$ . In particular,  $\Omega^4$  represents a hyperbolic surface of genus one, two cusps, and three conformal holes with lengths of their closed geodesics  $2\kappa_1, 2\kappa_5$ , and  $2\kappa_5$ .

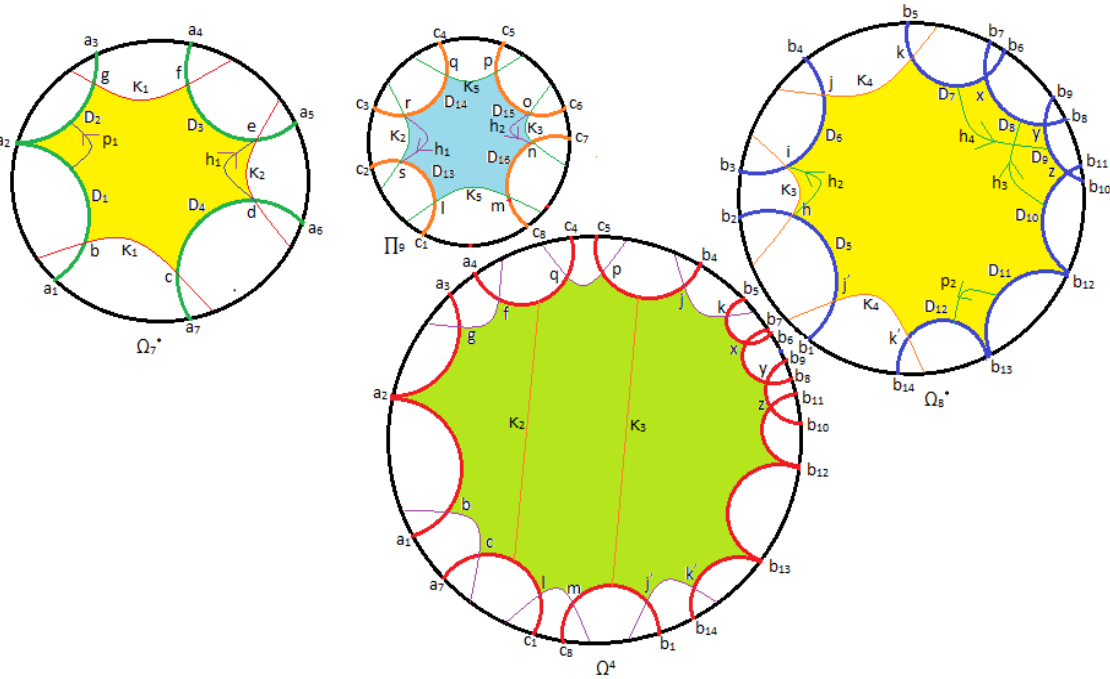


Figure : 7.1

*Proof.* Let  $D_1, D_2, D_3$ , and  $D_4$  be four semi-circles within the Poincaré disk model. In particular,  $D_1$  is paired with  $D_2$  by the parabolic Möbius transformation  $p_1$ , whereas  $D_3$  twins with  $D_4$  by the hyperbolic Möbius transformation  $h_1$ . Assume that these four semi-circles meet the boundary of the disk at the points  $a_1, a_2; a_2, a_3; a_4, a_5; a_6, a_7$  respectively. Now, on the boundary of the disk, we take six points, one between  $a_1$  and  $a_2$ , viz.,  $\tilde{b}$  (near the point  $a_1$ ), one between  $a_2$  and  $a_3$ , viz.,  $\tilde{g}$  (near the point  $a_3$ ), two between  $a_4$  and  $a_5$ , viz.,  $\tilde{f}$  (near the point  $a_4$ ) and  $\tilde{e}$  (near the point  $a_5$ ), and finally two, between the points  $a_6$  and  $a_7$ , viz.,  $\tilde{d}$  (near the point  $a_6$ ) and  $\tilde{c}$  (near the point  $a_7$ ). We draw the non-Euclidean curve lines by joining the points  $\tilde{b}$  to  $\tilde{c}$ ,  $\tilde{g}$  to  $\tilde{f}$ , and  $\tilde{e}$  to  $\tilde{d}$ . Notice that these four curves intersect in the four semi-circles  $D_1, D_2, D_3$ , and

$D_4$  at the points  $b; g; f; e; d$ , and  $c$ , respectively. Suppose the length of the curve line  $\overline{bc}$  (or  $\overline{fg}$ ) is  $\kappa_1$ , and  $\overline{de}$  is  $\kappa_2$ . Observe that the region (denoted by  $F_6$ ) joining the points  $a_2, b, c, d, e, f, g$ , and  $a_2$  represents the fundamental domain of a Fuchsian group ( $\Omega_7 = \langle p_1, h_1 \rangle$ ) with signature  $(0, 0, 2, 1, 0)$ . In particular, the Fuchsian group  $\Omega_7$  provides a hyperbolic Riemann surface with one cusp and two conformal holes where the lengths of the closed geodesic boundaries are  $2\kappa_1$  and  $\kappa_2$ . Now, we aim to consider another group  $\Omega_8$  of signature  $(1, 0, 2, 1, 0)$  in the following way.

Let  $D_5, D_6, D_7, D_8, D_9, D_{10}, D_{11}$ , and  $D_{12}$  be eight semi-circles within the Poincaré disk model. In fact,  $D_5$  is linked with  $D_6$  by the hyperbolic Möbius transformation  $h_2$ , whereas  $D_7$  twins with  $D_9$  by the hyperbolic Möbius transformation  $h_4$ , and  $D_{10}$  is paired with  $D_8$  by the hyperbolic Möbius transformation  $h_3$ ; On the other hand,  $D_{12}$  is linked with  $D_{11}$  by the parabolic Möbius transformation  $p_2$ . Suppose these eight semi-circles  $D_5, D_6, D_7, D_8, D_9, D_{10}, D_{11}$  and  $D_{12}$  meet the boundary of the disk at the points  $b_1, b_2; b_3, b_4; b_5, b_6; b_7, b_8, b_9, b_{10}; b_{11}, b_{12}; b_{12}, b_{13}; b_{13}, b_{14}$  respectively. Now, on the boundary of the disk, we take six points, one between  $b_{14}$  and  $b_{13}$ , viz.,  $\tilde{k}'$  (near the point  $b_{14}$ ), one between  $b_5$  and  $b_6$ , viz.,  $\tilde{k}$  (near the point  $b_5$ ), two between  $b_1$  and  $b_2$ , viz.,  $\tilde{j}'$  (near the point  $b_1$ ) and  $\tilde{h}$  (near the point  $b_2$ ), and finally two, between the points  $b_3$  and  $b_4$ , viz.,  $\tilde{i}$  (near the point  $b_3$ ) and  $\tilde{j}$  (near the point  $b_4$ ). We draw three non-Euclidean curve lines by joining the points  $\tilde{k}'$  to  $\tilde{j}'$ ,  $\tilde{h}$  to  $\tilde{i}$ , and  $\tilde{j}$  to  $\tilde{k}$ . Clearly, these three curves intersect in the four semi-circles  $D_{12}, D_5, D_6$ , and  $D_7$  at the points  $k'; j', h; i, j$ ; and  $k$  respectively. We assume that the length of the curve line  $\overline{j'k'}$  (or  $\overline{jk}$ ) is  $2\kappa_4$ , and  $\overline{hi}$  is  $\kappa_3$ . Furthermore, we suppose that the semi-circles  $D_7$  and  $D_8$  intersect within the Poincaré disk at the point 'x', whereas  $D_8$  and  $D_9$  meet at 'y'; on the other hand,  $D_9$  and  $D_{10}$  intersect at the point 'z'. Now, we assert that the region (denoted by  $F_7$ ) joining the points  $h, j', k', b_{13}, b_{12}, z, y, x, k, j, i$ , and  $h$  represents the fundamental domain of a Fuchsian group ( $\Omega_8 = \langle h_2, h_3, h_4, p_2 \rangle$ ) with signature  $(1, 0, 2, 1, 0)$ . In particular, the group  $\Omega_8$  provides a hyperbolic Riemann surface with one genus, one cusp end, and two conformal holes where the lengths of the closed geodesic boundaries are  $2\kappa_4$  and  $\kappa_3$ .

Now, if we wish to attach the groups,  $\Omega_7$  and  $\Omega_8$ , two cases will arise:

- (i) If possible, let  $\kappa_2 = \kappa_3$ . Then, one can easily glue these two groups,  $\Omega_7$  and  $\Omega_8$  in a direct way. This act indicates a Fuchsian group with signature  $(1, 0, 2, 2, 0)$ . Suppose this will not happen (say). Then, one can move forward to the second case.
- (ii) Assume that  $\kappa_2 \neq \kappa_3$ . So, in this process, we can not attach  $\Omega_7$  and  $\Omega_8$  directly. Now, to glue these two groups, we are going to fabricate a new group in the subsequent way:

Let  $D_{13}$ ,  $D_{14}$ ,  $D_{15}$ , and  $D_{16}$  be four semi-circles within the Poincaré disk model. In fact,  $D_{13}$  is linked with  $D_{14}$  by the hyperbolic Möbius transformation  $h_4 (= h_1^{-1})$ , on the other hand,  $D_{16}$  is paired with  $D_{15}$  by the hyperbolic Möbius transformation  $h_5 (= h_2^{-1})$ . Suppose these four semi-circles  $D_{13}$ ,  $D_{14}$ ,  $D_{15}$ , and  $D_{16}$  meet the boundary of the disk at the points  $c_1, c_2; c_3, c_4; c_5, c_6; c_7, c_8$  respectively. Now, on the boundary of the disk, we take eight points, two between  $c_1$  and  $c_2$ , viz.,  $\tilde{l}$  (near the point  $c_1$ ) and  $\tilde{s}$  (near the point  $c_2$ ), two between  $c_3$  and  $c_4$ , viz.,  $\tilde{r}$  (near the point  $c_3$ ) and  $\tilde{q}$  (near the point  $c_4$ ), two between  $c_5$  and  $c_6$ , viz.,  $\tilde{p}$  (near the point  $c_5$ ) and  $\tilde{o}$  (near the point  $c_6$ ), and finally two between  $c_7$  and  $c_8$ , viz.,  $\tilde{n}$  (near the point  $c_7$ ) and  $\tilde{m}$  (near the point  $c_8$ ). Now, we draw four non-Euclidean lines by joining the points  $\tilde{l}$  to  $\tilde{m}$ ,  $\tilde{s}$  to  $\tilde{r}$ ,  $\tilde{q}$  to  $\tilde{p}$ , and  $\tilde{o}$  to  $\tilde{n}$ . Suppose these four curves intersect in the four semi-circles  $D_{13}$ ,  $D_{14}$ ,  $D_{15}$ , and  $D_{16}$  at the points  $l, s; r, q; p, o; n$ , and  $m$ . Let the lengths of the curve lines  $\overline{r\tilde{s}}$ ,  $\overline{o\tilde{n}}$ , and  $\overline{l\tilde{m}}$  (or  $\overline{p\tilde{q}}$ ) be  $\kappa_2$ ,  $\kappa_3$ , and  $2\kappa_4$ , respectively. The region (denoted by  $F_8$ ) joining the points  $s, l, m, n, o, p, q, r$ , and  $s$  represents the fundamental domain of the Fuchsian group  $(\Pi_9 = \langle h_1, h_2 \rangle)$  with signature  $(0, 0, 3, 0, 0)$ .

Now, to glue the two groups  $\Omega_7$  and  $\Omega_8$ , we just need to introduce another group,  $\Pi_9$ , in the scenario. First, we take two groups  $\Omega_7$  and  $\Pi_{10}$  in such a way that the curve lines (i.e.,  $\overline{de}$  in  $\Omega_7$  and  $\overline{r\tilde{s}}$  in  $\Pi_9$ ) can match, i.e, their lengths are the same (i.e.,  $\kappa_2$ ) but directions are opposite to each other. After this operation, clearly, by utilizing the combination theorem, we obtain a new group  $\langle \Omega_7, \Pi_9 \rangle = \langle p_1, h_2 \rangle = \Gamma_1$  with signature  $(0, 0, 3, 1, 0)$ . Furthermore, we will proceed to glue the groups  $\Gamma_1$  and  $\Omega_8$  on the analogy of the strategy that we have just carried out in the above (i.e., by matching the curve lines  $\overline{o\tilde{n}}$  in  $\Gamma_1$  and  $\overline{hi}$  in  $\Omega_8$ ). Now, again, by using the combination theorem, we achieve a new group  $\Gamma_2 = \langle \Gamma_1, \Omega_8 \rangle = \langle p_1, p_2, h_3, h_4 \rangle (= \Omega^4, \text{ say})$  with signature  $(1, 0, 3, 2, 0)$ . Therefore, the fundamental region of the group  $\Gamma_2 (= \Omega^4, \text{ say})$  is the domain bounded by the non-Euclidean lines joining the points  $a_2, b, c, l, m, j', k', b_{13}, b_{12}, z, y, x, k, j, p, q, f, g$ , and  $a_2$ . This completes the proof of Lemma 7.1.  $\square$

**Remark 5.** Notice that the neck lengths  $\kappa_2$  and  $\kappa_3$  are arising from the axes of the hyperbolic elements  $h_1$  and  $h_2$ , respectively. Furthermore, for the group  $\Omega_7$  (or  $\Omega_7^*$ ), the neck length  $\kappa_1$  comes from the product of the two elements  $p_1$  and  $h_1$ , whereas for the group  $\Pi_9$ , the neck length  $\kappa_5$  emerges from the product of the axes of the two hyperbolic elements  $h_1$  and  $h_2$ , on the other hand, for the group  $\Omega_8$  (or  $\Omega_8^*$ ), the neck lengths  $\kappa_3$  and  $\kappa_4$  are created from the axis of the hyperbolic element  $h_2$  and the product of the two elements  $p_2$  and  $h_2$ , respectively.

Moreover, the lengths of the curve lines  $\overline{xy}$  and  $\overline{yz}$  are basically the hyperbolic lengths of the axes of  $h_3$  and  $h_4$ , respectively.

## 8. PROOF OF THE SECOND AND THIRD MAIN THEOREMS

In this section, we formulate a general strategy for constructing an arbitrary finite type hyperbolic Riemann surface (for instance, a surface of type  $(2, 0, 1, 2, 8)$ ) by proving Theorem 1.2. After that, we have initiated the structures of non-compact hyperbolic Riemann surfaces of types  $(0, 0, 0, 0, \infty)$ ,  $(0, 0, 0, \infty, \infty)$ ,  $(0, 0, \infty, 0, \infty)$ , and  $(0, 0, \infty, \infty, \infty)$  via fundamental groups by establishing Theorem 1.3.

**Proof of Theorem 1.2.** Step 1. Let  $\Omega_1$  and  $\Omega_2$  be two Fuchsian groups with signatures  $(1, 0, 0, 0, 2)$  and  $(0, 0, 1, 0, 2)$  respectively. Let  $SC_1, SC_2, SC_3, SC_4, SC_5, SC_6, SC_7,$  and  $SC_8$  be eight semi-circles within the Poincaré disk model. In particular,  $SC_1$  is paired with  $SC_3$  by the hyperbolic Möbius transformation  $h_1$ ,  $SC_4$  is linked with  $SC_2$  by the hyperbolic Möbius transformation  $h_2$ ,  $SC_8$  twins with  $SC_5$  by the hyperbolic Möbius transformation  $h_3$ , and  $SC_7$  is paired with  $SC_6$  by the hyperbolic Möbius transformation  $h_4$ . We assume that these eight semi-circles meet the boundary of the disk at the points  $a_1, a_2; a_3, a_4; a_5, a_6; a_7, a_8; a_8, a_9; a_{10}, a_{11}; a_{12}, a_{13}; a_{14}, a_1$  respectively. Suppose within the Poincaré disk,  $SC_1$  meets  $SC_2$  at the point  $x_1$ ,  $SC_2$  intersects  $SC_3$  at  $x_2$ , and  $SC_3$  touches  $SC_4$  at the point  $x_3$ . Now, on the boundary of the disk, we consider two points, one between  $a_{12}$  and  $a_{13}$ , viz.,  $\tilde{A}_1$  (near the point  $a_{12}$ ), and one between  $a_{10}$  and  $a_{11}$ , viz.,  $\tilde{A}_2$  (near the point  $a_{11}$ ). We draw the non-Euclidean curve line by joining points  $\tilde{A}_1$  and  $\tilde{A}_2$ . The curve line  $\overline{\tilde{A}_1\tilde{A}_2}$  intersects the two semi-circles  $SC_6$  and  $SC_7$  at the points  $A_1$  and  $A_2$ . Let the lengths of the curve line  $\overline{\tilde{A}_1\tilde{A}_2}$  be  $\kappa_1$ . Furthermore, we assume that the axis of the hyperbolic element  $h_3$  intersects at the point  $x_4$  at  $SC_8$  and  $x_5$  at  $SC_5$ . Now, the region (denoted by  $R_1$ ) joining the points  $a_1, x_1, x_2, x_3, a_8, x_5, a_9, a_{10}, A_2, a_{11}, a_{12}, A_1, a_{13}, a_{14}, x_4,$  and  $a_1$  represents the fundamental domain of a Fuchsian group (denoted by  $\Omega_1 = \langle h_1, h_2, h_3, h_4 \rangle$ ) with signature  $(1, 0, 0, 0, 2)$ . Now, we consider the other group  $\Omega_2$  with signature  $(0, 0, 1, 0, 2)$ . We proceed in a manner similar to that described in the above construction. The region (denoted by  $R_2$ ) joining the points  $a_{16}, A_3, A_7, A_8, A_6, a_{21}, a_{20}, A_5, A_{10}, A_9, A_4, a_{17},$  and  $a_{16}$  represents a fundamental domain of a Fuchsian group (denoted by  $\Omega_2 = \langle h_5, h_6 \rangle$ ) with signature  $(0, 0, 1, 0, 2)$  (see Figure 8(a)). Note that  $\Omega_2$  is a Fuchsian group with signature  $(0, 0, 1, 0, 2)$  consisting of one closed geodesic boundary of length  $2\kappa_2\kappa_3$ , and two funnels with neck lengths  $\kappa_2$  and  $\kappa_3$ . Let us consider a group  $\Omega_3 = \langle h_4, h_5 \rangle$  with signature  $(0, 0, 2, 0, 1)$  consisting of one funnel end of neck length  $2\kappa_1\kappa_2$  and two conformal holes with lengths of their closed geodesic

boundaries  $\kappa_1$  and  $\kappa_2$ . Now, to glue  $\Omega_1$  and  $\Omega_2$  with signatures  $(1, 0, 0, 0, 2)$  and  $(0, 0, 1, 0, 2)$ , we cut the two corresponding surfaces, one funnel end each, along the necks of their funnels. After this operation, two groups are formed: one is  $\Omega_1^*$  and the other is  $\Omega_2^*$ . Then, a group  $\Omega_3$  is added such that the neck lengths can match  $\Omega_1^*$  and  $\Omega_2^*$ . Note that when we paste with  $\Omega_1^*$  to  $\Omega_3$ , the curve lines (i.e.,  $\overline{A_1A_2}$  in  $\Omega_1^*$  and  $\overline{A_{12}A_{11}}$  in  $\Omega_3$ ) can match, that is, their lengths are the same and directions are opposite to each other. Now, using the combination theorem, we obtain a Fuchsian group  $\Omega_{13} = \langle \Omega_1^*, \Omega_3 \rangle = \langle h_1, h_2, h_3, h_5 \rangle$  with signature  $(1, 0, 1, 0, 2)$ . Subsequently, we will paste the groups  $\Omega_{13}$  and  $\Omega_2^*$  on the analogy of the technique that we mention just before (i.e., matching the curve lines  $\overline{A_{13}A_{14}}$  in  $\Omega_{13}$  and  $\overline{A_3A_4}$  in  $\Omega_2$ ). Consequently, one can again utilize the combination theorem to obtain a Fuchsian group  $\Omega_{132} = \langle \Omega_{13}, \Omega_2^* \rangle = \langle h_1, h_2, h_3, h_6 \rangle$  with signature  $(1, 0, 1, 0, 3)$ . Therefore, the fundamental domain of the arising group  $\Omega_{132}$  is the region bounded by the non-Euclidean lines joining the points  $a_1, x_2, x_3, a_8, x_5, a_9, a_{10}, A_2, a_{26}, a_{27}, A_4, A_9, A_{10}, a_{20}, a_{21}, A_8, A_7, A_3, a_{30}, a_{23}, A_1, a_{13}, a_{14}, x_4$ , and  $a_1$ .

Step 2. Let us consider two Fuchsian groups,  $\Omega_{132}$  and  $\Omega_4 = \langle p_1, h_7, h_8 \rangle$  with signatures  $(1, 0, 1, 0, 3)$  and  $(0, 0, 1, 1, 2)$  respectively. To glue these two groups, we apply the same technique as we have operated in the preceding, that is, cutting one funnel end each for the groups  $\Omega_{132}$  and  $\Omega_4$  along the neck of their corresponding funnels and adding a new group  $\Omega_5$  with signature  $(0, 0, 2, 0, 1)$ . Note that the region joining the points  $B_1, B_1^*, b_1, b_8, B_2^{**}, B_4, B_3, B^{**}, b_5, b_4, B^*, B_2$ , and  $B_1$  represents the fundamental domain of a Fuchsian group (denoted by  $\Omega_5 = \langle h_6, h_7 \rangle$ ) with signature  $(0, 0, 2, 0, 1)$ . The domain joining the points  $B_5, B_7', b_9, b_{18}, B_8', b_{17}, b_{16}, B_{10}, B_9, B_8, b_{13}, b_{12}, B_7, B_6$  and  $B_5$  represents the fundamental domain of a Fuchsian group (denoted by  $\Omega_4^* = \langle p_1, h_7, h_8 \rangle$ ) with signature  $(0, 0, 2, 1, 1)$  (see Figure 8(a)). Now, to glue two Fuchsian groups,  $\Omega_{132}$  and  $\Omega_4$ , with signatures  $(1, 0, 1, 0, 3)$  and  $(0, 0, 1, 1, 2)$ , as we have mentioned earlier, we cut the two Fuchsian surface groups, one funnel end each, along the neck of their funnels. After this action, it reduces two groups, one is  $\Omega_{132}^*$  and the other is  $\Omega_4^*$ . Then a Fuchsian group  $\Omega_5$  is added in such a way that the neck length can match the groups  $\Omega_{132}^*$  and  $\Omega_4^*$ ; when we attach  $\Omega_{132}^*$  to  $\Omega_5$ , the curve lines (i.e.,  $\overline{A_5A_6}$  in  $\Omega_{132}^*$  and  $\overline{B_1B_2}$  in  $\Omega_4^*$ ) can match, that is, their lengths are the same, and directions are opposite to each other. Now, we use the combination theorem to obtain a Fuchsian group  $\Omega_{1325} = \langle \Omega_{132}^*, \Omega_5 \rangle = \langle h_1, h_2, h_3, h_7 \rangle$  with signature  $(1, 0, 2, 0, 3)$ . Subsequently, we will glue the groups  $\Omega_{1325}$  and  $\Omega_4^*$  on the harmony of the technique that we have mentioned just before (i.e., the direction of the curve lines  $\overline{B_3B_4}$  in  $\Omega_{1325}$  and  $\overline{B_5B_6}$  in  $\Omega_4^*$ ). Again, one can utilize the combination theorem to obtain a Fuchsian group



group  $\Omega_{132546}$  is the region bounded by curve lines joining the points  $a_1, x_4, a_{14}, a_{13}, A_1, a_{23}, a_{30}, A_3, A_7, A_8, A_6, b_1, b_8, B_5, B'_7, b_9, b_{18}, B'_8, b_{17}, B^*, b_{16}, C_1, c_1, c_{12}, C_3, C'_5, c_{11}, c_{10}, C'_6, C_8, c_9, c_8, C_6, c_7, c_6, C_5, C_4, c_5, c_4, B_9, b_{13}, b_{12}, B_6, b_5, b_4, A_5, A_{10}, A_9, A_4, a_{27}, a_{26}, A_2, a_{10}, a_9, x_5, a_8, x_3, x_2, x_1$ , and  $a_1$ .

Step 4. Finally, we take two Fuchsian groups,  $\Omega_{132546}$  and  $\Omega_7 (= \langle h_{11}, h_{12}, h_{13}, h_{14}, p_2 \rangle)$  with signatures  $(1, 0, 1, 1, 7)$  and  $(1, 0, 0, 1, 2)$  respectively. To attach these two groups, we apply the same technique as we have performed in the overhead construction, i.e., cutting one funnel end each for the groups  $\Omega_{132546}$  and  $\Omega_7$  along the neck of their corresponding funnels and adding a new group  $\Omega_8 = \langle h_{10}, h_{11} \rangle$  with signature  $(0, 0, 2, 0, 1)$ . We take the region joining the points  $D_1, d_1, d_8, D_4, D_3, d_5, d_4, D_2$ , and  $D_1$  represents the fundamental domain of a Fuchsian group  $\Omega_8$  with signature  $(0, 0, 2, 0, 1)$  (see Figure 8(a)). The domain joining the points  $e_2, E_1, e_1, e_{16}, E_3, e_{15}, e_{14}, e_{13}, z^*, y^*, x^*, e_6, E_4, e_5, e_4, E_2, e_3$ , and  $e_2$  represents a fundamental domain of a Fuchsian group with signature  $(1, 0, 0, 1, 2)$  (see Figure 8(a)). Now, to glue two Fuchsian groups  $\Omega_{132546}$  and  $\Omega_7$  with signatures  $(1, 0, 1, 1, 7)$  and  $(1, 0, 0, 1, 2)$ , as usual, we cut the two Fuchsian surface groups at one funnel end each, along the neck of their funnels. After this operation, it reduces two groups, one is  $\Omega_{132546}^*$  and the other is  $\Omega_7^*$ . Then, adding a Fuchsian group  $\Omega_8$  in such a way that the neck length can match the groups  $\Omega_{132546}^*$  and  $\Omega_7^*$ . Note that, when we paste the group  $\Omega_{132546}^*$  with  $\Omega_8$  the curve lines (i.e.,  $\overline{C_7C_8}$  in  $\Omega_{132546}^*$  and  $\overline{D_1D_2}$  in  $\Omega_8^*$ ) can match. Now, we again use the combination theorem to obtain a Fuchsian group  $\Omega_{1325468} = \langle \Omega_{132546}^*, \Omega_8^* \rangle = \langle h_1, h_2, h_3, h_9, h_{11}, p_1 \rangle$  with signature  $(1, 0, 1, 1, 8)$ . After that, we will attach the groups  $\Omega_{1325468}$  and  $\Omega_7^*$  to the harmony of the technique that we have mentioned in the above construction (i.e., the direction of the curve lines  $\overline{D_3D_4}$  in  $\Omega_{1325468}$  and  $\overline{E_1E_2}$  in  $\Omega_7^*$ ). Finally, by utilizing the combination theorem, we achieve a Fuchsian group  $\Omega_{13254687} = \langle \Omega_{1325468}, \Omega_7^* \rangle = \langle h_1, h_2, h_3, h_9, h_{12}, h_{13}, h_{14}, p_1, p_2 \rangle$  (say,  $\Omega^*$ ) with signature  $(2, 0, 1, 2, 8)$ . Hence, the fundamental domain of the arising group  $\Omega_{13254687}$  is the region bounded by curve lines joining the points  $a_1, x_4, a_{14}, a_{13}, A_1, a_{23}, a_{30}, A_3, A_7, A_8, A_6, b_1, b_8, B_5, B'_7, b_9, b_{18}, B'_8, b_{17}, B^*, b_{16}, C_1, c_1, c_{12}, C_3, c_{11}, c_{10}, D_1, d_1, d_8, E_1, e_1, e_{16}, E_3, E_{15}, E^*, e_{14}, E^{*'}, e_{13}, z^*, y^*, x^*, e_6, E_4, e_5, e_4, E_2, d_5, d_4, D_2, c_7, c_6, C_5, C_4, c_5, c_4, C_2, b_{13}, b_{12}, B_6, b_5, b_4, A_5, A_{10}, A_9, A_4, a_{27}, a_{26}, A_2, a_{10}, a_9, x_5, a_8, x_3, x_2, x_1$ , and  $a_1$ .

So, Theorem 1.1 is proven. This establishes the second main theorem of this article.  $\square$

Now, our goal is to construct four kinds of infinite type generalized non-tight flute surfaces and four kinds of infinite type generalized tight flute surfaces via fundamental groups. At first, we look into non-tight surfaces. After that, we have investigated the tight ones.

**Proof of Theorem 1.3 (a).** (Constructing infinite type generalized non-tight flute surface: Type 1, see Figure 8 (b)).

Let,  $\Gamma_1 (= \langle h_1, h_2, h_3 \rangle)$  and  $\Gamma_2 (= \langle h_4, h_5, h_6 \rangle)$  be two Fuchsian surface groups with signature  $(0, 0, 0, 0, 4)$  consisting of neck lengths  $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ , and  $\kappa_5, \kappa_6, \kappa_7, \kappa_8$  respectively, where  $h_i$ 's are hyperbolic elements. We now wish to attach these two groups. For this purpose, we cut the two corresponding surfaces of the groups  $\Gamma_1$  and  $\Gamma_2$  (reduced to  $\Gamma_1^*$  and  $\Gamma_2^*$ ), one funnel end each along the neck of their funnels, where the lengths of their necks are  $\kappa_4$  and  $\kappa_5$ , respectively. Subsequently, we add a group  $\Gamma_3 = \langle h_3, h_7, h_4 \rangle$  of signature  $(0, 0, 2, 0, 2)$  consisting of two funnel ends of neck lengths  $\kappa_9, \kappa_{10}$ , and two conformal holes with lengths of their closed geodesic boundaries  $\kappa_4$  and  $\kappa_5$ . In particular, we take  $\Gamma_3$  such that it can match with groups  $\Gamma_1^*$  and  $\Gamma_2^*$ . Therefore, taking the notion of gluing Fuchsian groups from the above sections, we use the combination theorem to obtain a Fuchsian group  $\Gamma_{132} = \langle \Gamma_1^*, \Gamma_3, \Gamma_2^* \rangle = \langle h_1, h_2, h_5, h_6, h_7 \rangle$  of signature  $(0, 0, 0, 0, 8)$ .

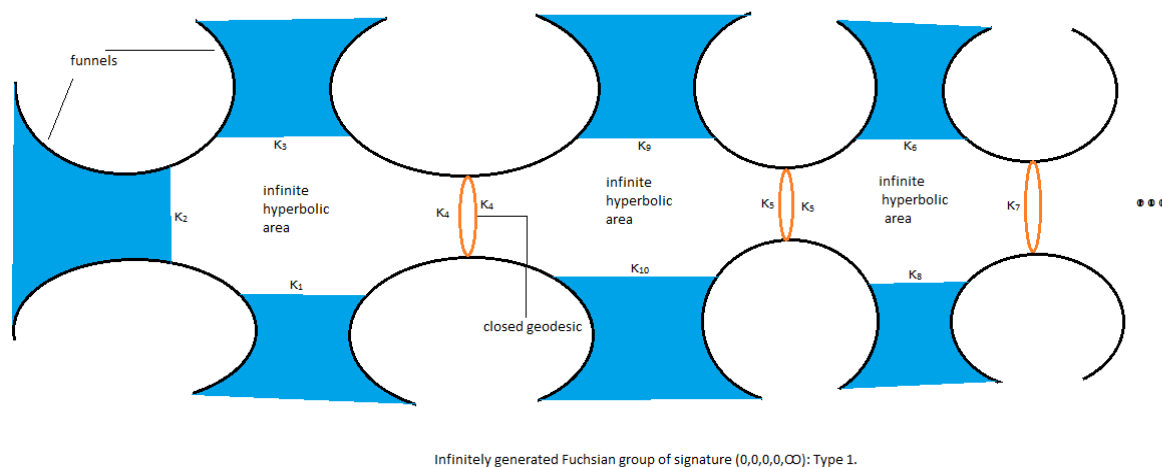


Figure : 8(b)

In the second step, we consider two groups, one is  $\Gamma_{132}$  and the other is  $\Gamma_4 = \langle h_8, h_9, h_{10} \rangle$  of signature  $(0, 0, 0, 0, 4)$ , consisting of four funnel ends of neck lengths  $\kappa_{11}, \kappa_{12}, \kappa_{13}$ , and  $\kappa_{14}$ . Then, we proceeded in a similar way as we have carried out in the preceding construction, that is, cut the two corresponding surfaces of the groups  $\Gamma_{132}$  and  $\Gamma_4$  (transformed to  $\Gamma_{132}^*$  and  $\Gamma_4^*$ ), one funnel end each along the neck of their funnels, where the lengths of their necks are  $\kappa_7$  and  $\kappa_{11}$  respectively. Furthermore, we add a group  $\Gamma_5 = \langle h_7, h_{11}, h_8 \rangle$  of signature  $(0, 0, 2, 0, 2)$  with neck lengths  $\kappa_{15}, \kappa_{16}$ , and closed geodesic boundary lengths are  $\kappa_7$  and  $\kappa_{11}$ . Now, we again apply the combination theorem to obtain a group  $\Gamma_{13254}$



and the other is  $\Delta_4$  of signature  $(0, 0, 0, 0, 3) = \langle h_5, h_6 \rangle$ . After that, we move forward in an analogous way as we have carried out earlier, that is, cut the two corresponding surfaces of the groups  $\Delta_{132}$  and  $\Delta_4$  (transformed to  $\Delta_{132}^*$  and  $\Delta_4^*$ ), one funnel end each along the neck of their funnels, where the lengths of their necks are  $\alpha_6$  and  $\alpha_8$ , respectively. Further, we add a group  $\Delta_5 = \langle h_4, h_5 \rangle$  of signature  $(0, 0, 2, 0, 1)$  with neck length  $\alpha_{11}$ , and closed geodesic boundary lengths are  $\alpha_6$  and  $\alpha_8$ . Now, we again utilize the combination theorem to obtain  $\Delta_{13254} = \langle \Delta_{132}^*, \Delta_5, \Delta_4^* \rangle = \langle h_1, h_6 \rangle$  of signature  $(0, 0, 0, 0, 7)$ . So, in the  $n$ th stage, we will get  $\Delta_{1325467\dots(2n+1)(2n)} = \langle \Delta_1^*, \Delta_3, \Delta_2^{**}, \Delta_5, \Delta_4^{**}, \Delta_7, \Delta_6^{**}, \dots, \Delta_{(2n+1)}, \Delta_{(2n)}^{**} \rangle = \langle h_1, h_{(2n+2)} \rangle$  of signature  $(0, 0, 0, 0, (2n + 3))$ . Therefore, in this way, one can continue this process for an infinite number of times to obtain an infinitely generated group  $\Delta_{13254\dots} = \langle \Delta_1^*, \Delta_3, \Delta_2^{**}, \Delta_5, \Delta_4^{**}, \Delta_7, \Delta_6^{**}, \dots \rangle = \langle h_1, \dots \rangle$ . This infinitely generated group is also of type  $(0, 0, 0, 0, \infty)$  that supplies an infinite type of generalized non-tight flute surfaces, where each block is of infinite hyperbolic area, but each block of the Fuchsian surface group  $\Delta_{132547698\dots}$  contains one funnel end (apart from the first one), whereas for the group  $\Gamma_{13254\dots}$ , two funnels appear in each block.  $\square$

**Proof of Theorem 1.3 (c).** (Constructing infinite type generalized non-tight flute surface: Type 3, see Figure 8(d)).

Let  $\Omega_1 = \langle h_1, h_2 \rangle$  and  $\Omega_2 = \langle h_3, h_4 \rangle$  be two Fuchsian surface groups with signature  $(0, 0, 0, 0, 3)$  consisting of neck lengths  $\beta_1, \beta_2, \beta_3$ , and  $\beta_4, \beta_5, \beta_6$ , respectively. Now, to glue these two groups  $\Omega_1$  and  $\Omega_2$ , we cut the two corresponding surfaces (reduced to  $\Omega_1^*$  and  $\Omega_2^*$ ), one funnel end each, along the neck of their funnels, where the lengths of their necks are  $\beta_3$  and  $\beta_4$ , respectively. After this operation, we add a group  $\Omega_3 = \langle h_2, h_3 \rangle$  of signature  $(0, 0, 3, 0, 0)$  consisting of three conformal holes with lengths of their closed geodesic boundaries being  $\beta_3, \beta_4$ , and  $\beta_7$ . In fact, we take  $\Omega_3$  in such a way that it can match with the groups  $\Omega_1^*$  and  $\Omega_2^*$ . After that, we use the combination theorem to obtain a new Fuchsian group  $\Omega_{132} = \langle \Omega_1^*, \Omega_3, \Omega_2^* \rangle = \langle h_1, h_4 \rangle$  of signature  $(0, 0, 1, 0, 4)$ .

In the next stage, we take two groups, one is  $\Omega_{132}$  and the other is  $\Omega_4 = \langle h_5, h_6 \rangle$  of signature  $(0, 0, 0, 0, 3)$ . Then, we proceeded along a similar path as we have done in the earlier construction, that is, cut the two corresponding surfaces of the groups  $\Omega_{132}$  and  $\Omega_4$  (transformed to  $\Omega_{132}^*$  and  $\Omega_4^*$ ), one funnel end each along the neck of their funnels, where the lengths of their necks are  $\beta_6$  and  $\beta_8$ , respectively. Subsequently, we add a group  $\Omega_5 = \langle h_4, h_5 \rangle$  of signature  $(0, 0, 3, 0, 0)$  with closed geodesic boundary lengths are  $\beta_6, \beta_8$ , and  $\beta_{11}$ . So, again utilizing the combination theorem, we obtain  $\Omega_{13254} = \langle \Omega_{132}^*, \Omega_5, \Omega_4^* \rangle = \langle h_1, h_6 \rangle$  of signature  $(0, 0, 2, 0, 5)$ . So, in the

$n$ th stage, we will get  $\Omega_{1325467\dots(2n+1)(2n)} = \langle \Omega_1^*, \Omega_3, \Omega_2^{**}, \Omega_5, \Omega_4^{**}, \Omega_7, \Omega_6^{**}, \dots, \Omega_{(2n+1)}, \Omega_{(2n)}^{**} \rangle = \langle h_1, h_{(2n+2)} \rangle$  of signature  $(0, 0, n, 0, (n+3))$ . Consequently, in this fashion, one can continue this process for an infinite number of times to obtain an infinitely generated group  $\Omega_{13254\dots} = \langle \Omega_1^*, \Omega_3, \Omega_2^{**}, \Omega_5, \Omega_4^{**}, \Omega_7, \Omega_6^{**}, \dots \rangle = \langle h_1, \dots \rangle$ . This infinitely generated group is of type  $(0, 0, \infty, 0, \infty)$  that indicates an infinite type of generalized non-tight flute surface, where 1st, 3rd, 5th, 7th, ... blocks are of infinite hyperbolic area, but 2nd, 4th, 6th, 8th, ... blocks are of finite area (i.e.,  $\pi$ ),  $h_i$ 's are hyperbolic elements,  $i \in \mathbb{N}$ .

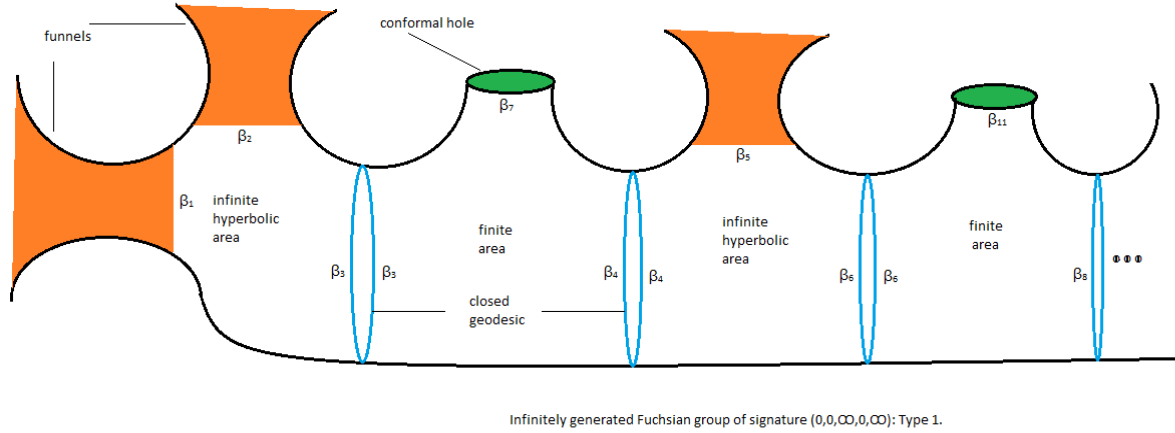


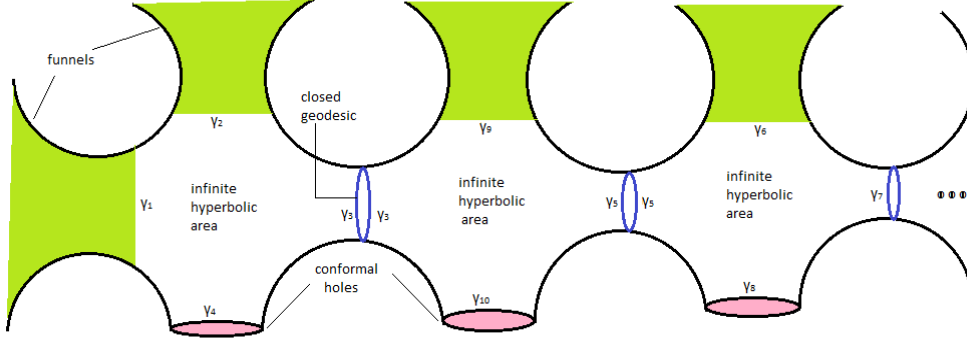
Figure : 8(d)

□

**Proof of Theorem 1.3 (d).** (Constructing infinite type generalized non-tight flute surface: Type 4, see Figure 8(e)).

Let,  $\Phi_1 (= \langle h_1, h_2, h_3 \rangle)$  and  $\Phi_2 (= \langle h_4, h_5, h_6 \rangle)$  be two Fuchsian surface groups with signature  $(0, 0, 1, 0, 3)$  consisting one conformal hole each with lengths of their closed geodesics being  $\gamma_4$  and  $\gamma_8$  respectively and three funnel ends (each) of neck lengths  $\gamma_1, \gamma_2, \gamma_3$ , and  $\gamma_5, \gamma_6, \gamma_7$  respectively. Now, we wish to glue these two groups, but assume that  $\gamma_4 \neq \gamma_8$ . So, we cut the two corresponding surfaces of the groups  $\Phi_1$  and  $\Phi_2$  (changed to  $\Phi_1^*$  and  $\Phi_2^*$ ), one funnel end each, along the neck of their funnels, where the lengths of their necks are  $\gamma_3$  and  $\gamma_5$  respectively. Subsequently, we add a group  $\Phi_3 (= \langle h_3, h_7, h_4 \rangle)$  of signature  $(0, 0, 3, 0, 1)$  consisting of one funnel end of neck length  $\gamma_9$ , and three conformal holes with lengths of their closed geodesic boundaries are  $\gamma_3, \gamma_5$ , and  $\gamma_{10}$ . In particular, we take  $\Phi_3$  in such a way that it can match with

the groups  $\Phi_1^*$  and  $\Phi_2^*$ . So, by the combination theorem, we obtain  $\Pi_{132} = \langle \Phi_1^*, \Phi_3, \Phi_2^* \rangle = \langle h_1, h_2, h_5, h_6, h_7 \rangle$  of signature  $(0, 0, 3, 0, 5)$ .



Infinitely generated Fuchsian group of signature  $(0, 0, \infty, 0, \infty)$ : Type 2.

Figure : 8(e)

In the second stage, we consider two groups: one is  $\Pi_{132}$ , and the other is  $\Phi_4 (= \langle h_8, h_9, h_{10} \rangle)$  of signature  $(0, 0, 1, 0, 3)$  with neck length  $\gamma_{12}$ , and the lengths of the closed geodesic boundaries are  $\gamma_{11}$ ,  $\gamma_{13}$ , and  $\gamma_{14}$ . Then, we proceeded in a similar way as we have operated in the earlier construction, that is, cut the two corresponding surfaces of the groups  $\Pi_{132}$  and  $\Phi_4$  (reduced to  $\Pi_{132}^*$  and  $\Phi_4^*$ ), one funnel end each along the neck of their funnels, where the lengths of their necks are  $\gamma_7$  and  $\gamma_{11}$  respectively. After that, we add  $\Phi_5 (= \langle h_6, h_{11}, h_8 \rangle)$  of signature  $(0, 0, 3, 0, 1)$  with neck length  $\gamma_{15}$ , and closed geodesic boundaries lengths are  $\gamma_7$ ,  $\gamma_{11}$ , and  $\gamma_{16}$ . Now, again utilizing the combination theorem, we obtain a Fuchsian group  $\Pi_{13254} = \langle \Pi_{132}^*, \Phi_5, \Phi_4^* \rangle = \langle h_1, h_2, h_5, h_7, h_9, h_{10}, h_{11} \rangle$  of signature  $(0, 0, 5, 0, 7)$ . Therefore, in the  $n$ th stage, we have  $\Pi_{1325467\dots(2n+1)(2n)} = \langle \Phi_1^*, \Phi_3, \Phi_2^{**}, \Phi_5, \Phi_4^{**}, \Phi_7, \Phi_6^{**}, \dots, \Phi_{(2n+1)}, \Phi_{(2n)}^{**} \rangle = \langle h_1, h_{(2n+2)} \rangle$  of signature  $(0, 0, (2n + 1), 0, (2n + 3))$ . So, in this way, we can continue this process for an infinite number of times to obtain an infinitely generated group  $\Pi_{13254\dots} = \langle \Phi_1^*, \Phi_3, \Phi_2^{**}, \Phi_5, \Phi_4^{**}, \Phi_7, \Phi_6^{**}, \dots \rangle = \langle h_1, h_2, \dots \rangle$ . This infinitely generated group is also of type  $(0, 0, \infty, 0, \infty)$  that provides an infinite type of generalized non-tight flute surface, but here each block carries an infinite hyperbolic area,  $h_i$ 's are hyperbolic elements,  $i \in \mathbb{N}$ .  $\square$

In the following, we have constructed four new kinds of infinite type generalized tight flute surfaces via fundamental (non-compact Fuchsian) groups.

**Proof of Theorem 1.3 (e).** (Constructing infinite type generalized tight flute surface: Type 1, see Figure 8(f)).

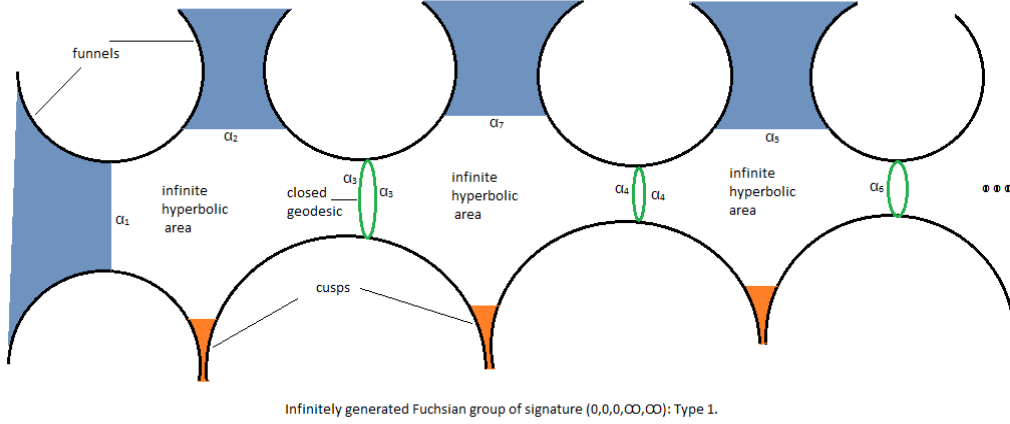


Figure : 8(f)

Let,  $\Xi_1 (= \langle h_1, h_2, p_1 \rangle)$  and  $\Xi_2 (= \langle h_3, h_4, p_2 \rangle)$  be two Fuchsian surface groups with signature  $(0, 0, 0, 1, 3)$  consisting neck lengths  $\alpha_1, \alpha_2, \alpha_3$ , and  $\alpha_4, \alpha_5, \alpha_6$ , where  $h_i$ 's are hyperbolic elements and where  $p_i$ 's are parabolic elements respectively. Now, to obtain a new surface, we will glue these two groups. For this reason, we cut the two corresponding surfaces of the groups  $\Xi_1$  and  $\Xi_2$  (reduced to  $\Xi_1^*$  and  $\Xi_2^*$ ), one funnel end each, along the neck of their funnels, where the lengths of their necks are  $\alpha_3$  and  $\alpha_4$ , respectively. After cutting, we add  $\Xi_3 (= \langle h_2, h_3, p_3 \rangle)$  of signature  $(0, 0, 2, 1, 1)$  consisting of one funnel end of neck length  $\alpha_7$ , one cusp, and two conformal holes with lengths of their closed geodesic boundaries are  $\alpha_3$  and  $\alpha_4$ . In particular, we consider  $\Xi_3$  in such a way that it helps to match with groups  $\Xi_1^*$  and  $\Xi_2^*$ . Now, we can use the combination theorem to obtain a group  $\Xi_{132} = \langle \Xi_1^*, \Xi_3, \Xi_2^* \rangle = \langle h_1, h_4, p_1, p_2, p_3 \rangle$  of signature  $(0, 0, 0, 3, 5)$ .

In the next stage, we take two groups, one is  $\Xi_{132}$ , and the other is  $\Xi_4 (= \langle h_5, h_6, p_4 \rangle)$  of signature  $(0, 0, 0, 1, 3)$  with neck lengths  $\alpha_8, \alpha_9$ , and  $\alpha_{10}$ . After that, we proceeded in a similar way as we have approached in the construction just above, that is, cut the two corresponding surfaces of the groups  $\Xi_{132}$  and  $\Xi_4$  (reduced to  $\Xi_{132}^*$  and  $\Xi_4^*$ ), one funnel end each along the neck of their funnels, where the lengths of their necks are  $\alpha_6$  and  $\alpha_8$ , respectively. Now, we move forward on a similar path as we have operated in the preceding framework, that is, cutting the corresponding

surfaces that arise from the groups  $\Xi_{132}$  and  $\Xi_4$  (reduced to  $\Xi_{132}^*$  and  $\Xi_4^*$ ), one funnel end each along the neck of their funnels, where the lengths of their necks are  $\alpha_6$  and  $\alpha_8$  respectively. After that, we add  $\Xi_5 (= \langle h_4, h_5, p_5 \rangle)$  of signature  $(0, 0, 2, 1, 1)$  with neck length  $\alpha_{11}$ , and the closed geodesic boundary lengths are  $\alpha_6$  and  $\alpha_8$ . Now, again applying the combination theorem, we obtain  $\Xi_{13254} = \langle \Xi_{132}^*, \Xi_5, \Xi_4^* \rangle = \langle h_1, h_6, p_1, p_2, p_3, p_4, p_5 \rangle$  of signature  $(0, 0, 0, 5, 7)$ . So, in the  $n$ th stage, we will get  $\Xi_{1325467\dots(2n+1)(2n)} = \langle \Xi_1^*, \Xi_3, \Xi_2^{**}, \Xi_5, \Xi_4^{**}, \Xi_7, \Xi_6^{**}, \dots, \Xi_{(2n+1)}, \Xi_{(2n)}^{**} \rangle = \langle h_1, h_{(2n+2)}, p_1, p_2, \dots, p_{(2n+1)} \rangle$  of signature  $(0, 0, 0, (2n+1), (2n+3))$ . Hence, in this way, we can continue this process for an infinite number of times to get an infinitely generated group  $\Xi_{13254\dots} = \langle \Xi_1^*, \Xi_3, \Xi_2^{**}, \Xi_5, \Xi_4^{**}, \Xi_7, \Xi_6^{**}, \dots \rangle = \langle h_1, \dots; p_1, \dots \rangle$ . This infinitely generated group is of type  $(0, 0, 0, \infty, \infty)$  that gives rise to an infinite type of generalized tight flute surface, where each block is of infinite hyperbolic area,  $h_i$ 's and  $p_i$ 's are hyperbolic and parabolic elements respectively,  $i \in \mathbb{N}$ .  $\square$

**Proof of Theorem 1.3 (f).** (Constructing infinite type generalized tight flute surface: Type 2, see Figure 8(g)).

Let  $\Psi_1 (= \langle h_1, h_2 \rangle)$  and  $\Psi_2 (= \langle h_3, h_4 \rangle)$  be two Fuchsian surface groups with signature  $(0, 0, 0, 0, 3)$  consisting neck lengths  $\beta_1, \beta_2, \beta_3$ , and  $\beta_4, \beta_5, \beta_6$ , respectively. Now, we aim to glue these two groups to obtain a new surface. So, we cut the two corresponding surfaces of the groups  $\Psi_1$  and  $\Psi_2$  (reduced to  $\Psi_1^*$  and  $\Psi_2^*$ ), one funnel end each, along the neck of their funnels, where the lengths of their necks are  $\beta_3$  and  $\beta_4$ , respectively. After this operation, we add a Fuchsian group  $\Pi_3 (= \langle h_2, p_1 \rangle)$  of signature  $(0, 0, 2, 1, 0)$  consisting of one cusp end with two conformal holes, where the lengths of their closed geodesic boundaries are  $\beta_3$  and  $\beta_4$ . In fact, we take  $\Pi_3$  such that it fits for both groups  $\Psi_1^*$  and  $\Psi_2^*$ . Therefore, applying the combination theorem, we get  $\Psi_{132} = \langle \Psi_1^*, \Pi_3, \Psi_2^* \rangle = \langle h_1, h_4, p_1 \rangle$  of signature  $(0, 0, 0, 1, 4)$ .

In the second stage, we consider two groups, one is  $\Psi_{132}$ , and the other is  $\Psi_4 (= \langle h_5, h_6 \rangle)$  with a signature of  $(0, 0, 0, 0, 3)$  for neck lengths  $\beta_7, \beta_8$ , and  $\beta_9$ . Now, we can continue in an analogous way as we have performed in the preceding formation, that is, cutting the corresponding surfaces arise from the groups  $\Psi_{132}$  and  $\Psi_4$  (reduced to  $\Psi_{132}^*$  and  $\Psi_4^*$ ), one funnel end each along the neck of their funnels, where the lengths of their necks are  $\beta_6$  and  $\beta_7$  respectively. Then, we add a group  $\Pi_5 (= \langle h_4, p_2 \rangle)$  of signature  $(0, 0, 2, 1, 0)$ , where the closed geodesic boundaries lengths are  $\beta_6$  and  $\beta_7$ . Now, again we use the combination theorem and we obtain  $\Psi_{13254} = \langle \Psi_{132}^*, \Pi_5, \Psi_4^* \rangle = \langle h_1, h_6, p_1, p_2 \rangle$  of signature  $(0, 0, 0, 2, 5)$ . Therefore, in the  $n$ th stage, we will get  $\Psi_{1325467\dots(2n+1)(2n)} = \langle \Psi_1^*, \Pi_3, \Psi_2^{**}, \Pi_5, \Psi_4^{**}, \Pi_7, \Psi_6^{**}, \dots, \Pi_{(2n+1)}, \Psi_{(2n)}^{**} \rangle = \langle h_1, h_{(2n+2)}, p_1, p_2, \dots, p_{(2n+1)} \rangle$  of signature  $(0, 0, 0, n, (n+3))$ . Hence, in this way, one

can continue this process for an infinite number of times to get an infinitely generated group  $\Psi_{13254\dots} = \langle \Psi_1^*, \Pi_3, \Psi_2^{**}, \Pi_5, \Psi_4^{**}, \Pi_7, \Psi_6^{**}, \dots \rangle = \langle h_1, \dots; p_1, p_2, p_3, \dots \rangle$ . This infinitely generated group also is of type  $(0, 0, 0, \infty, \infty)$  which produces an infinite type of generalized tight flute surface, but, unlike the previous case, here 1st, 3rd, 5th, 7th, ... blocks are of infinite hyperbolic area, whereas 2nd, 4th, 6th, 8th, ... blocks are of finite area (i.e.,  $\pi$ ),  $h_i$ 's and  $p_i$ 's are hyperbolic and parabolic elements respectively,  $i \in \mathbb{N}$ .

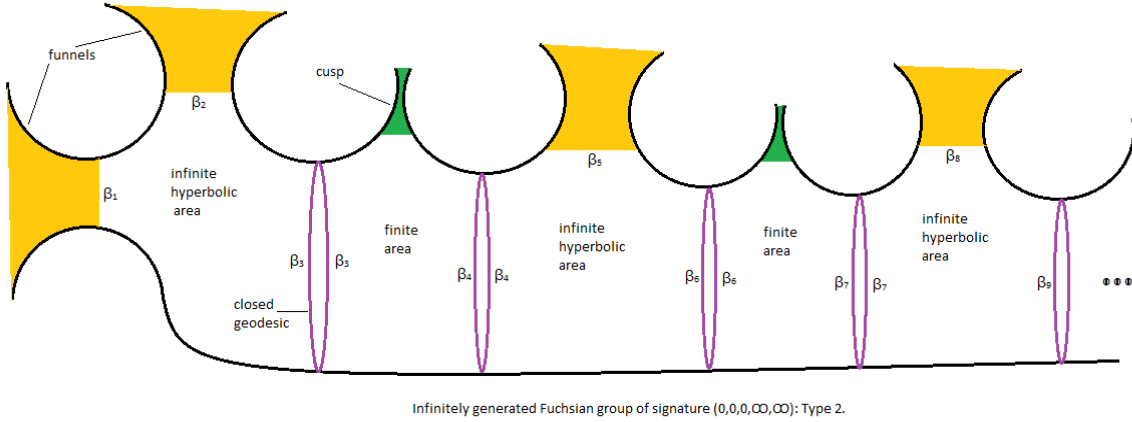


Figure :  $8(g)$

□

**Remark 6.** Note that a group generated by  $\langle h_2, p_1 \rangle$  with signature  $(0, 0, 2, 1, 0)$  consists of one cusp end and two conformal holes. Here, we take  $p_1$  parabolic,  $h_2$  hyperbolic, and the product  $h_2 p_1$  hyperbolic. In particular,  $p_1$  provides the cusp end, whereas the two corresponding lengths of the closed geodesics of the conformal holes arise from the hyperbolic elements  $h_2$  and  $h_2 p_1$ .

**Proof of Theorem 1.3 (g).** (Constructing infinite type generalized tight flute surface: Type 3, see Figure  $8(h)$ ).

Let  $\Sigma_1 (= \langle h_1, h_2 \rangle)$  and  $\Upsilon_1 (= \langle h_3, p_1 \rangle)$  be two Fuchsian surface groups with signatures  $(0, 0, 0, 0, 3)$  and  $(0, 0, 0, 1, 2)$  consisting of neck lengths  $\gamma_1, \gamma_2, \gamma_3$ , and  $\gamma_4, \gamma_5$ , respectively. Now, our goal is to attach these two groups to obtain a new group. So, we cut the two corresponding surfaces of groups  $\Sigma_1$  and  $\Upsilon_1$  (reduced to  $\Sigma_1^*$  and  $\Upsilon_1^*$ ), one funnel end each along the neck of

their funnels, where the lengths of their necks are  $\gamma_3$  and  $\gamma_4$  respectively. After cutting these surfaces, we add a  $\Pi_1$  ( $=\langle h_2, h_3 \rangle$ ) of signature  $(0, 0, 3, 0, 0)$  consisting of three conformal holes, where lengths of their closed geodesic boundaries are  $\gamma_3$ ,  $\gamma_4$ , and  $\gamma_6$ . In particular, we take  $\Pi_1$  such that it can match with groups  $\Sigma_1^*$  and  $\Upsilon_1^*$ . So, by utilizing the combination theorem, we obtain  $\Omega_{111} = \langle \Sigma_1^*, \Pi_1, \Upsilon_1^* \rangle = \langle h_1, p_1 \rangle$  of signature  $(0, 0, 1, 1, 3)$ .

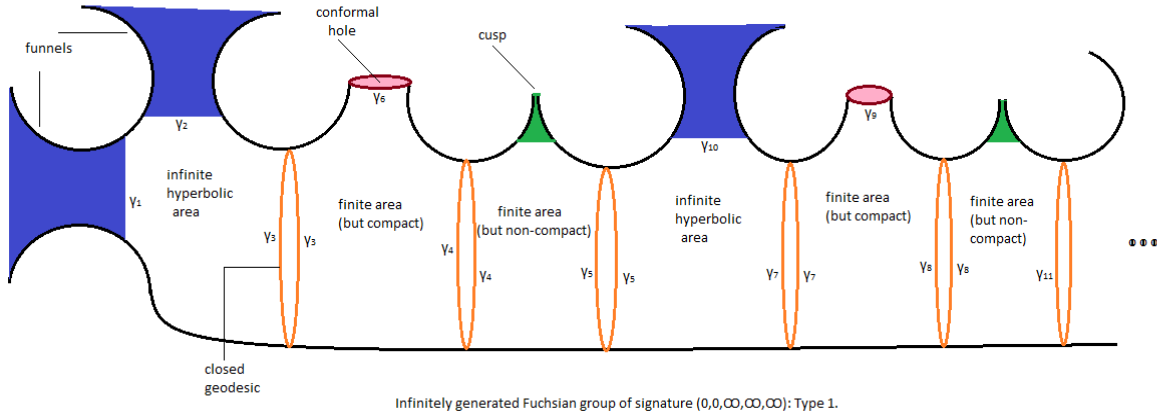


Figure : 8(h)

In the second step, we consider two groups: one is  $\Omega_{111}$ , and the other is  $\Pi_2$  ( $=\langle h_4, h_5 \rangle$ ) of signature  $(0, 0, 1, 0, 2)$  consisting of one conformal hole with length of the closed geodesic is  $\gamma_9$ , and two funnels end of neck length  $\gamma_7$  and  $\gamma_8$ . After that, we will continue in a similar fashion as we have performed in the overhead construction, that is, cut the two corresponding surfaces of the groups  $\Omega_{111}$  and  $\Pi_2$  (transformed to  $\Omega_{111}^*$  and  $\Pi_2^*$ ), one funnel end each, along the neck of their funnels, where the lengths of their necks are  $\gamma_5$  and  $\gamma_7$ , respectively. Furthermore, we add  $\Sigma_2$  ( $=\langle h_6, h_4 \rangle$ ) of signature  $(0, 0, 2, 0, 1)$  with neck length  $\gamma_{10}$ , and the lengths of closed geodesic boundaries are  $\gamma_5$  and  $\gamma_7$ . Now, by applying the combination theorem, we obtain  $\Omega_{11122} = \langle \Omega_{111}^*, \Sigma_2, \Pi_2^* \rangle = \langle h_1, h_5, h_6, p_1 \rangle$  of signature  $(0, 0, 2, 1, 4)$ . In the third step, we further take two groups, one is  $\Omega_{11122}$ , and the other is  $\Sigma_3$  ( $=\langle h_7, h_8 \rangle$ ) of signature  $(0, 0, 0, 0, 3)$  consisting three funnel ends with neck lengths  $\gamma_{11}$ ,  $\gamma_{12}$ , and  $\gamma_{13}$ . Furthermore, we proceeded in a similar way as we have carried out in the preceding formation, that is, cut the two corresponding surfaces of the groups  $\Omega_{11122}$  and  $\Sigma_3$  (converted to  $\Omega_{11122}^*$  and  $\Sigma_3^*$ ), one funnel end each along the neck of their funnels, where the lengths of their necks are  $\gamma_8$  and  $\gamma_{11}$  respectively. In this

situation, we again add a group  $\Upsilon_2 (= \langle h_6, p_2 \rangle)$  of signature  $(0, 0, 2, 1, 0)$ , where the length of the closed geodesic boundaries are  $\gamma_8$  and  $\gamma_{11}$ . So, we can again utilize the combination theorem to obtain a new group  $\Omega_{1112223} = \langle \Omega_{11122}^*, \Upsilon_2, \Sigma_3^* \rangle = \langle h_1, h_5, h_8, p_1, p_2 \rangle$  of signature  $(0, 0, 2, 2, 5)$ . In the fourth step, we will again consider two groups, one is  $\Omega_{1112223}$ , and the other is  $\Upsilon_3 (= \langle h_9, p_3 \rangle)$  of signature  $(0, 0, 0, 1, 2)$ , consisting of one cusp end and two funnel ends with neck lengths  $\gamma_{14}$  and  $\gamma_{15}$ . Furthermore, we cut the two corresponding surfaces of the groups  $\Omega_{1112223}$  and  $\Upsilon_3$  (reduced to  $\Omega_{1112223}^*$  and  $\Upsilon_3^*$ ), one funnel end each along the neck of their funnels, where the lengths of their necks are  $\gamma_{13}$  and  $\gamma_{14}$ , respectively, to glue them to each other. Finally, we again add  $\Pi_3 (= \langle h_8, h_9 \rangle)$  of signature  $(0, 0, 2, 0, 1)$  with neck length  $\gamma_{16}$ , and the lengths of their closed geodesic boundaries are  $\gamma_{13}$ ,  $\gamma_{14}$ , and  $\gamma_{16}$ . Hence, by using the combination theorem, we get  $\Omega_{111222333} = \langle \Omega_{1112223}, \Pi_3, \Upsilon_3^* \rangle = \langle h_1, h_5, p_1, p_2, p_3 \rangle$  of signature  $(0, 0, 3, 3, 5)$ . Observe that this fourth step is basically a repetition of our (previous) first step. Therefore, in this way, one can continue this process for an infinite number of times to get an infinitely generated group  $\Upsilon = \Omega_{111222333\dots} = \langle \Sigma_1^*, \Pi_1, \Upsilon_1^{**}, \Sigma_2, \Pi_2^{**}, \Upsilon_2, \Sigma_3^{**}, \Pi_3, \Upsilon_3^{**}, \dots \rangle = \langle h_1, h_5, \dots; p_1, p_2, p_3, \dots \rangle$ . This group is of type  $(0, 0, \infty, \infty, \infty)$  which provides an infinite type of generalized tight flute surface, where 2nd, 5th, 8th, ... blocks are of non-compact infinite hyperbolic areas, whereas 2nd, 4th, 6th, ... blocks are of compact and hence finite areas (more precisely,  $\pi$ ), on the other hand, 3rd, 6th, 9th, ... blocks are non-compact but finite hyperbolic areas,  $h_i$ 's and  $h_i$ 's are hyperbolic and parabolic elements respectively,  $i \in \mathbb{N}$ .  $\square$

**Remark 7.** Note that a group generated by  $\langle h_3, p_1 \rangle$  with signature  $(0, 0, 0, 1, 2)$  consists of one cusp end and two funnel ends. In fact, we take  $p_1$  parabolic,  $h_3$  hyperbolic, and the product  $h_3 p_1$  hyperbolic. In particular,  $p_1$  provides the cusp end, whereas for the two funnel components, the corresponding two neck lengths are inherent from the axes of two hyperbolic elements  $h_3$  and  $h_3 p_1$ .

**Proof of Theorem 1.3 (h).** (Constructing infinite type generalized tight flute surface: Type 4, see Figure 8(i)).

Let  $\Lambda_1 (= \langle h_1, h_2, h_3 \rangle)$  and  $\Omega_1 (= \langle h_4, h_5, p_1 \rangle)$  be two Fuchsian surface groups with signatures  $(0, 0, 0, 0, 4)$  and  $(0, 0, 0, 1, 3)$  consisting neck lengths  $\delta_1, \delta_2, \delta_3, \delta_4$ , and  $\delta_5, \delta_6, \delta_7$  respectively. Now, we aim to glue these two groups to obtain a new group. So, we cut the two corresponding surfaces of groups  $\Lambda_1$  and  $\Omega_1$  (reduced to  $\Lambda_1^*$  and  $\Omega_1^*$ ), one funnel end each, along the neck of their funnels, where the lengths of their necks are  $\delta_4$  and  $\delta_5$ , respectively. After cutting these surfaces, we add  $\Theta_1$  of signature  $(0, 0, 3, 0, 1)$  consisting of three conformal holes, where the lengths of their closed geodesic boundaries are  $\delta_4, \delta_5$ , and  $\delta_8$ . In particular, we take

$\Theta_1 (= \langle h_3, h_6, h_4 \rangle)$  such that it can match with groups  $\Lambda_1$  and  $\Omega_2$ . Therefore, by utilizing the combination theorem, we get  $\Pi_{132} = \langle \Lambda_1^*, \Theta_1, \Omega_1^* \rangle = \langle h_1, h_2, h_5, h_6, p_1 \rangle$  of signature  $(0, 0, 1, 1, 6)$ .

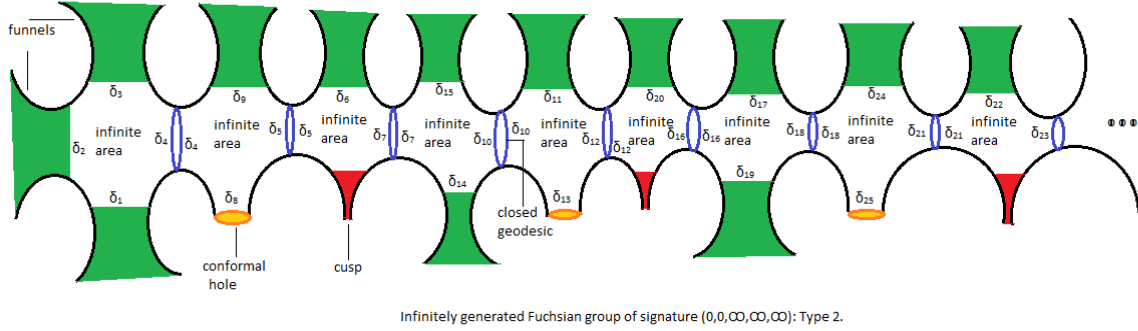


Figure : 8(i)

In the second step, we consider two groups: one is  $\Pi_{111}$ , and the other is  $\Theta_2 (= \langle h_7, h_8, h_9 \rangle)$  of signature  $(0, 0, 3, 0, 1)$  consisting of three conformal holes with lengths of their closed geodesics  $\delta_{10}, \delta_{12}, \delta_{13}$ , and one funnel end of neck length  $\delta_{11}$ . After that, we proceeded in a similar way as we have carried out in the preceding construction, that is, cut the two corresponding surfaces of the groups  $\Pi_{111}$  and  $\Theta_2$  (converted to  $\Pi_{111}^*$  and  $\Theta_2^*$ ), one funnel end each along the neck of their funnels, where the lengths of their necks are  $\delta_7$  and  $\delta_{10}$ , respectively. Furthermore, we add  $\Lambda_5 (= \langle h_6, h_{10}, h_7 \rangle)$  of signature  $(0, 0, 2, 0, 2)$  with neck lengths  $\delta_{14}, \delta_{15}$ , and closed geodesic boundaries lengths are  $\delta_7$  and  $\delta_{10}$ . Now, we again use the combination theorem to obtain a new group  $\Pi_{11122} = \langle \Pi_{111}^*, \Lambda_2, \Theta_2^* \rangle = \langle h_1, h_2, h_5, h_8, h_9, h_{10}, p_1 \rangle$  of signature  $(0, 0, 2, 1, 9)$ .

In the third step, we further take two groups, one is  $\Pi_{11122}$ , and the other is  $\Lambda_3 (= \langle h_{11}, h_{12}, h_{13} \rangle)$  of signature  $(0, 0, 0, 0, 4)$  consisting four funnel ends with neck lengths  $\delta_{16}, \delta_{17}, \delta_{18}$ , and  $\delta_{19}$ . Furthermore, we proceeded in a similar way as we have done in the overhead formation, that is, cut the two corresponding surfaces of the groups  $\Pi_{11122}$  and  $\Lambda_3$  (transformed to  $\Pi_{11122}^*$  and  $\Lambda_3^*$ ), one funnel end each along the neck of their funnels where the lengths of their necks are  $\delta_{12}$  and  $\delta_{16}$  respectively. Now, again, we add a group  $\Omega_2 (= \langle h_{10}, h_{11}, p_2 \rangle)$  of signature  $(0, 0, 2, 1, 1)$  with neck length  $\delta_{20}$ , and closed geodesic boundary lengths are  $\delta_{12}$  and

$\delta_{16}$ . Therefore, one can again utilize the combination theorem to obtain a new group  $\Pi_{1112223} = \langle \Pi_{11122}^*, \Omega_2, \Lambda_3^* \rangle = \langle h_1, h_2, h_5, h_8, h_9, h_{12}, h_{13}, p_1, p_2 \rangle$  of signature  $(0, 0, 2, 2, 12)$ . In the fourth step, we again consider two groups, one is  $\Pi_{1112223}$ , and the other is  $\Omega_3 (= \langle h_{14}, h_{15}, p_3 \rangle)$  of signature  $(0, 0, 0, 1, 3)$ , consisting of one cusp end and three funnel ends with neck lengths  $\delta_{21}$ ,  $\delta_{22}$ , and  $\delta_{23}$ . Further, we cut the two corresponding surfaces of groups  $\Pi_{1112223}$  and  $\Omega_3$  (reduced to  $\Pi_{1112223}^*$  and  $\Omega_3^*$ ), one funnel end each along the neck of their funnels, where the lengths of their necks are  $\delta_{18}$  and  $\delta_{21}$ , respectively, to glue each other. Finally, again, we add a group  $\Theta_3 (= \langle h_{13}, h_{14}, h_{16} \rangle)$  of signature  $(0, 0, 3, 0, 1)$  with neck length  $\delta_{24}$ , and closed geodesic boundary lengths are  $\delta_{18}$ ,  $\delta_{21}$ , and  $\delta_{25}$ . Therefore, by the combination theorem, we obtain  $\Pi_{111222333} = \langle \Pi_{111222333}^*, \Theta_3, \Omega_3^* \rangle = \langle h_1, h_2, h_5, h_8, h_9, h_{12}, h_{15}, h_{16}, p_1, p_2, p_3 \rangle$  of signature  $(0, 0, 3, 3, 14)$ . Note that this fourth step is basically a repetition of our previous first step. Therefore, in this way, one can continue this process for an infinite number of times to get an infinitely generated group  $\Psi_{111222333\dots} = \langle \Lambda_1^*, \Theta_1, \Omega_1^{**}, \Lambda_2, \Theta_2^{**}, \Omega_2, \Lambda_3^{**}, \Theta_3, \Omega_3^{**}, \dots \rangle = \langle h_1, h_2, h_5, h_8, \dots; p_1, p_2, p_3, p_4, \dots \rangle$ . This group also is of type  $(0, 0, \infty, \infty, \infty)$ , but unlike the previous case, each block is of infinite hyperbolic area,  $h_i$ 's and  $h_i$ 's are hyperbolic and parabolic elements respectively,  $i \in \mathbb{N}$ .

This completes the proof of Theorem 1.3. In particular, our third main theorem of this article is established.  $\square$

**Remark 8.** By using the procedure carried out in Theorems 1.2 and 1.3, one can easily construct any infinitely generated Fuchsian groups of signatures  $(\infty, 0, 0, 0, \infty)$ ,  $(\infty, 0, 0, \infty, \infty)$ ,  $(\infty, 0, \infty, 0, \infty)$ , and  $(\infty, 0, \infty, \infty, \infty)$ .

**Notion of limit sets.** Limit sets of our aforementioned constructed infinitely generated groups of types  $(0, 0, 0, 0, \infty)$  and  $(0, 0, \infty, 0, \infty)$  that give rise to infinite type generalized non-tight flute surfaces are Cantor sets, whereas for the types  $(0, 0, 0, \infty, \infty)$  and  $(0, 0, \infty, \infty, \infty)$  lead to infinite types generalized tight flute surfaces, are the whole circle at infinity. Indeed, this type of situation occurred due to the existence of parabolic elements in the infinitely generated Schottky groups of signatures  $(0, 0, 0, \infty, \infty)$  and  $(0, 0, \infty, \infty, \infty)$ . In essence, the infinitely generated groups of signatures  $(0, 0, 0, 0, \infty)$  and  $(0, 0, \infty, 0, \infty)$  are of Fuchsian groups of the second kind, whereas the infinitely generated groups of signatures  $(0, 0, 0, \infty, \infty)$  and  $(0, 0, \infty, \infty, \infty)$  are of Fuchsian groups of the first kind.

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## 9. DATA AVAILABILITY STATEMENT

Our manuscript has no associated data.

## 10. STATEMENTS AND DECLARATIONS

The authors declare that they have no conflict of interest. The authors have no relevant financial or non-financial interests to disclose.

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