# Revealing a Singularity in Collatz Sequences

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## Abstract

The Collatz conjecture, first proposed by Lothar Collatz in 1937, has captivated generations of mathematicians due to its deceptive simplicity and its enduring resistance to proof. Also referred to as the 3n+1 problem, the Syracuse problem, the Ulam conjecture, or the Hailstone sequence, it has spread informally across academic communities, often through oral tradition and recreational mathematics. Its basic rule can be explained to a child, yet its resolution has defied the most brilliant minds in mathematics. As Shizuo Kakutani noted in 1960, “For about a month everyone at Yale worked on it, with no result... A joke was made that this problem was part of a conspiracy to slow down mathematical research in the U.S.” Paul Erdős, in 1983, famously declared that “Mathematics is not yet ready for such questions.” More recently, in 2010, Jeffrey Lagarias [1] described it as “an extraordinarily difficult problem, completely out of reach of present day mathematics. The conjecture sits at the intersection of several mathematical fields, including number theory, dynamical systems, and the study of chaotic behavior. Despite vast numerical evidence and partial results, a general proof remains elusive. In this work, we use the Hidden Order method which reveals many patterns of the Collatz sequences and, more importantly, a singularity that will radically change the understanding of the Collatz dynamic.

# 1 Introduction

Understanding chaotic dynamical systems often begins with an effort to impose order — not to oversimplify, but to illuminate the hidden structures that chaos may conceal. The Collatz sequence, long considered erratic and unpredictable, is no exception. In this work, we construct a more organized and representative tree structure of the Collatz sequences. This restructured tree is not merely a visual aid; it is a strategic reformulation of the problem that allows underlying patterns to emerge more clearly. The act of ordering brings forth what appears to be a singularity, a critical and recurring structure within the dynamics of the Collatz process. This singularity transforms our understanding of how the sequence behaves. Rather than relying on brute-force computation or stochastic models, this approach highlights deterministic structures encoded within the sequence itself; while attempts to prove the Collatz conjecture have so far culminated in the work of Terence Tao (2019), which establishes "Almost all orbits of the Collatz map attain almost bounded values" [2], in this work we are not providing a proof of Collatz conjecture.

# 2 Definition of the Collatz Conjecture

## 2.1 The Collatz Sequence

$$\text{Let } \mathbb{N} = \{0, 1, 2, \dots\}, \text{ and } \qquad \mathbb{N}^* = \mathbb{N} \setminus \{0\}$$

The Collatz sequence is a recursive sequence defined from $n \in \mathbb{N}^*$.

This sequence is constructed by applying the following rule:

- If the current term is odd, multiply it by **3** and add **1**.
- If the current term is even, divide it by **2**.

Thus, starting from an initial term, we obtain the next term by applying the rule according to the parity of the current term.

## 2.2 Mathematical Formulation of the Collatz Sequence

We first define the two auxiliary functions:

$$f_2 : 2\mathbb{N}^* \to \mathbb{N}^*$$
$$f_2(n) = \frac{n}{2} \quad \text{where } 2\mathbb{N}^* \text{ is the set of } n \in \mathbb{N}^*$$

$$f_3 : \{n \in \mathbb{N}^* \mid n \equiv 1 \pmod 2\} \to 2\mathbb{N}^*$$
$$f_3(n) = 3n + 1$$

The Collatz sequence is then defined as follows:

$$C : \mathbb{N}^* \to \mathbb{N}^*$$

$$C_{n+1} = \begin{cases} f_3(C_n) & \text{if } C_n \equiv 1 \pmod 2 \\ f_2(C_n) & \text{if } C_n \equiv 0 \pmod 2 \end{cases}$$

### 2.2.1 Statement of the Collatz Conjecture

The Collatz conjecture states that, for every initial $C_0 \in \mathbb{N}^*$, the Collatz sequence always reaches the value **1** after a finite number of iterations.

It can be formally expressed as:

$$\forall C_0 \in \mathbb{N}^*,\ \exists k \in \mathbb{N}^* \text{ such that } C_k = 1,$$

where $k$ denotes a strictly positive number of iterations.

To this day, the conjecture remains unproven.

# 3 Properties of the Collatz Sequence

## 3.1 Trajectory in the Collatz Sequence

The trajectory of a $C_0 \in \mathbb{N}^*$ is the ordered sequence $(C_n)_{n\geq 0}$ obtained by iterating the Collatz map. It starts at $C_0$ and stops at the first occurrence of 1, when such an occurrence exists.

## 3.2 The Trivial Cycle {1,2,4}

Let: $C_0 = 1$, then:

$$C_1 = f_3(C_0) = 3 \times 1 + 1 = 4$$
$$C_2 = f_2(C_1) = \frac{4}{2} = 2$$
$$C_3 = f_2(C_2) = \frac{2}{2} = 1 = C_0$$

Let: $C_0 = 2$, then:

$$C_1 = f_2(C_0) = \frac{2}{2} = 1$$
$$C_2 = f_3(C_1) = 3 \times 1 + 1 = 4$$
$$C_3 = f_2(C_2) = \frac{4}{2} = 2 = C_0$$

Let: $C_0 = 4$, then:

$$C_1 = f_2(C_0) = \frac{4}{2} = 2$$
$$C_2 = f_2(C_1) = \frac{2}{2} = 1$$
$$C_3 = f_3(C_2) = 3 \times 1 + 1 = 4 = C_0$$

Thus, there exists a cycle $\{1, 2, 4\}$, **called the trivial cycle** $1 \to 4 \to 2 \to 1$.
Since **1** belongs to the cycle, then the set $\{1, 2, 4\}$ satisfies the Collatz conjecture.

Moreover, among the elements of the cycle $\{1, 2, 4\}$, only **4** admits a predecessor outside the trivial cycle.:

$4 = f_2(8) = 8/2$, $8 \notin \{1, 2, 4\}$.

The element 4 is called ***the Gateway*** to the trivial cycle.

## 3.3 Theorem

The Collatz conjecture is valid if and only if the Collatz sequence converges to **4** for every $n \in \mathbb{N}^*$ that does not belong to the trivial cycle, that is to say, for all $n \in \mathbb{N}^* \setminus \{1, 2, 4\}$. This can be expressed as:

$$\forall\ C_0 \in \mathbb{N}^*,\ \exists\ k \in \mathbb{N}^*\ :\ C_k = 1 \quad \Leftrightarrow \quad \forall\ C_0 \in \mathbb{N}^* \setminus \{1, 2, 4\}, \exists\ k \in \mathbb{N}^*\ :\ C_k = 4.$$

## 3.4 Property

It can be shown that every $n \in \mathbb{N}^*$ converges to **4** before reaching **1**. To demonstrate this, we define the *inverse Collatz sequence* $C^{-1}$.

### 3.4.1 Inverse Functions

- $f_2^{-1} : \mathbb{N}^* \to 2\mathbb{N}^*$

$$f_2^{-1}(n) = 2n.$$

- $f_3^{-1} : \{n \in \mathbb{N}^* \mid n \equiv 1 \mod 3, n \equiv 0 \mod 2\} \to 2\mathbb{N}^* + 1$

$$f_3^{-1}(n) = \frac{n-1}{3}$$

### 3.4.2 Inverse Collatz Sequence

$C_0^{-1} : \mathbb{N}^* \to \mathbb{N}^*$

$$C_{n+1}^{-1} = \begin{cases} f_3^{-1}(C_n^{-1}) & \text{if } C_n^{-1} \equiv 1 \mod 3 \text{ and } C_n^{-1} \equiv 0 \mod 2 \\ f_2^{-1}(C_n^{-1}) & \end{cases}$$

Proof of property 4.3:

Let $C_0 \in \mathbb{N}^* \setminus \{1, 2, 4\}$

If $C_0$ satisfies the Collatz conjecture, then:

$$\exists\, k \in \mathbb{N}^* : \quad C_k = 1 \quad \Rightarrow \quad C_{k-1} = f_2^{-1}(C_k) = 2 \times 1 = 2,$$
$$\Rightarrow \quad C_{k-2} = f_2^{-1}(C_{k-1}) = 2 \times 2 = 4.$$

Therefore:

$$\exists\, m \in \mathbb{N}^* : \quad C_m = 4 \quad \text{with} \quad m < k$$

### 3.4.3 Lemma

Every $C_0 \in \mathbb{N}^* \setminus \{1, 2, 4\}$, if it converges to **1**, then it first converges to **4**.

Formal statement:

$$\forall\, C_0 \in \mathbb{N}^* \setminus \{1, 2, 4\}, \quad \exists\, k \in \mathbb{N}^* : C_k = 1 \Rightarrow \exists\, m = k - 2 \ : \ C_m = 4 \quad \text{with} \quad m < k$$

## 3.5 Property

The Collatz conjecture is false in the following cases:

**Case 1:** If there exists $n \in \mathbb{N}^*$ such that the sequence starting at **n** does not reach **1**, that is, it diverges to infinity.

**Case 2:** If there exists a non-trivial cycle, that is, another cycle distinct from the trivial cycle $\{1, 2, 4\}$.

## 3.6 Property

- $\forall\ C_0 \in \mathbb{N}^*$ and $\forall\ k \in \mathbb{N}$ we have :

$$C_k \equiv 1 \pmod 2 \quad \Rightarrow \quad C_{k+1} \equiv 0 \pmod 2$$

  That is to say, every odd term is followed by an even term.

- $\forall\ C_0^{-1} \in \mathbb{N}^*$ and $\forall\ k \in \mathbb{N}$ we have:

$$C_k^{-1} \equiv 1 \pmod 2 \quad \Rightarrow \quad C_{k+1}^{-1} \equiv 0 \pmod 2$$

  That is to say, every odd term in the inverse Collatz sequence is followed by an even term.

- $\forall\ C_0^{-1} \in \mathbb{N}^*$ and $\forall\ k \in \mathbb{N}$ we have:

$$C_k^{-1} \equiv 0 \pmod 2 \ \Rightarrow \begin{array}{l} C_{k+1}^{-1} = f_2^{-1}(C_k^{-1}) \equiv 0 \pmod 2, \\ C_k^{-1} \equiv 1 \pmod 3 \ \Rightarrow \ C_{k+1}^{-1} = f_3^{-1}(C_k^{-1}) \equiv 1 \pmod 2. \end{array}$$

  That is to say, every even term in the inverse Collatz sequence is followed by an even term, but in case the term is congruent to 1 modulo 3, it is followed also by an odd term.

- $\forall\ C_0 \in \mathbb{N}^*$ and $\forall\ k \in \mathbb{N}$ we have:

$$C_k \equiv 0 \pmod 2 \quad \Rightarrow \quad \begin{cases} C_{k+1} = f_2(C_k) \equiv 0 \pmod 2 \text{ if } C_k \equiv 0 \pmod 4 \\ C_{k+1} = f_2(C_k) \equiv 1 \pmod 2 \text{ if } C_k \equiv 2 \pmod 4 \end{cases}$$

  Every even term divisible by 4 is followed by an even term in the Collatz sequence, and every even term that is not divisible by 4 is followed by an odd term.

### 3.6.1 Remark:

Convergence to 1 for $C_0$, $C_k$, $C_{k+1}$, $C_0^{-1}$, $C_k^{-1}$, $C_{k+1}^{-1}$ is not a necessary condition for the property 3.6 to hold.

# 4 Construction of a More Representative Collatz Tree

## 4.1 Division of $\mathbb{N}^*$ into Subsets

- If the $n \in \mathbb{N}^*$, $n \geq 1$ is a powers of **2** then $n$ belongs to the subset:

  $$\mathcal{A} = \left\{ n \in \mathbb{N}^* \,\middle|\, \exists\, k \in \mathbb{N},\ n = 2^k \right\}$$

  Here, **1** is included in $\mathcal{A}$, because $1 = 2^0$.

- If the $n \in \mathbb{N}^*$, $n \geq 1$ is odd then $n$ belongs to the subset

  $$\mathcal{B} = \{ n \in \mathbb{N}^* \mid n \equiv 1 \pmod 2 \}.$$

  We have

  $$\mathcal{A} \cap \mathcal{B} = \{1\}.$$

- If the $n \in \mathbb{N}^*$, $n \geq 1$ is even, but not a power of 2, then **n** belongs to the subset:

  $$\mathcal{M} = \left\{ n \in \mathbb{N}^* \,\middle|\, n = p \cdot 2^k,\ p > 1,\ p \equiv 1 \pmod 2,\ k \geq 1 \right\}$$

  $$\text{with} \quad \mathcal{M} \cap \mathcal{A} = \mathcal{M} \cap \mathcal{B} = \emptyset$$

  Thus, we have the following partition:

  $$\mathbb{N}^* = \mathcal{A} \cup \mathcal{B} \cup \mathcal{M}$$

## 4.2 Description of the Collatz Tree to Be Constructed

The Collatz tree we are about to construct represents all $n \in \mathbb{N}^*$ that satisfy the Collatz conjecture. The representation of these numbers will be based on the **rank** of **n**.

The **rank** of $n \in \mathbb{N}^*$, denoted $\text{rank}(n)$, is defined as the number of $f_3^{-1}$ iterations required to reach a value **n**, using the inverse Collatz sequence $C^{-1}$, and starting from $C_0^{-1} = 1$ as the initial term.

To determine the rank of $n \in \mathbb{N}^*$ and its representation in the Collatz tree, it is not possible to deduce it directly from **n** using $C^{-1}$, instead it is necessary to:

- Verify whether **n** satisfies the Collatz conjecture (otherwise, it cannot be represented in the tree);
- Determine the rank of **n**, rank($n$), as well as the number of successive $f_2$ iterations required to go from one odd number to the next odd number, rank($n$), is the number of $f_3$ iterations required to reach 1, using the Collatz sequence $C$, and starting from **n** as the initial termr.

According to the Collatz sequence, for each term, there is only one possible next term. This implies that every $n \in \mathbb{N}^*$ has a unique trajectory in the Collatz sequence $C$. Therefore, every $n \in \mathbb{N}^*$ that satisfies the Collatz conjecture can only have one unique representation in the Collatz tree.

#### 4.2.1 Properties of the Elements in Subset $\mathcal{A}$ (Powers of 2)


Figure 1: Trunk of the Collatz Tree

$$\forall\ C_0 \in \mathcal{A} :\ C_0 > 1 \quad \text{then} \quad C_0 = 2^k : \quad k \in \mathbb{N}^*$$

$$\Rightarrow \quad C_1 = f_2(C_0) = \frac{C_0}{2} = \frac{2^k}{2} = 2^{k-1}$$

$$\Rightarrow \quad C_k = \frac{2^k}{2^k} = 1$$

Therefore, for any $C_0 \in \mathcal{A} : C_0 > 1$, $C_0$ satisfies the Collatz conjecture.

For $C_0 = 1$, we have previously shown that **1** satisfies the Collatz conjecture.

Thus:

$$\forall\ C_0 \in \mathcal{A}, \quad C_0 \text{ satisfies the Collatz conjecture.}$$

The elements of subset $\mathcal{A}$ will be represented by a vertical line, called the ***trunk*** of the Collatz tree (Figure 1).

### 4.2.2 Remark:

The elements of subset $\mathcal{A}$ exempt 1 have rank zero.

Let $n \in \mathcal{M}$

$$\text{We have: } \mathcal{M} = \left\{ n \in \mathbb{N}^* \,\middle|\, n = p \cdot 2^k,\ p > 1,\ p \equiv 1 \pmod 2,\ k \geq 1 \right\}$$

Thus,

$$f_2^k(n) = f_2^k(p \cdot 2^k) = p$$

where $f_2$ is the division by 2 auxiliary function.

$p \in \mathcal{B}$, so any $n \in \mathbb{N}^*$ $n \in \mathcal{M}$ will be transformed into an element of $\mathcal{B}$ in the Collatz sequence by applying $f_2$ $k$ times.

Therefore, if all the elements of $\mathcal{B}$ satisfy the Collatz conjecture, then all the elements of $\mathcal{M}$ also satisfy the conjecture.

This is expressed as:

$$\forall\ n \in \mathcal{B},\ \exists\ k \in \mathbb{N}^*,\ C_k = 1 \quad \Rightarrow \quad \forall\ m \in \mathcal{M},\ \exists\ j \in \mathbb{N}^*,\ C_j = 1$$

### 4.2.3 Conclusion

We have shown that:

- All elements of the subset $\mathcal{A}$ satisfy the Collatz conjecture.
- If all elements of the subset $\mathcal{B}$ satisfy the conjecture, then all elements of the subset $\mathcal{M}$ also satisfy the conjecture.

And since:

$$\mathcal{A} \cup \mathcal{B} \cup \mathcal{M} = \mathbb{N}^*$$

We derive the following lemma:

### 4.2.4 Lemma

The validity of the Collatz conjecture is equivalent to its validity over the subset $\mathcal{B}$ of all odd $n \in \mathbb{N}^*$.

### 4.2.5 Lemma

We have the subset $\mathcal{B}$ is included in the set of $n \in \mathbb{N}^*$ and does not contain the two elements of the trivial cycle $\{4, 2\}$. and since the sequence starting from 1 reaches 4 before the 1

Then, according to **lemma 3.4.3**, for every odd initial term $C_0 \in \mathcal{B}$, the Collatz sequence always reaches a term $C_k = 4$ before reaching **1**. Therefore, for every $C_0 \in \mathcal{B}$, there exists a term:

$$C_j = 4 \cdot 2^p = 2^{2+p} \quad \text{with} \quad p \in \mathbb{N},$$

and then the term:

$$C_k = 4 = f_2^p(C_j),$$

after $p$ iterations via the function $f_2$.

### 4.2.6 Representation of Elements from $\mathcal{B}$ and $\mathcal{M}$

Each element $n \in \mathcal{B}$ that satisfies the Collatz conjecture will be represented by a distinct point on the Collatz tree.
Any element $m \in \mathcal{M}$ is reducible to an element $n \in \mathcal{B}$, which means $m = n \cdot 2^k \quad \text{with} \quad n > 1,\ n \equiv 1 \pmod 2,\ k \geq 1$, will be represented on a half-line starting from its original point **n**.

This half-line is called a ***Branch*** (Figure 2) and the origin point **n** is called the **Beginning of a branch**.

A branch that is emerging directly from the trunk is called a **Daughter Branch**. If a branch have a Daughter Branch emerging from one of its $\mathcal{M}$ elements, we call it **Mother Branch**.

### 4.2.7 Exception

For the number **1**, it will be represented on the trunk as an element of $\mathcal{B}$, but since rang(1) = 1 we will represent it also as a beginning of a branch without it's element of $\mathcal{M}$, other ways it will generate a second Trunk.

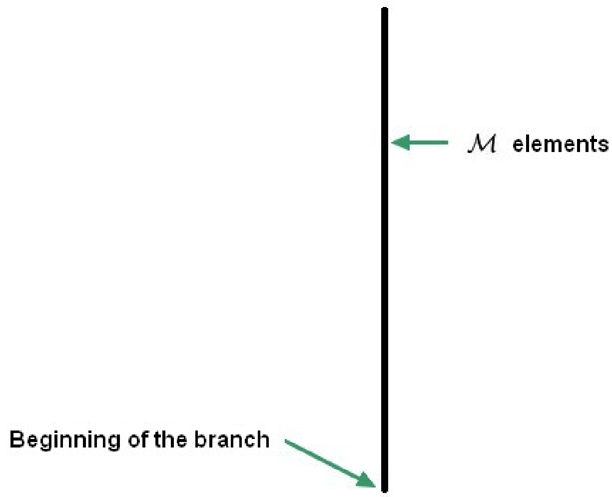


Figure 2: Diagram of a Branch

### 4.2.8 Branch Generator

A **branch generator** (Figure 3) is an even $n \in \mathbb{N}^*$ that belongs to a mother branch or the trunk, which means, it is an element of $\mathcal{A}$ or $\mathcal{M}$.

A $g \in \mathbb{N}^*$ is a branch generator if and only if $f_3^{-1}(g) = n$, with $n$ being a branch beginning.

$$f_3^{-1}(g) = n \quad \Rightarrow \quad n = \frac{g-1}{3} \quad \Rightarrow \quad g \equiv 1 \pmod 3 \quad \text{and} \quad g \equiv 0 \pmod 2$$

Therefore, **g** is a branch generator if and only if:

$$\exists\, n \in \mathcal{B}, g = 3n + 1$$

We represent the connection between a branch and its mother branch (or the trunk) by an inclined **line segment**, running from the top (the branch generator) down to the bottom (the branch beginning).

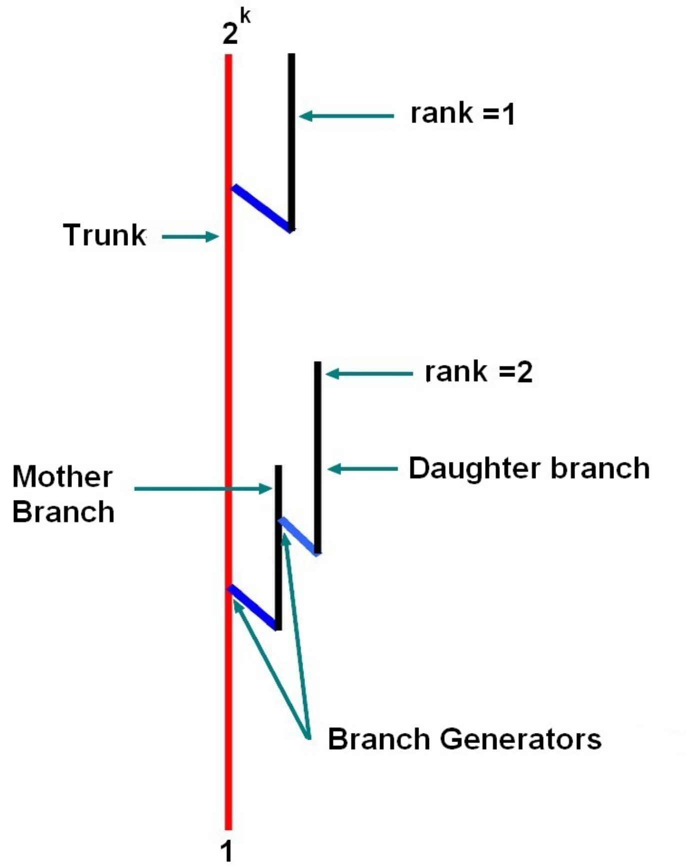


Figure 3: Branch Generator

### 4.2.9 Remark:

Convergence to 1 for the beginning of the mother branch is not a condition to define the daughter branch generator.

### 4.2.10 Sister Branches

Two branches are **sister branches** (Figure 4) if and only if they share the same mother branch or they are daughter branches emerging from the trunk.

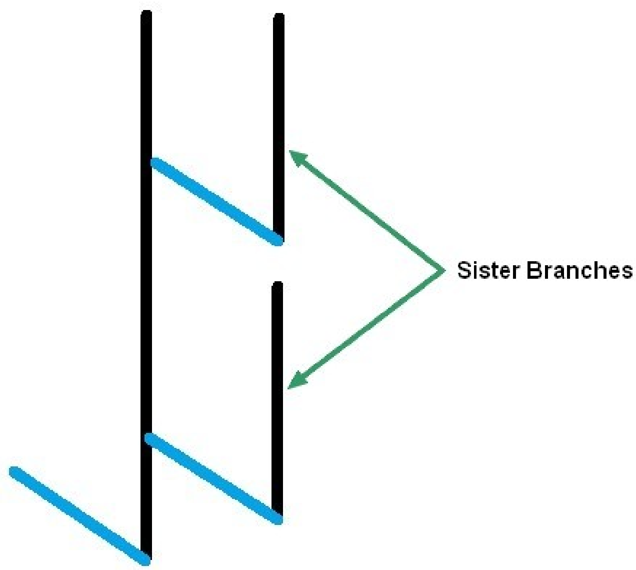


Figure 4: sister branches

### 4.2.11 Remark:

Convergence to 1 for the beginning of the mother branch **is not** a condition to define sister branches.

# 5 Recognition of Structural Patterns in the Collatz Dynamics

## 5.1 The Form of a Branch Generator

Let $g \in \mathbb{N}^*$

$g$ is a branch generator if and only if:

$$\exists\ k \in \mathbb{N}^* \quad \text{such that} \quad k \equiv 1 \pmod 2 \quad \text{and} \quad g = 3k + 1$$

$$k > 0 \Rightarrow g \geq 4$$

$$\text{and } k \equiv 1 \pmod 2 \Rightarrow g \equiv 0 \pmod 2$$

### 5.1.1 Lemma

A $g \in \mathbb{N}^*$ is a branch generator if and only if:

- $g \geq 4$,
- $g \equiv 0 \pmod 2$,
- $g - 1 \equiv 0 \pmod 3$.

### 5.1.2 Remark:

Convergence to 1 for **g** is not a condition for Lemma 5.1.1 to hold.

## 5.2 The First Branch Generator

Now we ask the question: how many times we must multiply by **2** a branch beginning or the beginning of the trunk to obtain the first branch generator?

Let $n \in \mathcal{A} \cup \mathcal{B}$ be the beginning of a branch or the trunk.

$$\text{Then:} \quad \exists\, k \in \mathbb{N},\ n = \begin{cases} 3k, & k \text{ odd} \\ 3k+1, & k \text{ even} \\ 3k+2, & k \text{ odd} \end{cases}$$

**Case 1:** $n = 3k$, $k$ **odd**

Let $p \in \mathbb{N}^*$ such that $2^p \cdot n$ is the first branch generator. So $2^p \cdot n = (3k) \cdot 2^p$ with $k$ odd.

Then the beginning of the branch that is generated from $(3k) \cdot 2^p$ is:

$$d = f_3^{-1}(3k \cdot 2^p) = \frac{(3k \cdot 2^p) - 1}{3} = k \cdot 2^p - \frac{1}{3}$$

$d$ cannot be in $\mathbb{N}^*$. Thus, a branch with a branch beginning with the form $3k$ cannot have a branch generator.

We call a ***dead branch*** (Figure 5) any branch that has a beginning of the form $3k$.

**Case 2:** $n = 3k+1$, $k$ **even**

If we multiply **n** once by **2**, we obtain $2n$.

So:

$$d = f_3^{-1}(2n) = \frac{2n-1}{3} = \frac{2(3k+1)-1}{3} = 2 \cdot k + \frac{1}{3}$$

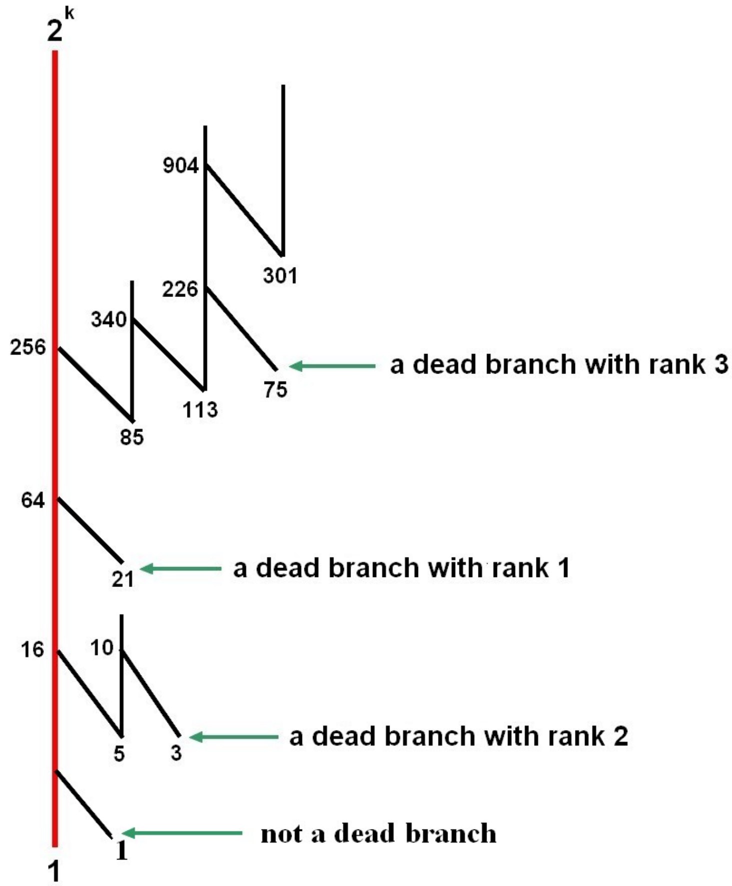


Figure 5: Dead Branches

$d$ cannot be in $\mathbb{N}^*$.

If we multiply **n** twice by **2**, we get $n \cdot 2^2$.
Therefore:

$$d = f_3^{-1}(2^2 n) = \frac{n \cdot 2^2 - 1}{3} = \frac{(3k+1) \cdot 2^2 - 1}{3} = \frac{12k+3}{3} = 4k+1$$

$d$ is odd, so $n \cdot 2^2$ is the first branch generator.

**Case 3:** $n = 3k + 2$, $k$ **odd**
If we multiply **n** once by **2**, we obtain $2n$.
So:

$$d = f_3^{-1}(2n) = \frac{2n-1}{3} = \frac{2(3k+2)-1}{3} = \frac{6k+3}{3} = 2k+1$$

So the first branch generator is $n \cdot 2$.

### 5.2.1 Lemma

For any beginning of a branch or of the trunk **n**:

- If **n** is of the form $3k$ with $k$ odd, then **n** is the beginning of a dead branch that has no daughter branch generator.
- If **n** is of the form $3k+1$ with $k$ even, then the first daughter branch generator is $n \cdot 2^2$.
- If **n** is of the form $3k+2$ with $k$ odd, then the first daughter branch generator is $n \cdot 2^1$.

### 5.2.2 Remark:

Convergence to 1 for **n** is not a condition for Lemma 5.2.1 to hold.

## 5.3 The smallest successor of a branch generator

Let $g$ be a branch generator. Then $g \geq 4$, $g \equiv 0 \pmod 2$, $g \equiv 1 \pmod 3$.

Let $d = f_3^{-1}(g)$ be the beginning of the branch generated by $g$ so $g = 3d+1$ with $d$ odd.

- If we multiply $g$ once by **2** we get:

$$2g = 2(3d+1) = 6d+2$$

  Let $d'$ be the beginning of the branch generated by $2g$.

$$d' = \frac{2g-1}{3} = \frac{2(3d+1)-1}{3} = \frac{6d+1}{3} = 2d - \frac{1}{3}$$

  $d' \notin \mathbb{N}^*$.

- If we multiply $g$ twice by **2**, we get:

$$g \cdot 2^2 = 4(3d+1) = 12d+4$$

Let $d'$ be the beginning of the branch generated by $g2^2$.

$$d' = \frac{4g-1}{3} = \frac{4(3d+1)-1}{3} = \frac{12d+3}{3} = 4d+1$$

with $d$ odd.

Therefore, the smallest successor of a branch generator $g$ is $g \cdot 2^2$.

### 5.3.1 Lemma

*The smallest successor of a branch generator $g$ is $g \cdot 2^2$.*

### 5.3.2 Remark:

Convergence to 1 for $g$ is not a condition for Lemma 5.3.1 to hold.

## 5.4 Expression of the beginning of the sister branch that follows a branch

Let $d_2$ be the beginning of the sister branch that follows immediately the sister branch whose beginning is $d_1$, $g_1$ is the generator of $d_1$ and $g_2$ the generator of $d_2$ (Figure 6).

We will express $d_2$ as a function of $d_1$.

We have:

$$d_2 = \frac{g_2-1}{3} \quad \text{and from } \textbf{Lemma 5.3.1} \text{ we have:} \quad g_2 = 2^2 \cdot g_1$$

We also have:

$$g_1 = 3d_1 + 1$$

Therefore:

$$g_2 = (3d_1+1) \cdot 2^2 \quad \Rightarrow \quad d_2 = \frac{g_2-1}{3} = \frac{(3d_1+1)\cdot 2^2 - 1}{3} = \frac{12d_1+3}{3} = 4d_1+1$$

### 5.4.1 Remark:

Convergence to 1 for $g_1$, $g_2$, $d_1$, $d_2$ is not a condition.

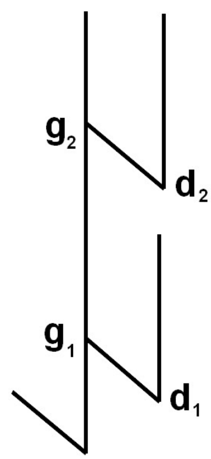


Figure 6: Diagram of sister branches

### 5.4.2 Lemma

*The beginning of the sister branch that follows a branch whose beginning is $d_1$ is $4d_1 + 1$.*

## 5.5 The congruence modulo 3 of branch beginnings and the beginnings of daughter branches.

Let $d_1 = 3k + 1$ with $k$ even, be the beginning of a mother branch. Then, according to **lemma 5.2.1** the first generator of the daughter branch is:

$$g = 2^2 \cdot d_1 = 4(3k + 1)$$

and the beginning of the daughter branch is:

$$d_2 = \frac{g - 1}{3} = \frac{4(3k + 1) - 1}{3} = \frac{12k + 3}{3} = 4k + 1 \quad \text{with} \quad k \text{ even.}$$

## Exemples

| $d_1 = 3k+1$ | $g = d_1 \cdot 2^2$ | $d_2 = 4k+1$ |
|---|---|---|
| 7 | 28 | $9 \equiv 0 \pmod 3$ |
| 13 | 52 | $17 \equiv 2 \pmod 3$ |
| 19 | 76 | $25 \equiv 1 \pmod 3$ |

Let $d_1 = 3k+2$, $k$ odd, be the beginning of a mother branch. This implies that the first generator of the daughter branch is:

$$g = 2^1 \cdot d_1 = 2(3k+2)$$

and the beginning of the daughter branch is :

$$d_2 = \frac{g-1}{3} = \frac{2(3k+2)-1}{3} = \frac{6k+3}{3} = 2k+1$$

## Example

| $d_1 = 3k+2$ | $g = 2^1 \cdot d_1$ | $d_2 = 2k+1$ |
|---|---|---|
| 5 | 10 | $3 \equiv 0 \pmod 3$ |
| 11 | 22 | $7 \equiv 1 \pmod 3$ |
| 17 | 34 | $11 \equiv 2 \pmod 3$ |

Let $d_1$ be the beginning of a mother branch, not necessarily the first daughter branch of the Trunk or another mother branch.

## case 1: $d_1 = 3k+1$, $k$ even

What will be $d_2$ mod 3, with $d_2$ being the beginning of the sister branch that follows immediately the branch having $d_1$ as its beginning?

From **lemma 5.4.2** we have:

$$d_2 = 4d_1 + 1 = 4(3k+1) + 1 = 12k + 5 = 3(4k+1) + 2$$

We observe that $4k+1$ is odd.

Let $4k+1 = k'$, then:

$$d_2 = 3k' + 2$$

## case 2: $d_1 = 3k + 2$, $k$ odd

According to **lemma 5.4.2** :

$$d_2 = 4d_1 + 1 = 4(3k + 2) + 1 = 12k + 9 = 3(4k + 3)$$

We observe that $4k + 3$ is odd. Let $4k + 3 = k'$, then:

$$d_2 = 3k'$$

with $k'$ odd.

## case 3: $d_1 = 3k$, $k$ odd

According to **lemma 5.4.2** :

$$d_2 = 4d_1 + 1 = 4(3k) + 1 = 12k + 1 = 3(4k + 1) + 1$$

We observe that $4k + 1$ is odd. so let $4k + 1 = k'$, then:

$$d_2 = 3k' + 1$$

### 5.5.1 Lemma

Let $d_1 = 3k + a$ with $a \in \{0, 1, 2\}$ be the beginning of a branch, Then the beginning of the sister branch that succeeds immediately is of the form $3k + b$ such that:

$$a + 1 \equiv b \pmod 3$$

### 5.5.2 Remark:

Convergence to 1 for $d_1$ , $d_2$ is not a condition for Lemma 5.5.1 to hold.

## 5.6 Consequence of lemma 5.5.1

Whatever the rank **r**>0, there is always a branch of rank **r**, that is not a dead branch, and it can therefore generate daughter branches. Thus, we can always have branches of rank **r+1** regardless of the rank **r** of a branch.

### 5.6.1 Theorem

$$\forall\ r \in \mathbb{N}, \quad \exists\ d \in \mathcal{B} \cup \mathcal{A} \quad \text{such that} \quad \text{rang}(d) = r$$

where $d$ is the beginning of a branch or of the trunk that satisfies the Collatz conjecture.

$\mathcal{B} \cup \mathcal{A}$ : the union of the two subsets of $\mathbb{N}^*$.

## 5.7 Second Consequence of Lemma 5.5.1

Whatever the beginning of a branch or the trunk, we can always generate a daughter branch whose beginning is a multiple of **3**. That is to say, any beginning of a branch or of the trunk belonging to the trajectory of an initial term which is an odd multiple of **3**. Therefore, the assertion that all elements of subset $\mathcal{B}$ which are multiples of **3** satisfy the Collatz conjecture is equivalent to the assertion that all elements of $\mathcal{B}$ satisfy the Collatz conjecture. Thus, according to **lemmas 5.5.1** and **4.2.4**, we have the following lemma:

### 5.7.1 Lemma

The validity of the Collatz conjecture is equivalent to its validity for every $n \in \mathcal{B}$ such that $n \equiv 0 \pmod 3$.

$\mathcal{B}$ is the subset of all odd $n \in \mathbb{N}^*$ .

## 5.8 The formula for branch generators on the trunk

According to **lemma 5.1.1** : A $g \in \mathbb{N}^*$ is a branch generator if and only if:

$$g \geq 4, \quad g \equiv 0 \pmod 2, \quad g - 1 \equiv 0 \pmod 3$$

Therefore, the first number to test is **4**.

We have : $4 \geq 4$, $4 \equiv 0 \pmod 2$ et $4 - 1 \equiv 0 \pmod 3$.

Thus **4** is the first branch generator on the trunk.

According to **lemma 5.3.1**, the branch generator that succeeds the trunk branch generator $g_1 = 4$ is:

$$g_2 = g_1 \cdot 2^2 = 4 \cdot 2^2 = 2^2 \cdot 2^2 = 2^{2(1+1)}$$

The third branch generator on the trunk is:

$$g_3 = g_2 \cdot 2^2 = 2^{2(1+1)} \cdot 2^2 = 2^{2(1+1+1)}$$

So, we deduce that the **n**-th branch generator on the trunk is $2^{2n}$.

### 5.8.1 Lemma

The $n$-th branch generator on the trunk is $2^{2n}$, with $n \in \mathbb{N}^*$.

## 5.9 The form of rank 1 branch beginnings

Let $d_n$ be the **n**-th beginning of a rank **1**, branch, which is generated by the **n**-th branch generator on the trunk $g_n$ such that:

$$\text{According to Lemma 5.8.1 we have: } g_n = 2^{2n}$$

$$d_n = \frac{g_n - 1}{3}$$

We start with $n = 1$, so the first beginning of a rank **1** branch is:

$$d_1 = \frac{g_1 - 1}{3} = \frac{2^{2\times 1} - 1}{3} = \frac{2^2 - 1}{3} = \frac{4 - 1}{3} = 1$$

Expanding in binary:

$$g_n - 1 = (4 - 1) = (100)_2 - (1)_2 = (11)_2$$

Thus:

$$\begin{aligned}(11)_2 &= (1)_2 + (10)_2 \\ &= (1)_2 + (1)_2 \cdot (10)_2 \\ &= (1)_2 \cdot (11)_2\end{aligned}$$

Hence:

$$d_1 = \frac{4 - 1}{3} = \frac{(1)_2 \cdot (11)_2}{(11)_2} = (1)_2$$

So:

$$d_1 = 2^0 = \sum_{i=0}^{0} 2^{2i}$$

We propose the following formula to be proven by induction, which gives the **n**-th beginning of a rank **1** branch:

$$d_n = \sum_{i=0}^{n-1} 2^{2i} \quad \text{to be proven.}$$

We now formulate the recurrence property $P(n)$ :

$$P(n) \Leftrightarrow d_n = \sum_{i=0}^{n-1} 2^{2i}, \quad n \in \mathbb{N}^*$$

## Initialization

For the initial index $n_0 = 1$, we have:

$$d_1 = \sum_{i=0}^{0} 2^{2i}$$

This is true because:

$$d_1 = 1 = 2^0 = \sum_{i=0}^{1-1} 2^{2i}$$

So $P(1)$ is true.

## Induction

We suppose that $P(k)$ is true for some $k \in \mathbb{N}^*$, that is:

$$d_k = \sum_{i=0}^{k-1} 2^{2i}.$$

We will now show that this implies $P(k+1)$ is also true, that is:

$$d_{k+1} = \sum_{i=0}^{k} 2^{2i}.$$

According to **lemma 5.4.2** we have : $d_{k+1} = 4 \cdot d_k + 1$

So:

$$d_{k+1} = 4 \cdot \left( \sum_{i=0}^{k-1} 2^{2i} \right) + 1 = 2^2 \cdot \left( \sum_{i=0}^{k-1} 2^{2i} \right) + 2^0$$
$$= \left( \sum_{i=0}^{k-1} 2^{2i+2} \right) + 2^0 = \left( \sum_{i=0}^{k-1} 2^{2(i+1)} \right) + 2^0$$

We set j=i+1, then we have:

$$d_{k+1} = \left( \sum_{j=1}^{k} 2^{2j} \right) + 2^{2\times 0} = \left( \sum_{j=0}^{k} 2^{2j} \right)$$

Thus: $P(k+1)$ is true.

## Conclusion

By the principle of mathematical induction, we have thus proven that for all $n \in \mathbb{N}^*$ :

$$d_n = \sum_{i=0}^{n-1} 2^{2i}$$

### 5.9.1 Theorem

The **n**-th beginning of a rank **1** branch is given by:

$$d_n = \sum_{i=0}^{n-1} 2^{2i} : \quad n \in \mathbb{N}^*.$$

## 5.10 Singularity of the Collatz sequence

According to **lemma 4.2.5** and the **lemma 5.8.1**, any $C_0 \in \mathbb{N}^*$, with $C_0 \in \mathcal{B}$ chosen as the first term of the Collatz sequence $C$, and which satisfies the Collatz conjecture, will always yield a term $C_k = 2^{2p}$, $p \in \mathbb{N}^*$, before reaching the term $C_i = 4$, and then the term $C_j = 1$.

So all the elements of $\mathcal{B}$ that satisfy the Collatz conjecture will be transformed by the Collatz sequence into a branch generator located on the trunk:

$$C_k = 2^{2p}, \quad p \in \mathbb{N}^*$$

But before reaching the term $C_k$, we will always have the term:

$$C_{k-1} = \sum_{j=0}^{p-1} 2^{2j} \quad \text{(beginning of a rank } \mathbf{1} \text{ branch).}$$

Having the first term $C_0$ or reaching a term that is the beginning of a rank **1** branch is reaching the **singularity of the Collatz sequence**, because:

- Only the branch beginnings of rank **1** are directly transformed into a power of **2**, $2^{2p} \in \mathcal{A}$, by a single iteration via auxiliary function $f_3$, all other branch beginnings are transformed through iterations via $f_3$ into an element of a subset $\mathcal{M}$.

### 5.10.1 Lemma

All branch beginnings $C_0 \in \mathcal{B}$ that satisfy the Collatz conjecture, if they are not of rank 1, they will be transformed into a branch beginning of a rank **1** by the Collatz sequence. And all branch beginnings of rank 1 will be transformed, via a single $f_3$ iteration, into a power of 2 (branch generator on the trunk), and they are the only ones that allow reaching a power of 2 via a single iteration of $f_3$.
The branch beginnings of rank 1, are the **singularity of the Collatz sequence**.

## 5.11 Consequence of Lemmas 5.10.1 and 5.7.1

The Collatz conjecture is valid if and only if for every $C_0 \in \mathcal{B}$.

$$\exists\ k \in \mathbb{N},\ C_k = \sum_{i=0}^{n} 2^{2i}, \quad n \in \mathbb{N}$$

That is $C_k$ is a branch beginning of a rank 1 ($C_k$ is a Singularity).

And according to **lemma 5.7.1**, the Collatz conjecture is valid if and only if for every $C_0 \in \mathcal{B}$ such that $C_0 \equiv 0 \pmod 3$.

$$\exists\ k \in \mathbb{N},\ C_k = \sum_{i=0}^{n} 2^{2i}, \quad n \in \mathbb{N}$$

That is $C_k$ is a branch beginning of a rank 1 ($C_k$ is a Singularity).

### 5.11.1 Theorem

The Collatz conjecture is true **if and only if**:

$$\forall\, C_0 \in \mathcal{B}, \quad C_0 \equiv 0 \pmod 3.$$

then either $C_0 = \sum_{i=0}^{n} 2^{2i}$, with $n \in \mathbb{N}$, or the Collatz sequence starting from $C_0$ will reach a term on the form :

$$\sum_{i=0}^{n} 2^{2i}, \quad n \in \mathbb{N}$$

that is to say, a branch beginning of a rank 1.

### 5.11.2 The binary structure of the Collatz singularity

We have, any branch beginning of a rank 1 can be written in the form:

$$d = \sum_{i=0}^{n} 2^{2i}, \quad n \in \mathbb{N}$$

where $d$ is the sum of powers of 2, $2^{2i}$ with even exponents.

We know that every $n \in \mathbb{N}$ can be written in binary as a sequence of bits:

$$n = (b_k, b_{k-1}, \ldots, b_1, b_0)_2$$

where each bit $b_i \in \{0, 1\}$. and $b_0$ is the least significant bit (furthest to the right).

We can write:

$$n = b_k 2^k + b_{k-1} 2^{k-1} + \ldots + b_1 2^1 + b_0 2^0$$

with $b_k = 1$, Thus, for $d$, we have:

$$d = \sum_{i=0}^{n} 2^{2i}, \quad n \in \mathbb{N} \implies d = (1_{2n}, 0, 1_{2(n-1)}, 0, \ldots 1_{2\times 2}, 0, 1_{2\times 1}, 0, 1_{2\times 0})_2$$

And therefore, in the binary representation of the beginnings of rank 1, branches, all the bits with an **even index** are equal to **1**, and all the bits with an **odd index** are equal to **0**.

### 5.11.3 Theorem

A $d \in \mathbb{N}^*$, with $d \in \mathcal{B}$ is a Singularity (branch beginning of a rank 1) that is:

$$d = \sum_{i=0}^{n} 2^{2i}, \quad n \in \mathbb{N}$$

*if and only if in its binary representation:*

- all bits with an even index are equal to **1** and
- and all bits with an odd index are equal to **0**.

In other words, the singularity term begins and ends with a bit equal to **1**, and each pair of consecutive bits equal to **1** are separated by one bit equal to **0**.
The number of bits equal to **1** in the binary representation of the term is greater than or equal to 1.

### 5.11.4 Theorem

The Collatz conjecture holds if and only if every $n \in \mathbb{N}^*$ is either a singularity or eventually reaches a singularity under the Collatz iteration.

# 6 Conclusion

The Hidden Order approach used in this work offers a fresh structural perspective on the Collatz problem. By reordering the landscape through a refined Collatz tree, we have uncovered a singularity — a recurring pattern deeply embedded in the dynamics — that challenges the prevailing view of the sequence as chaotic and patternless. This discovery not only reframes our understanding of Collatz sequences but also opens new methodological directions for tackling the conjecture. Currently,i am exploring several additional structural patterns, aiming to expose the underlying deterministic framework. These patterns may provide the necessary scaffolding to move beyond empirical observation and toward formal proof. The results so far are promising, and I believe that a clearer understanding of these hidden regularities will play a key role in resolving this long-standing mathematical enigma.

# References


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