On Some Programs in Mathematical-Physics

Galois, Klein, Lie, Noether & Langlands & more

Lucian M. Ionescu, ISU Oct. 24

(Take note / digest later :)

Goals for Teacher & Students *[Work in progress … "en passant"]*

- Math "at large" can be learned via Math and Its History ("*Math Embryology*" approach).

- Featuring "*breakthrough moments*": Programs in Math-Physics …

- Color Codes: RED / concepts to be taught; GREEN / hints (Examples/History); BLUE: facts (Portals) ...

Zooming out …

- Did you ever walk by foot in Illinois? … not a “scenic drive”, is it? This is how Math “feels” sometimes … so, let’s Zoom-out:

Pic: Illinois, US, Earth …

Rocky Mountains are beautiful …
can’t see this? fly to Denver CO etc.

- With Math is similar: for details (Def., Th. etc.) take a class of Abs. Algebra or Algebraic-Geometry, or Topology … not offered? well … there is Wikipedia anytime/anywhere :)

Programs in Mathematics

(know some Math History & Math Geography)

- The development of Mathematics was marked by *far-reaching new ideas*, known as "Programs", and associated to top Mathematicians, e.g. Galois, Klein, Lie, Noether, Hilbert, Langlands and *much more* …

- They become "Principles" that *changed the landscape of Mathematics* and led to *breakthroughs in Sciences*, notably in Physics, Chemistry and Biology …

Part I: Galois, Klein & Lie, Einstein-Grossman & Noether

L. M. Ionescu, ISU Undergraduate Colloquium - Oct.10, 2024

Evariste Galois & Galois' Principle

Evariste Galois *(Read intro!)* is a "good example" of what a bright young man, passionate about Math can achieve … and change Math forever!

- Problem: What polynomial equations have algebraic formulas for their roots? (Solvable by radicals). Example: quadratic equations have Quadratic Formula …
- Brilliant Idea => Galois Principle: To study a Math Object/Problem associate & study its group of Symmetries":

Object → Group of Symmetries.

hmm … so we need to learn more Abstract Alg. :)

Yes, following the learning path from numbers, operations (2+3=?), functions ($f(x)=x^2$:), operators $\int f(x)dx$ … we have to "group objects" by Categories (of Math structures), look at how objects compare with one another (Morphisms), what "standard constructions" are there (Functors, e.g. Group Ring and Group of Units ;) and natural pairings (Nat. Tr.) etc.:

Add "big words", like salt, from 1st grade!

… when you "cook math", "plate it" & serve it in the classroom … You don't have to **define** the ionic structure of $Na^{\pm}Cl$- nor s,p,d, … orbital theory!

Felix Klein & Abstract Geometry (*Erlangen Program*)

- *Unified Geometries* (Euclidean, Non-Euclidean, projective, Affine etc.) by defining "Abstract Geometries" (113): a group acting on a set (to be generalized to Vector Spaces, Modules etc.).

- In particular, the above groups of transformations are subgroups of Projective Transformations: that's why "Physics is conformal" (Mobius T. <-> Lorentz T.); quantization: finite subgroups of Modular Group SL(2;Z) *[Not Langland's SL2(R)!]*

Sophus Lie & Differential Equations

- Now Klein & Lie were good friends talking Math all the time: a Math Team to understand!

[... *Descartes* played cards with *Pascal* … & “voila”: Probability Theory emerged (put back the fun in Math ;)]

… *Klein unified Geometries & Sophus Lie Diff. Eqs*, by using the Galois Principle: Theory of Lie Groups and Lie algebras emerged (big moment also!).

Who said “We need Limits”? Not Leibnitz!!

By the way … it’s all Algebra AND/OR Geometry! (Ok: & Dynamics :)

Derivation Laws (Leibnitz/ “Power rule” in Calculus) rule Math Models for Natural Systems which are “quantum” and there is no “continuum” (Democrit, Zeno, Tao, Planck etc.).

- Sir Isaac Newton’s “dream”: “All functions are power series” … then Diff. / Int. term-by-term etc.

Indeed: “all natural” ones, “organic” by Mother Nature … but this is another “story” …

Artin’s Formulation of Galois Correspondence

But we need linearization! So first thing first, “relocate” Galois Theory in Abs. Alg. where Linear Algebra is the language (Z-Modules, F-Modules, R-Modules, D-Modules ;)

- Emile Artin formulated the Galois Correspondence between Field Extensions (*Number Systems*: adjoining solutions as needed :) and Galois Groups (their Symmetries) … *But that’s quite general! “GP@work”*

Example Q(√2)

- Galois approach: *adjoin √2 symbol* to Q: a+b√2 and group of permutations of roots +/- √2.

- Artin's approach: consider the field generated by these roots and field automorphisms Gal(Q(√2)/Q). Then study its subfields and subgroups correspondence.

- For details on *operator approach and adapted basis*, in connection to $SL_2(Z)$ representation of real numbers, see L.I. ISU presentation Spring 2024, Slide 12 (Understanding "Number Systems").

Sophus Lie & Differential Galois Theory

Lie and Klein were friends and collaborators …

It was natural for Lie to develop Klein's ideas and extend Galois Principle to the study of Differential Equations, associating to Linear DEs a group of symmetries (Lie group). The modern theory was later developed by Picard and Vessiot …

Differential Fields *[Number Systems & Calculus: the Alg. way!]*

- In a nutshell, differential equations [& D-Modules etc.] are abstracted as Differential Fields (Rings with a derivation), in parallel with the theory of Algebraic Field extensions.

- Differential Algebra is the *"study of Diff. Eqs. & Diff. Operators as algebraic objects, without computing solutions"* [Wiki] (abstractly; no approximations, power series solutions etc.).

… but this is creating "new functions" from "old" [Like Calculu I style: integrate to gate new functions … & behold the power of Log & exp!]

… hmmm … is this introducing "imaginary DE solutions"? adjoining new "symbols"?

Time-out!

… this approach to learning Math, is like learning Spanish using an audiobook: episodes of "real life" use ("To the grocery store", "To the Sport Stadium" etc.).

Learning *Natural Language* is mimicked by Computer Scientists *creating AI* (Deep Learning & Zoom-in/out Hierarchies etc.) … *[& Abstract-Concrete duality … "story"]*

If we don't "upgrade" our methods for teaching students, we'll get behind … my "opinion" ;) … deeply rooted in CS expertise …

Picard-Vessiot Theory (cont. GP@Work)

The modern approach in studying DEs consists in studying **differential fields** (New Category of Objects) and their **differential Galois group** (Aut(Obj) in such a category):

DE $\longrightarrow$ field of solutions & Diff. Gal. group

… i.e. *"just" add a derivation law to Alg. Number Theory* … formal solutions are adjoined to the field of functions defining coefficients, like for algebraic field extensions.

- This is an abstraction of Lie Theory, as Artin did for Galois' approach to the study of polynomial eq. & Number Fields.

Differential Fields and D-Modules (cont.)

- Abelian groups are Z-Modules (Gen. V. Sp.); Galois' groups of permutation of roots of a polynomial equation, now become *Lie groups associated to a Diff. Eq*.

- Def. differential field (DF)   [-> Sheafs & D-Modules]

- Now we study Extensions of Diff. Fields by adjoining formal solutions (Kolchin: "solvable by quadratures", clarifying and extending the work of Picard-Vessiot).

Emmy Noether and a Math-Physics Bridge

- Emmy Noether: father of Modern Algebra (Abstract) and “most influential women in Physics” (einstein).
- Noether Theorems: Lagrangian Symmetries <-> Conservation Laws (GP@ Tangent Bundle level); ex: energy/time invariance, momentum/space tr. & rot.
- Picard-Vessiot ver.: Diff. Galois Group <-> Invariant Integrals (DE solutions; Periods) … Project!

… plus “Noether’s Idea” => Unification Math-Phys

- There is plenty of evidence that measurable Physics quantities are Algebraic Periods (values of algebraic integrals), e.g. Feynman Integrals etc.

- *It is natural to apply the Alg-Geom correspondence to Dynamics (Quantum Physics), as Noether did*: a far-reaching Math-Phys-Chem-Bio-Program emerges (2023): “*Natural Laws are Math Period Laws*” [1], [2], [3] etc. answering “*Wigner’s puzzle*” … (TBC :)

The
Algebra-Geometry
Bridge

… and Dynamics!

… and Langlands Program (1967)

- Later, holomorphic functions with divisors (poles & zeros, i.e. "roots") invariant under such group actions (e.g. by $SL_2(R)$ etc.) become known as automorphic functions & automorphic forms … leading Langlands to formulate his Program in Math *(see Part II of the talk: PAMS Oct. 24)*.

- It's the "same" bridge Algebra-Geometry, started with the work of Descartes, but built with "new (Math) technology"!

… and a bit of History *(to be continued in Part II)*

- Langlands Program focuses on SL2(R), but later developments (1990s Witten-Seiberg Eq., Invariants of Knots, Exotic manifolds in 4D etc.) led to efforts to "*eliminate the continuum*", focusing on SL2(C) (Modular Forms) and Modular Group SL2(Z).

- L.I. 2005-2007 Unifying Connes-Kreimer + Kontsevich Formality Th. => Cohomology of Graphs, Feynman Legacy etc. & towards understanding Algebraic Periods … (and *Digital World Theory*: +CS).

- A New Program emerges: algebraic adeles, bring RH home, *understand Reals as perturbations of SL2Z* (Rethinking Reals) etc., and a new approach to understand Number Fields / Function Fields correspondence (*come to WZ AG Seminar*).

Group Acting on an Object: a unifying concept in Abstract Algebra

(If adding a generous pinch of Category Theory …)

Why is it a "Unifying approach"?

… because of:

1) Cayley Theorem; 2) Distributive Law for Rings; 3) Abelian Groups are Z-modules (Discrete spaces of vectors); 4) It's Klein's Abstract Geometry; 5) Reinterpret Vector Spaces as k-Modules; 6) R-Modules etc. … and in Physics elementary particles "are" irreducible reps etc.

"Rethinking" Abstract Algebra

The AA terms "ring, field, ideal" etc. have been introduced, historically, "as needed"; by now a new better, unifying approach, emerges, focusing on the concept of action, as a morphism from an object of a category to the endomorphism of an object of another category, following the natural progression (To be explained in another talk).

- Project: J. Silverman, L.I., Math-Ed. faculty & GS.

Part II: On Langlands Program and Beyond …

L. M. Ionescu, ISU PAM Seminar, Oct. 24, 2024

Goals

- Learn History of Math development of Modern Algebraic-Geometry
- Learn some key ideas and Math-Tools
- “Parse” Langlands Program: automorphic functions and forms …
- Learn what’s “next” (bridging Math & Physics)

Recall from Part I: Galois' Principle

- Galois: Polynomial eq. → Group of Symm.
- Klein: geometry → Group of symmetries

… and conversely: "*Group acting on a Space is an Abstract Geometry*"!

- Lie: same for Diff. Eqs.
- Noether: same for Dynamical Systems (DEs).

How to reach our goals? Zooming out …

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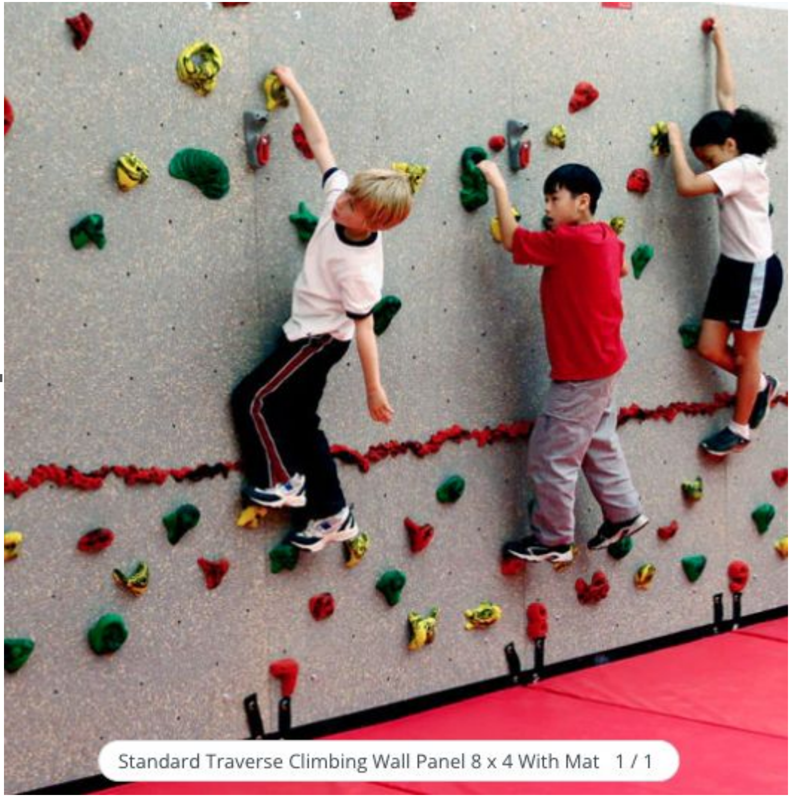
So, from Mounting Climbing to Climbing Wall

We'll take some technical problems to classroom as "toy / practice models" …

Ex. "actions", "reciprocity", "functoriality" etc.



What is *Langlands Program*? (Wiki study mode)

LP is a "*web of far-reaching and consequential conjectures about connections*" [Wiki] between *Algebraic Number Theory* & Algebraic Geometry.

- It tries to understand the historical analogy between Number Fields & Function Fields.

- Simply put, *Langlands Philosophy* allows to understand "*the abstraction of numbers*" [see Wiki], yet LP it is a "*a kind of Grand Unified Theory in Mathematics*" (E. Frenkel).

Number Fields vs. Functions Fields

A few recalls by example …

- Number Fields: Q(i), field extension of Q with roots of $x^2+1=0$ which defines an algebraic variety (sphere S^2) over C;

- Function fields: meromorphic functions on this Riemann Sphere P(x)/Q(x) … etc.

Relating Galois Groups in ANT to Automorphic Forms etc

- A key object of study in LP is automorphic form:

  "well-behaved function from a topological group *G* to the complex numbers (or complex vector space) which is invariant under the action of a discrete subgroup $\Gamma \subset G$ of the topological group"

$$f(g \cdot x) = j_g(x) f(x)$$

- Simpler object: automorphic function j=Identity, i.e. f(x) is *invariant under the action of the group*.

- Familiar case: complex functions are differentiable such that the *tangent map* (Jacobian matrix) is C*-invariant (CR-eq.); now generalize, e.g. from C*->SL_2(C) & to the group setup …

Conjectures & Toy Models

1) Reciprocity: *"L-functions (Geometric side) <-> Dirichlet L-series" (Algebraic side) [see Mellin transform / "Multiplicative Fourier series"]*

2) Functoriality: "*homomorphism of L-groups is expected to give a correspondence between automorphic forms*"

3) Geometric Langlands Conjectures (& history)

[General phenomenon: Deformation vs. Geometric Quantization]

What is “*Reciprocity*”?

- From Legendre symbol & Gauss Quadratic Reciprocity: “*What p is a square in Z/q*?” <-> “*What q is a square Z/p*”

  Example: QR in Z/5 are {1,4}; NR {2,3}

- It is about “*splitting polynomials*” X^2-p=0 ?

- “Reciprocity” (historic term) does not generalizes well; “splitting” does (Ramification Theory wiki): from Artin reciprocity to Langlands reciprocity (not now!)

What is “Functoriality”?

- Recall “Galois correspondence”:
    Galois: Polynomial -> Group (original);
    Artin: Field subExt. <-> Galois subGroup
- Now is this “functorial”? i.e. if G->G’ is a morphism of Galois groups will it define naturally a morphism between Galois correspondences?

- Example of functoriality: group of units of a ring is a functor U:Rings -> Groups; ex. Z-->>Z/6

    proof: ab=e => f(a)f(b)=f(e)=e’; *CATT Diagram*!

… and Beyond:
Unifying Math and Physics
“Natural Laws are Periods Laws”

On “The Fundamental Lemma” (central to LP)

- Concepts and ideas involved: orbital integrals, trace formulas, extensions of concept of Galois groups (Weil group, endoscopic groups etc.).

- These LP-concepts are *technical Math topics related to Quantum Physics* (e.g. Gauge Theory / Connections & Feynman Path Integrals/Periods) … but before “adding some Physics”!!

- N.B. *Neither Langlands nor Grothendieck were also Physicists* …

Grothendieck's Program 1960s too (EGA, SGA)

Modern Abstract Algebra pioneered by Emmy Noether led to Algebraic Topology, Category Theory (S. MacLane & Eilenberg), which enabled Grothendieck to "father" Modern Algebraic-Geometry: Theory of Schemes etc.

- This *categorified* the ideas of Weil & Krul, fathers of Commutative Algebra. In contrast, LP remained "classical", hindered by "plain Abstract Algebra" axiomatic approach, a mix of "cases" [e.g. *Classification of Simple Groups*].

Evolution of Math & Physics in the 80-90s

On the Mathematical-Physics front (where Math takes advantage of Physics intuition), “group theory & reps” evolved into the theory of “Quantum Groups (Defromations of Hopf algebras) …

- On the other hand, the presence of “Reals” R & diff. realm (not complex!) manifested as “exotic R^4-manifolds” (Witten-Seiberg Eq. etc.).

The 21st Century Paradigm

- If we can't "eliminate" the Reals (wrong deformations of Q), recast them via $SL_2(C)$-Continued Fractions representation (see Anurag-Ionescu work), as "place at p=infinity" (Projective space P^1Z<-Q; algebraic adeles etc.).

- Then "*bring RH home*" … *[too much to be sad here/now]*.

- Also view "Numbers" as AG-structures, e.g. periods, i.e. algebraic integrals, as "Fourier coefficients" of a Hodge-de Rham structure (e.g. Riemann-Belyi Theorem setup & Ramific. Th.). This requires "Gauge Theory & Elem. PP".

Beyond Noether with a CATT Mindset

Now applying the knowledge on how 3+1D REality emerges from Quantum Computing / Physics (see LI/vixra): from the fundamental rep. to adjoint rep. (GT & Wilson Loops/FPI), and the new insight into "fundamental constants", via non-dimensionalization, results a *far-reaching program in Math-Physics*:

"*Natural Physics Laws ARE AG-period Laws*"

… as Wigner was noticing(TBC).

(to be continued)

Thank you!

At working level: *It’s all about “Algebraic Periods”*

- Relate with Orbital Integrals (LP), e.g. in Kirillov Coadjoint Orbit method …
- On GT, Feynman Path Integrals, Green functions, convolution (Fourier Transform: d’s & e’s) etc.