Solving Polynomial Equations with Integer Sequences

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Abstract:

Finding a root of polynomial equations is one of basic problems in mathematics. And Galois theory restricts the general radical solution for the degree no higher than four. The series solution, besides the iterated, is regarded as final and universal method to general polynomial equations.

This paper reports a discovery of the standard form of polynomial equations and a class of integer sequences associated thereof, which is a kind of extended Catalan numbers. The solution of polynomial equations in the standard form has a precise and perfect series expression. The convergence condition of the series is clear.

For the general polynomial equations which may not be satisfied with the convergence condition, some proper transformations, like the Tschirnhaus transformation can be employed to guarantee the convergence.

Considering that up to the quintic, there definitely exists a normal form for general equations, and the normal form can easily be changed to the standard form, our method has established a general, universal and effective technique to the quintic, as well as the quartic, the cubic, and the quadratic, without the radicals.

Keywords:

polynomial equation, integer sequence, normal form, standard form, Mingantu-Catalan number, Tschirnhaus transformation

Finding a root of polynomial equation is one of basic problems in mathematics. And Galois theory restricts the general radical solution for the degree higher than four. The series solution, besides the iterated, is regarded as final and universal method to general polynomial equations. [1, 2]

This paper reports a discovery of the standard form of polynomial equations and a class of integer sequences associated thereof, which is a kind of extended Catalan numbers. The solution of polynomial equations in the standard form has a precise and perfect series expression.

1. Catalan numbers and quadratic equation

The Catalan numbers are an integer sequence, which was published by Catalan in 1838 and hence the name.[3,4] And in fact, Mingantu first discovered them no later than 1730's, more than 100 years before. [5,6] So we may call them Mingantu-Catalan numbers.

For $\mathrm{n} \geq 0$, the n-th Mingantu-Catalan number is defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\,n!} = \binom{2n}{n} - \binom{2n}{n-1} \qquad (1)$$

For n=0,1,2,..., the first numbers are

{1, 1, 2, 5, 14, 42, 132, 429, ... }

which is coded as A000108 in the OEIS.[7]

Obviously, the sequence (1) is an integer sequence, and it has an interest geometric meaning, C_n is the number of (n+2)-gon convex polygon non-crossingly divided by triangles.

If using the Mingantu-Catalan numbers to construct a series,

$$\mathrm{x} = \sum_{\mathrm{n}\geq 0} C_n p^{\mathrm{n}} \qquad (2)$$

Directly calculating shows

$$(1 + p + 2p^2 + 5p^3 + 14p^4 + \ldots)^2 = 1 + 2p + 5p^2 + 14p^3 + \ldots \qquad (3)$$

which means that Eq.(2) is a root of the quadratic equation

$$p\mathrm{x}^2 - \mathrm{x} + 1 = 0 \qquad (4)$$

This observation result in the following.

2. Standard form of polynomial equations and associated integer sequences

Based on the Eq.(4), we have

Definition 1: Standard form of polynomial equations
For any integer $D > 0$, the polynomial equation with the form

$$p\mathrm{x}^{\mathrm{D}} - \mathrm{x} + 1 = 0 \tag{5}$$

where p is a real or complex number, is called a standard form.

It has only three terms, and the only parameter p appears as the coefficient of the highest term.

Proposition 1: For polynomial equations expressed in the standard form Eq.(5), there exist an extended Mingantu-Catalan numbers,

$$G_{D,n} = \frac{1}{(D-1)n+1}\binom{Dn}{n} = \binom{Dn}{n} - (D-1)\binom{Dn}{n-1} \tag{6}$$

where: D is the degree; when $D = 2$, $G_{2,n} = C_n$.
Obviously, they are integer sequences and have similar geometric meaning like the Mingantu-Catalan numbers. For example, $\mathrm{G}_{3,\mathrm{n}}$ is the number of (n+3)-gon convex polygon non-crossingly divided by quadrilateral, etc. Generally, $\mathrm{G}_{\mathrm{D,n}}$ may be the number of (n+D)-gon convex polygon non-crossingly divided by (D+1)-gon.

When $D = 3,4,5$, the sequences are,

{1, 1, 3, 12, 55, 273, 1428, 7752, 43263, 246675, ... }
{1, 1, 4, 22, 140, 969, 7084, 53820, 420732, 3362260, ... }
{1, 1, 5, 35, 285, 2530, 23751, 231880, 23950355, 250543370, ... }

They are coded as A001764, A002493, A002294 respectively in the OEIS. One can find a code there up to D =11. But Eq.(6) gives an unlimited number of sequences, they may be included in the OEIS.

From above, we come to

Proposition 2: For the standard form of polynomial equations Eq.(5), its solution can formally be expressed by a series, using the extended Mingantu-Catalan numbers as coefficients,

$$\mathrm{x} = \sum_{\mathrm{n}\geq 0} G_{D,n} p^{\mathrm{n}} \tag{7}$$

If the series converges, it is a root of Eq.(5), which is real or complex depending on p.

Similar to Eq.(3), directly calculating shows,

$$\begin{aligned}
&(1 + p + 3p^2 + 12p^3 + 55p^4 + ...)^3 \ \ = 1 + 3p + 12p^2 + 55p^3 + ..\\
&(1 + p + 4p^2 + 22p^3 + 140p^4 + ...)^4 = 1 + 4p + 22p^2 + 140p^3 + ..\\
&(1 + p + 5p^2 + 35p^3 + 285p^4 + ...)^5 = 1 + 5p + 35p^2 + 285p^3 + ..\\
&\dots , \dots
\end{aligned}$$

Similarly, they contain the solution to the standard form in Eq.(5).

3. Convergence condition and polynomial transformations

For the series in Eq.(7), the convergence condition is obvious.

Proposition 3: The convergence condition for the series in Eq.(7) is

$$q < \frac{1}{\sqrt[\mathrm{n}]{\binom{Dn}{n-1}}} \tag{8}$$

where: $q > 0$ is the absolute value of or the modulus of p.

For $D =$ 2,3,4, and $5, q$ are $\frac{1}{4}, \frac{4}{27}, \frac{27}{256}, \frac{256}{3125}$ respectively.

When the parameter p in the standard form is not satisfied with the convergence condition, some proper polynomial transformations are needed.

There exists a "normal form" for a quintic [8],

$$\mathrm{x}^5 + \mathrm{sx} + \mathrm{t} = 0 \tag{9}$$

That is, for a general quintic,

$$\mathrm{x}^5 + \mathrm{ex}^4 + \mathrm{dx}^3 + \mathrm{cx}^2 + \mathrm{bx}^1 + \mathrm{a} = 0 \tag{10}$$

where: $a \neq 0, b, c, d, e$ are real or complex numbers.

It is always possible to transform the general quintic into the normal form by the Tschirnhaus transformation.[9] In fact, this result is also valid for the degree 2,3, and 4. That is, we have the normal form for the quartic, the cubic, as well as the quadric:

$$x^D + sx + t = 0 \tag{11}$$

where: $D = 2,3,4,5$.

To our knowledge, for the degree great than 5, the normal form is generally difficult or impossible.

Now the transformation from the normal form in Eq.(11) to our standard form in Eq.(5) is simple,

$$p = \frac{1}{t}(-\frac{t}{s})^D \tag{12}$$

Now it is clear that if $p$in the standard form is not satisfied with the convergence condition in Eq.(8), some polynomial transformations, which may include the Tschirnhaus transformation, can be employed to guarentee the transformed standard form consist with Eq.(8) and (12).

Proposition 4: For the general polynomial equations with degree$D = 2,3,4,5$, it is always possible to transform them first to the normal form, and then to the standard form to satisfy the convergence condition to find a root in the series expression.

4. Solving polynomial equations of degrees up to the quintic

Thus, we have

Algorithm 1: Finding a root of polynomial equations of degree D no more than five.

(1) For general polynomial equations of degree D,

$$x^D + a_{D-1}x^{D-1} + ... + a_1x^1 + a_0 = 0$$

where: $a_0 \neq 0$

Using the Tschirnhaus transformation to transform it into the normal form [9]

$$y^D + sy + t = 0$$

(2) Transform the normal form into the standard form by Eq.(12):

$$pz^D - z + 1 = 0$$

(3) Check the convergence condition with Eq.(8),

If the condition is satisfied,

Then calculating the value with Eq.(7) up to a proper integer $n = k$

where: k>0 integer, related to predetermined accuracy;

Quit;

Else

Re-transform the standard form by using proper transformations,

which may include the Tschirnhaus transformation,

into a new normal form,

$$\mathrm{Y^D + SY + T = 0}$$

and a new standard form,

$$P\mathrm{Z^D - Z + 1 = 0}$$

to consist with Eq.(8) and (12);

Calculating the value with Eq.(7) up to a proper integer $n = k$

where: k>0 integer, related to predetermined accuracy;

End if;

(4) Execute the inverse transformations to find a root in original variable x;

(5) Stop.

Some examples are given in the following.

4.1 The Quadratic

Example 1: $f1 \equiv x^2 - 3x - 5 = 0$

Let $x = -\frac{5}{3}y$

The equation is now

$$-\frac{5}{9}\ y^2 - y + 1\ =\ 0$$

Compared with the converged standard form,

$$|p| = \frac{5}{9} > \ \frac{1}{4}$$

out of the convergence interval, the transformation is needed. Let

$$\mathrm{x} = \frac{1}{5}\mathrm{z} + 4$$

we have

$$Pz^2 - z + 1\ =\ 0$$

where: $P = -\frac{1}{25}$, satisfied the convergence condition, calculating z by Eq.(2), with $n = 0,1,2,...,10$,

$$\mathrm{z} = 0.962912018$$
$$\mathrm{x} = 4.192582404$$

Directly solve x,

$$\mathrm{x} = \frac{3}{2} + \frac{\sqrt{29}}{2} \; = \; 4.192582404$$

4.2 The Cubic

Example 2: Wallis equation is [4]

$$f2 \equiv x^3 - 2x - 5 = 0$$

Its standard form is

$$p\ y^3 - y + 1 = 0$$

where: $p = \frac{25}{8} > \frac{4}{27};\ y = -\frac{2}{5}x.$

Let

$$g2 \equiv x^2 - \frac{8}{5}x + z - \frac{4}{3} = 0$$

Find the resultant of f2 and g2 in z, we get

$$z^3 + 17.54666667z - 5.245925917 = 0$$

The new standard form is

$$Pu^4 - u + 1 = 0$$

where: $P = -0.005094017450\ \ , u = -\frac{17.54666667}{5.245925917}z$

Find u by Eq.(7), $u = -0.9949822795$ and $z = 0.2974697944$, then solve equation g4 to get two roots of x: 2.094551482, -0.4945514818.

Compared with the direct solution of f3, the only real root is 2.094551482.

4.3 The Quartic

Example 3: $f3 \equiv x^4 - 2x - 5 = 0$

Its standard form is

$$py^4 - y + 1 = 0$$

where: $p = -\frac{125}{16}$, $|p| > \frac{27}{256}$, $y = -\frac{2}{5}x$.

Let

$$g3 \equiv -\frac{679}{125}x^2 + \frac{679}{75}x + z = 0$$

Find the resultant of f3 and g3 in z, we get

$$z^4 - 8061.489443z + 2687.163147 = 0$$

The new standard form is

$$Pu^4 - u + 1 = 0$$

where: $P = 0.0000045943$, $u = \frac{8061.489443}{2687.16314}z$

Find u by Eq.(7), $u = 1.000004594$ and $z = 0.3333348647$, then solve equation g3 to get two roots of x: 1.702706371, -0.036039704.

Compared with the direct solution of f3, the two real roots are 1.702706371, -1.255937548.

4.4 The Quintic

Example 4: $f4 \equiv x^5 - 2x - 5 = 0$
Its standard form is

$$py^5 - y + 1 = 0$$

where: $p = \frac{625}{32} > \frac{256}{3125}$, $y = -\frac{2}{5}x$.

Let

$$g4 = ax^4 + bx^3 + cx^2 + dx + e + z = 0$$

where:

$$a = -0.0108724526$$
$$b = 0.0003505098$$
$$c = -0.2121296878$$
$$d = 0.6629052744$$
$$e = 0.0173959241$$

Find the resultant of f4 and g4 in z, we get

$$z^5 + 1.166400976z + 0.5832004880 = 0$$

The new standard form is

$$Pu^5 - u + 1 = 0$$

where: $P = -0.05358363143$, $u = -\frac{1.166400976}{0.5832004880}z$.

Find u by Eq.(7), $u = 0.9569901196$ and $z = -0.4764950598$, then solve equation g4 to get two real roots of x: 1.517002600,1.130870670.

Compared with the direct solution of f4, the only real root is 1.517002965.

5. Discussions and conclusions

This paper reports a discovery of the standard form of polynomial equations and the extended Mingantu-Catalan numbers, a class of integer sequences associated with it, which result in a series solution to the equations. For the degree no great than five, it is always possible to transform a general equation into the normal form and to a converged standard form to find a root, this establishes the algorithm for all the quintic, the quartic, the cubic, as well as the quadratic.

For there is only a single coefficient in the standard form, one may design a table to look up for the convenience in applications, just like the trigonometric table.

Finding a way to get a root for the quintic is interesting, which may open a door to higher degree equation. Because a normal form could not be found for a general sextic equation, the current technique is still limited. More work is needed along this direction.

6. References

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