An Integral of the Ising class

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ABSTRACT: In this note, we consider an integral of the Ising class.

Keywords: Number Pi, Double Integrals, Series.

I. Introduction

In this note, we consider the formula:

$$I = \int_0^1 \int_0^1 \left(\frac{1-x}{1+x}\right)^2 \left(\frac{1-y}{1+y}\right)^2 \left(\frac{1-x\,y}{1+x\,y}\right)^2 dx\,dy = 5 - \pi^2 - 4\ln(2) + 16\,(\ln(2))^2 \tag{1}$$

for details see [1],[2].

In this note, we give some formulas related to (1).

Notations:

The number Pi is defined as

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2\,n+1}$$

The number ln(2) is defined as

$$\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

The Gauss hypergeometric function is defined as

$${}_2F_1\,(a,\,b,\,c,\,x) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n\,n!}\,x^n \quad , \quad |x| < 1$$

where $(a)_n = a\,(a+1)\,(a+2) \ldots (a+n-1)$, $(a)_0 = 1$.

II. Formulas related to (1)

Entry 1.

$$I = \int_0^1 \int_0^1 \left(\frac{x}{2-x}\right)^2 \left(\frac{y}{2-y}\right)^2 \left(\frac{x+y-x\,y}{2-x-y+x\,y}\right)^2 dx\,dy \tag{2}$$

$$I = 4\int_0^1 \int_0^1 \left(\frac{x+y}{1+x\,y}\right)^2 \left(\frac{x\,y}{(1+x)\,(1+y)}\right)^2 dx\,dy \tag{3}$$

$$I = 4 \int_0^1 \int_0^1 \left(\frac{1-x}{2-x}\right)^2 \left(\frac{1-y}{2-y}\right)^2 \left(\frac{2-x-y}{2-x-y+x\,y}\right)^2 d\,x\,d\,y \tag{4}$$

$$I = 4 \int_0^1 \int_0^1 \left(\frac{1-x}{(1+x)\,(1+3\,x)}\right)^2 \left(\frac{1-y}{(1+y)\,(1+3\,y)}\right)^2 \left(\frac{1+x+y-3\,x\,y}{1+x+y+5\,x\,y}\right)^2 d\,x\,d\,y \tag{5}$$

$$I = 4 \int_0^1 \int_0^1 \left(\frac{x}{(2-x)\,(4-3\,x)}\right)^2 \left(\frac{y}{(2-y)\,(4-3\,y)}\right)^2 \left(\frac{2\,x+2\,y-3\,x\,y}{8-6\,x-6\,y+5\,x\,y}\right)^2 d\,x\,d\,y \tag{6}$$

$$I = 2 \int_0^1 \int_0^1 \left(\frac{x}{1+x}\right)^2 \left(\frac{1-y}{1+y}\right)^2 \left(\frac{1+x-y+x\,y}{1+x+y-x\,y}\right)^2 d\,x\,d\,y \tag{7}$$

$$I = 2^2 \cdot 3^8 \int_0^1 \int_0^1 \left(\frac{1-x}{9-x^2}\right)^2 \left(\frac{1-y}{9-y^2}\right)^2 \left(\frac{3-x-y-x\,y}{9-3\,x-3\,y+5\,x\,y}\right)^2 d\,x\,d\,y \tag{8}$$

$$I = 36 \int_0^1 \int_0^1 \left(\frac{1-x}{(1+2\,x)\,(2+x)}\right)^2 \left(\frac{1-y}{(1+2\,y)\,(2+y)}\right)^2 \left(\frac{2+x+y-4\,x\,y}{2+x+y+5\,x\,y}\right)^2 d\,x\,d\,y \tag{9}$$

$$I = 2 \int_0^1 \int_0^1 \left(\frac{1-x\,y}{1+x\,y}\right)^2 \left(\frac{1-y}{1+y}\right)^2 \left(\frac{1-x\,y^2}{1+x\,y^2}\right)^2 y\,d\,x\,d\,y \tag{10}$$

Entry 2.

$$I = \sum_{n=0}^{\infty} (n+1)\,2^{-n-3} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(k) \tag{11}$$

where

$$f(k) = \frac{{}_2F_1(1,\,2,\,k+4,\,1/2)\;{}_2F_1(1,\,2,\,k+6,\,1/2)}{(k+3)\,(k+5)} + \left(\frac{{}_2F_1(1,\,2,\,k+5,\,1/2)}{k+4}\right)^2 \tag{12}$$

Entry 3.

$$I = \sum_{n=0}^{\infty} 2^{-n-6} \sum_{k=0}^{n} (k+1) \sum_{m=0}^{n-k} (m+1)\,(n-k-m+1) \sum_{s=0}^{n-k-m+2} \frac{(-1)^s \binom{n-k-m+2}{s}}{\binom{k+s+2}{s}(k+s+3)\binom{m+s+2}{s}(m+s+3)} \tag{13}$$

Entry 4.

$$I = \sum_{n=0}^{\infty} 2^{-n-6} \sum_{k=0}^{n} (k+1) \sum_{m=0}^{n-k} (m+1)\,(n-k-m+1) \sum_{s=0}^{m+2} (-1)^s \binom{m+2}{s} \sum_{r=0}^{m-s+2} \frac{1}{(k+s+r+3)\,(n-k-r+5)} \binom{m-s+2}{r} \tag{14}$$

Entry 5.

$$I = \sum_{n=0}^{\infty} (n+1)\,2^{-n-2} \sum_{k=0}^{n+2} (-1)^k \binom{n+2}{k} \left(\frac{3+k-2\,(1+k)\,{}_2F_1(1,\,1+k,\,4+k,\,-1)}{(1+k)\,(2+k)\,(3+k)}\right)^2 \tag{15}$$

Entry 6.

$$I=\int_0^\infty\int_0^\infty\left(\frac{\sinh(x/2)\sinh(y/2)}{\cosh(x/2)\cosh(y/2)}\right)^4\left(\frac{\cosh(x)\cosh(y)-1}{\cosh(x)\cosh(y)+1}\right)^2\left(\frac{\tanh(x)\tanh(y)}{\cosh(x)\cosh(y)}\right)dx\,dy \tag{16}$$

$$I=\int_0^\infty\int_0^\infty\left(\frac{\cosh(x-y)}{\cosh(x+y)}\right)^2\frac{e^{-4x-4y}}{(\cosh(x)\cosh(y))^2}\,dx\,dy \tag{17}$$

$$I=4\int_0^\infty\int_0^\infty(\tanh(x)\tanh(y)\tanh(x+y))^2\,e^{-2x-2y}\,dx\,dy \tag{18}$$

Entry 7.

$$I=\frac{1}{4}\int_0^1\int_0^1\left(\left(\frac{x}{4-x}\right)^2\left(\frac{y}{4-y}\right)^2\left(\frac{2x+2y-xy}{8-2x-2y+xy}\right)^2+\right.$$
$$\left.2\left(\frac{1+x}{3-x}\right)^2\left(\frac{y}{4-y}\right)^2\left(\frac{2+2x+y-xy}{6-2x-y+xy}\right)^2+\left(\frac{1+x}{3-x}\right)^2\left(\frac{1+y}{3-y}\right)^2\left(\frac{3+x+y-xy}{5-x-y+xy}\right)^2\right)dx\,dy \tag{19}$$

$$I=\frac{1}{4}\int_0^1\int_0^1\left(\left(\frac{2-x}{2+x}\right)^2\left(\frac{2-y}{2+y}\right)^2\left(\frac{4-xy}{4+xy}\right)^2+\right.$$
$$\left.2\left(\frac{1-x}{3+x}\right)^2\left(\frac{2-y}{2+y}\right)^2\left(\frac{4-y-xy}{4+y+xy}\right)^2+\left(\frac{1-x}{3+x}\right)^2\left(\frac{1-y}{3+y}\right)^2\left(\frac{3-x-y-xy}{5+x+y+xy}\right)^2\right)dx\,dy \tag{20}$$

III. Endnote

$$\pi^2=11-\int_0^1\int_0^1\left(\frac{1-xy}{1+xy}\right)^2\frac{(3+2x+3x^2)}{(1+x)^2}\,dx\,dy \tag{21}$$

$$\pi^2=11-\int_0^1\left(\frac{5+x}{1+x}-\frac{4\ln(1+x)}{x}\right)\frac{(3+2x+3x^2)}{(1+x)^2}\,dx \tag{22}$$

IV. References

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