PLATONIC SOLIDS AS STRUCTURED GEOMETRIC OBJECTS

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Abstract

The possibility of constructing Platonic solids from structural elements is shown – Kepler triangles (ratio of legs 1:√1.618..) and Fibonacci (ratio of legs 1:1.618...) – provided that the area of these elements remains unchanged. The number of elements (or pairs of elements) that make up the structure of the "tetrahedron", "octahedron", "cube" increases, thus, by two times, and the "icosahedron" – by five times in relation to the number of elements of the tetrahedron, while the indicator "area of all structural elements of the figure" and radius (r=3) remain unchanged inscribed in the Platonic bodies of the sphere. In addition, the area of the structural elements of two dodecahedra (S=√959325) is equal to the area of the structural elements of any 5 Platonic solids, for example, 5 tetrahedra (or octahedra, cubes, icosahedra) (S=√38373). The possibility shown is in accordance with the text of Plato's work Timaeus, according to which Platonic bodies can "transform into each other...".

1 Introduction

In one of his works, Timaeus [5], Plato develops the "atomistic" doctrine. Plato accepts two types of triangles as "primary atoms" – a rectangular isosceles triangle and a rectangular one with a smaller leg equal to half the hypotenuse, and the square of the larger leg is three times larger than the square of the smaller leg. According to Plato's description in Timaeus [5], four isosceles triangles folded into a square form a complex "atom". You can make a cube out of such squares. Six other shapes folded into an equilateral triangle form a face, from which, in turn, three regular polyhedra can be built: a tetrahedron (4 faces), an octahedron (8 faces) and an icosahedron (20 faces). Interpreting Plato's words – "... if someone chooses and names something even more beautiful ... we will obey him not as an enemy, but as a friend" – as not an accidental slip of the tongue, but as the presented possibility of an alternative way of constructing a tetrahedron, octahedron, cube and icosahedron based on other triangles (other than the two types described Plato in Timaeus), we made an attempt to construct the geometry of Platonic solids by means of two triangles – Kepler and Fibonacci [2].

Plato in his work describes a number of "rebirths" of bodies:

"...all four kinds [of bodies] can be successively reborn into each other… ...four genera are really born from the triangles we have chosen… ... not all genera can resolve into each other and be born from one another by combining a large number of small [quantities] into a small number of large ones, and vice versa"; "... water crushed by fire or air allows one body of fire and two air bodies to form, just as fragments of one dissected part of the air can give rise to It consists of two bodies of fire. But on the contrary, when a small fraction of fire, once in large masses of air, water or earth, is caught up by their movement, crushed in the struggle and crushed, two bodies of fire unite into a single kind of air; or when the air undergoes violence and destruction, of its two bodies and a half, one whole kind of water is formed." [5].

In this regard, we have made an assumption: for the transition (rebirth) of Platonic solids into each other, some constant value (parameter) is required for other variables. It is assumed that the parameter "area of structural elements" (Kepler and Fibonacci triangles) will be used as a similar value (parameter), and their multiple number in the "structure" of Platonic solids will be used as a variable.

2 The main part

To verify the assumption, the following tasks are formulated:

– to investigate the possibility of constructing Platonic solids from structural elements – Kepler and Fibonacci triangles – provided that the area of these elements remains unchanged;

– to show the possibility of constructing Platonic solids from structuring elements – the Kepler and Fibonacci triangles and the "transition" (interconversion) of some Platonic solids into others, in various variations.

To solve these problems, we used geometric two-dimensional planar construction or, in other words, geometric construction of flat shapes. The constructed structured geometric objects– the Platonic solids "tetrahedron" and "octahedron", differ from their traditional form (representation) in that they are not regular polyhedra (more on this in the article [2]).

To construct Platonic bodies "tetrahedron", "octahedron", "cube", "icosahedron" from structural elements – Kepler and Fibonacci triangles – the following conditions must be fulfilled (indicated by us earlier in the article [2]):

– the construction of geometric bodies (Platonic bodies) is allowed by Kepler and Fibonacci triangles of various sizes, but no more than three;

– the number of Kepler and Fibonacci triangles should be the same (represented in pairs in the geometry of bodies by their number and size);

– the Kepler and Fibonacci triangles should form the center of symmetry of the constructed geometric body;

– the Kepler and Fibonacci triangles should make it possible to form the structure-forming elements of a geometric body: its vertices, faces, and center of symmetry.

In addition to the above–mentioned conditions for constructing Platonic solids, we have identified another initial condition: for constructing Platonic solids from structural elements – Kepler and Fibonacci triangles - the parameters of mathematical numerical equality of the volume and area values of these geometric shapes should be used as the initial ones (Table 1) [4]. At the same time, conventional units of measurement (hereinafter referred to as units of measurement) should be used as units of measurement for the lengths and areas of structural elements. As a unit of measurement for a measure of length, we have adopted a measure equal to 1/3 (=1 cu) of the radius of a sphere inscribed in Platonic solids (the value of the parameter "radius of the inscribed sphere" is unchanged).

Table 1 – Mathematical equality of the volume and area values of a sphere and Platonic solids [4]

Parameter \ Figure	Three–dimensional figures – sphere and Platonic solids					
	Sphere	Tetrahedron	Octahedron	Cube	Icosahedron	Dodecahedron
Square , S	113,097… ≈36 π ≈√12791	374,123… ≈√139 968	187,061… ≈√34 992	216	136,459…	149,857…
Volume , V	113,097… ≈36 π ≈√12791	374,123… ≈√139 968	187,061… ≈√34 992	216	136,459…	149,857…
The length of an edge or circle , l	18,85… ≈6 π	14,6969… ≈√216	7,3484… ≈√54	6	3,9695…	2,69416…
The radius of an inscribed circle , r	3	3	3	3	3	3

Taking into account the above conditions, we investigated the possibility of constructing Platonic solids from structural elements – Kepler and Fibonacci triangles – provided that the area of these elements in each of the Platonic solids remains unchanged – and obtained the following variants of structured geometric objects (figures 1, 2, 3, 4, 5, 6, 7, 8, 9, 10). At the same time (as noted earlier [2]), among the four Platonic solids (irregular tetrahedron and octahedron, as well as regular cube and icosahedron) there is a certain numerical pattern: the ratio of pairs of Kepler and Fibonacci triangles in the geometric structure of these bodies (from tetrahedron, octahedron and cube) increases exactly by a factor of 2 (6 pairs of structural elements in a tetrahedron, 12 pairs of structural elements in an octahedron, 24 pairs of structural elements in a cube). And the increase in the number of pairs of structural elements between the tetrahedron and the icosahedron (30 pairs of structural elements) is exactly 5 times, and between the icosahedron and the dodecahedron (120 pairs of structural elements) is exactly 4 times. For the convenience of describing structured geometric objects, we have defined Kepler triangles as "structural elements of the 1st family", and Fibonacci triangles, respectively, as "structural elements of the 2nd family".

Construction of an irregular tetrahedron (12 structural elements in 6 pairs). To construct an irregular tetrahedron, we used six pairs of Kepler and Fibonacci triangles of various sizes (three large and three small Kepler and Fibonacci triangles each) (Table 2).

Table 2 – Structural elements of two families of the irregular polygon "tetrahedron"

Figure	The area of all structural elements of the shape, S	Structural elements 1st family						Number of elements	Structural elements 2nd family						Number of elements
		The smaller Triangle Kepler, conventional units *			The larger triangle Kepler, conventional units *				The smaller Triangle Fibonacci, conventional units *			The larger triangle Fibonacci, conventional units *			
		L_1	L_2	S	L_1	L_2	S	∑	L_1	L_2	S	L_1	L_2	S	∑
The tetrahedron	∑194,745	4,9	6,24	15,288	5,9	7,5	22,125	3/3	3,1	5,016	7,78	4,9	7,93	19,428	3/3

* Note: relative to the initial size of the figure with a sphere with a radius of 3 conventional units inscribed (in this figure)

The hypotenuses of the three smaller Kepler triangles are connected at one end to the vertex of the tetrahedron, and at the other end to the center of symmetry of the tetrahedron. In turn, the hypotenuses of the three large Fibonacci triangles (forming pairs with the previously named Kepler triangles) are connected at one end to the center of symmetry of the tetrahedron, and the other to its vertex (Figures 1, 2). The pairs of triangles formed in this way touch each other with smaller catheters. The hypotenuses of the three large Kepler triangles touch the hypotenuses of the three large Fibonacci triangles, and the hypotenuses of the three smaller Fibonacci triangles touch the smaller legs of the three large Kepler triangles at right angles (Figures 1, 2).

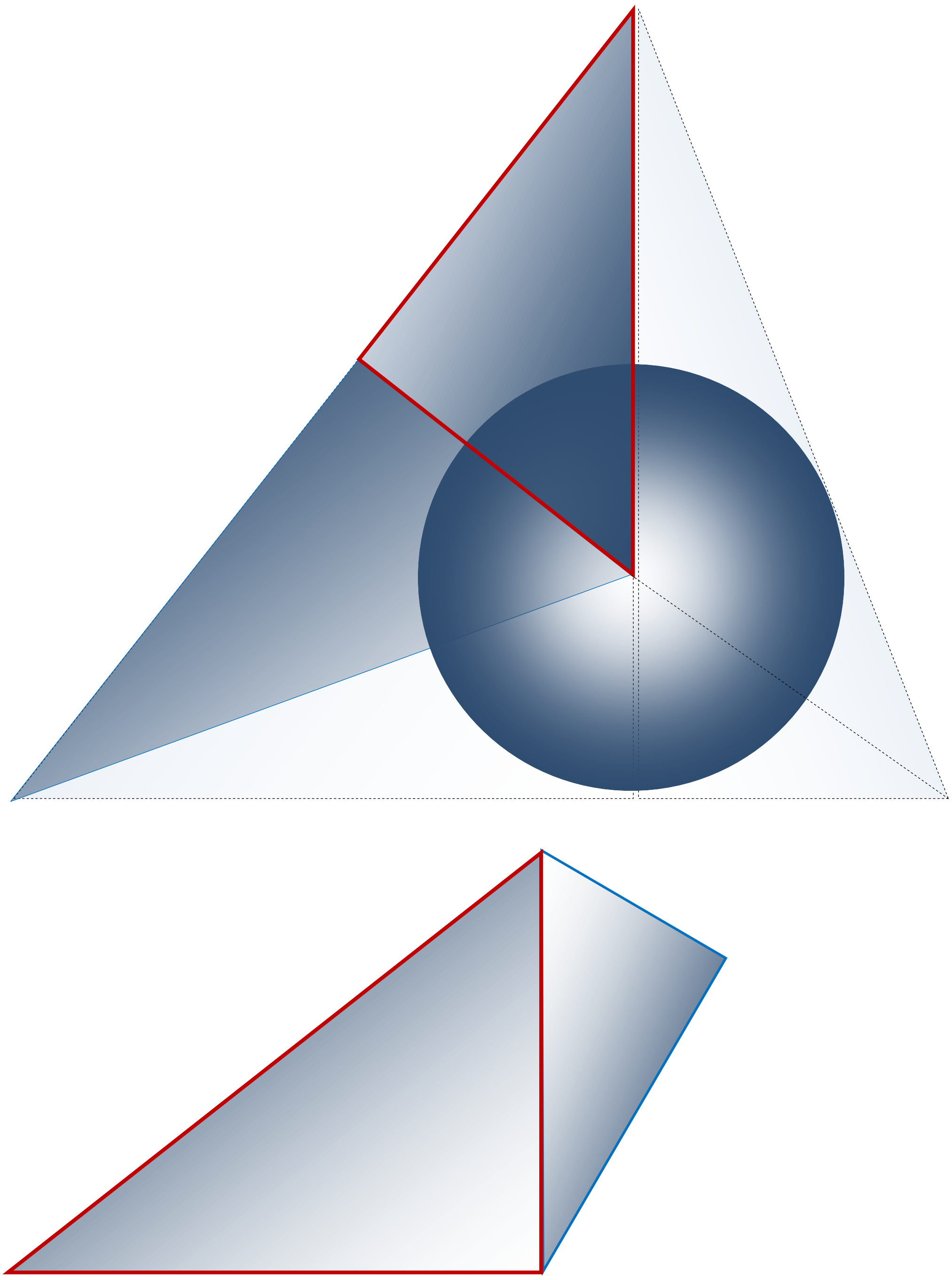

Figure 1 is an irregular tetrahedron. The section of an irregular tetrahedron passing through its vertex and the center of its base is an equilateral triangle (from above). Two-dimensional image of the larger Kepler triangle and the smaller Fibonacci triangle (bottom). The Kepler triangle is marked in red, the Fibonacci triangle in blue.

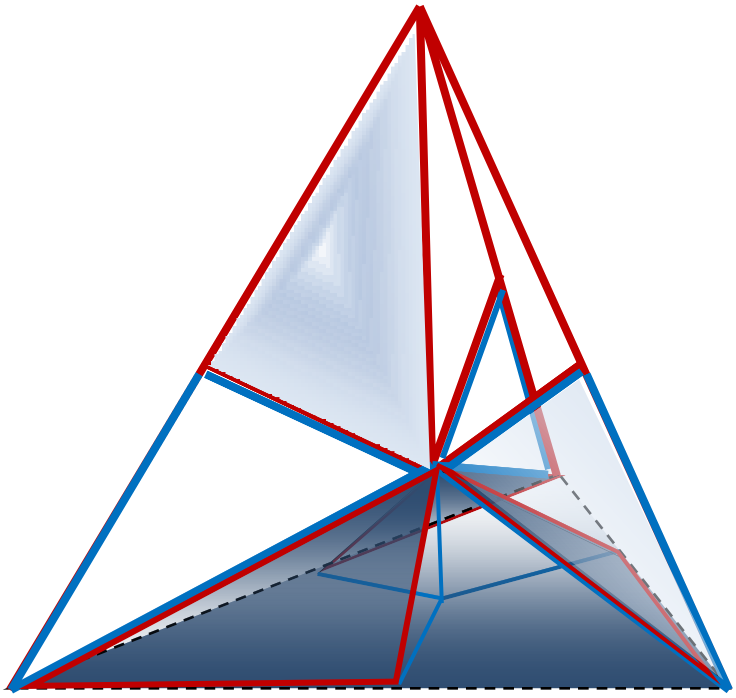

Figure 2 is an irregular tetrahedron. Top and side view. Kepler triangles are marked in red, Fibonacci triangles in blue.

Construction of an irregular octahedron (24 structural elements in 12 pairs). To construct an irregular octahedron, 12 pairs of Kepler and Fibonacci triangles were needed (Table 3), (Figures 3, 4). The "frame" of the body is formed by 8 large Kepler triangles** touching each other with large catheters in the vertical plane and smaller ones in the sagittal and horizontal planes (criss–crosswise) (Figure 3, bottom). The eight large Fibonacci triangles** form two opposite vertices of the body at the junctions of the hypotenuses and large catheters, and the perimeter of the middle part and four vertices of the body with their smaller catheters. The larger catheter of the large Fibonacci triangles** and the hypotenuses of the large Kepler triangles** form the apophemes of the two pyramidal structures of the geometric body. The "internal" structure of an irregular octahedron – the projection of the cross section in the horizontal and sagittal planes – is formed by two smaller and larger Kepler triangles, as well as two smaller and larger Fibonacci triangles (Figure 3, above).

Table 3 – Structural elements of two families of the irregular octahedron polygon

Figure	The area of all structural elements of the shape, S	Structural elements 1st family						Number of elements	Structural elements 2nd family						Number of elements
		The smaller Triangle Kepler, conventional units *			The larger triangle Kepler, conventional units *				The smaller Triangle Fibonacci, conventional units *			The larger triangle Fibonacci, conventional units *			
		L_1	L_2	S	L_1	L_2	S	∑	L_1	L_2	S	L_1	L_2	S	∑
The octahedron	∑195,704	1,77	2,25	1,99	2,8	3,56	4,984	2/2	1,73	2,8	2,422	2,25	3,64	4,095	2/2
		–	–	–	3,82**	4,86**	9,2826**	8	–	–	–	3,82**	6,182**	11,80762**	8

* Note: relative to the initial size of the figure with a sphere with a radius of 3 conventional units inscribed (in this figure), ** Note: the larger Kepler triangle** and the larger Fibonacci triangle **

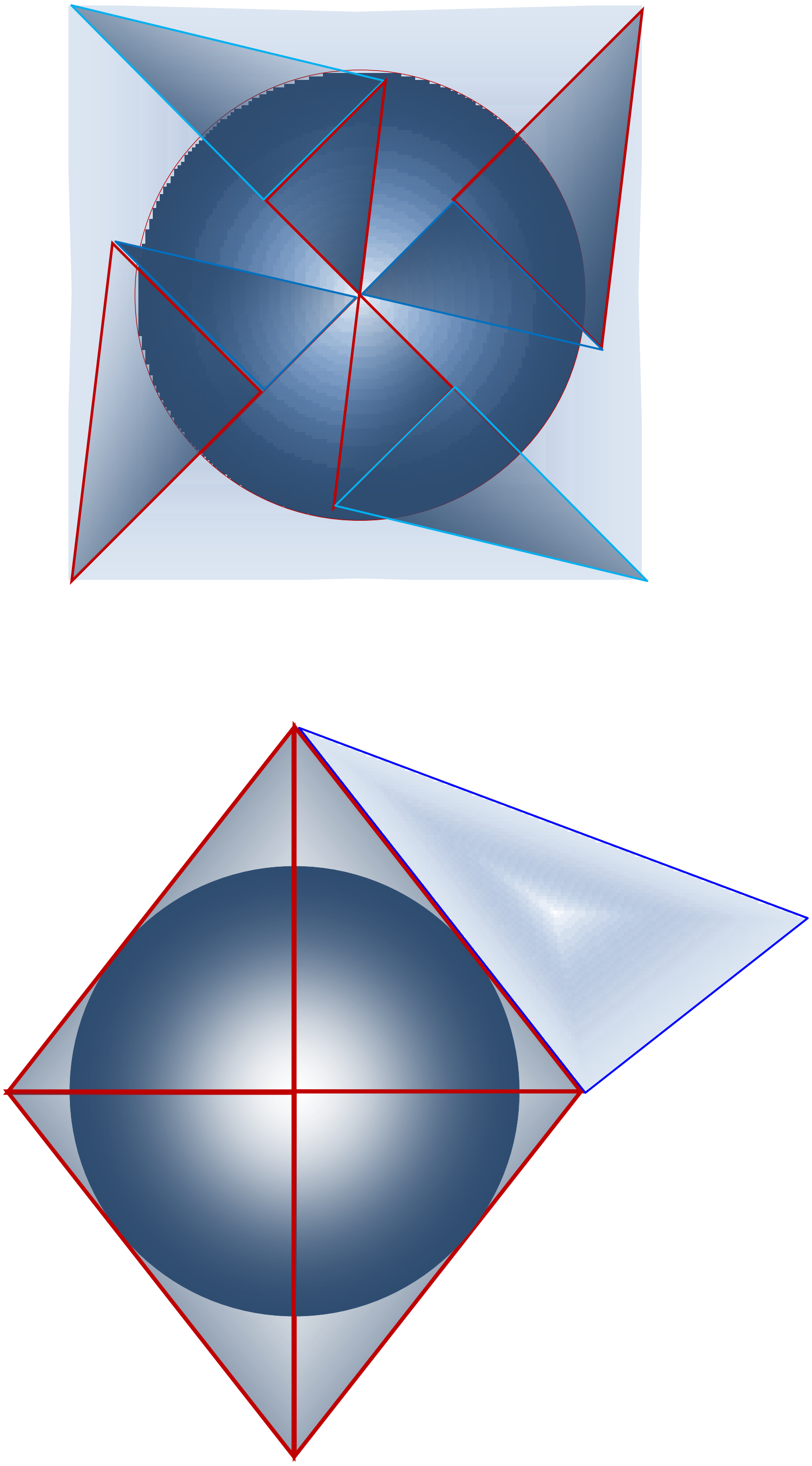

Figure 3 – Cross section (top) and longitudinal section (bottom) of an irregular octahedron passing through its center of symmetry. Kepler triangles are indicated in red, Fibonacci triangles in blue.

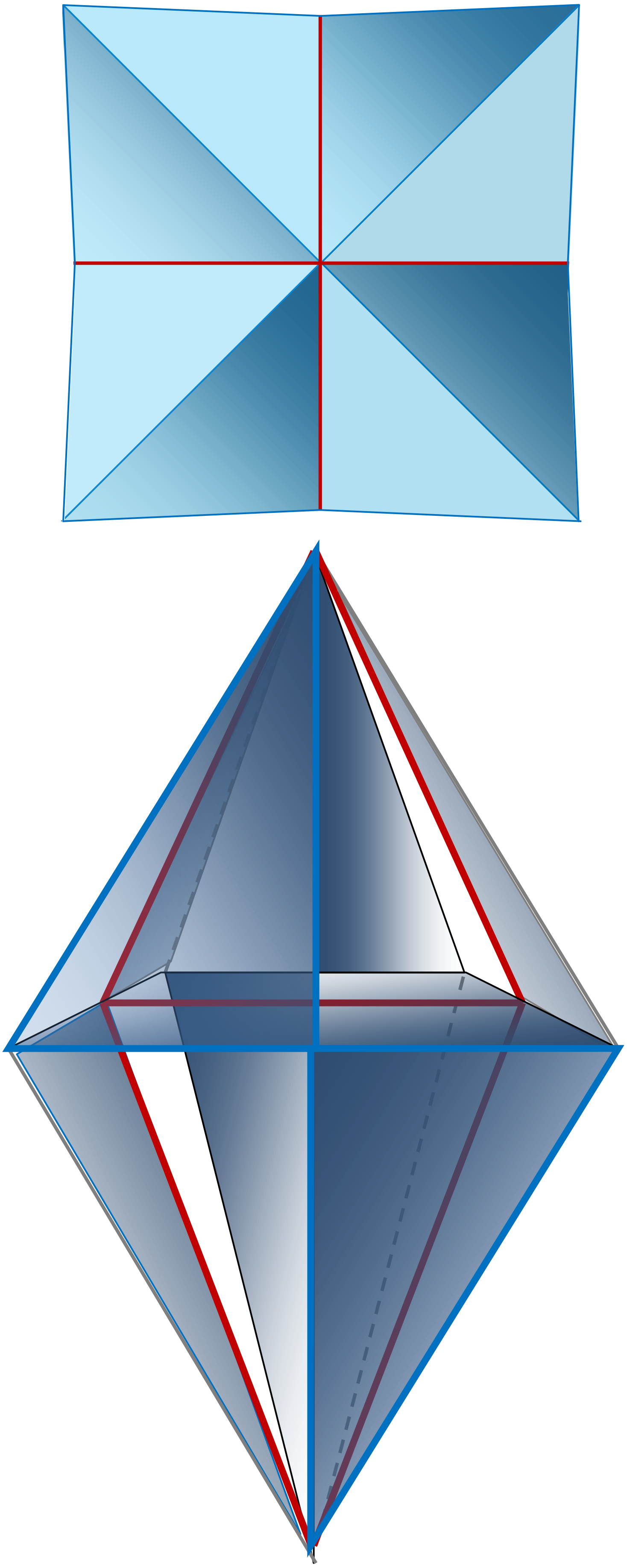

Figure 4 is an irregular octahedron. Top view (top), side view (bottom). Kepler triangles are indicated in red, Fibonacci triangles in blue.

Building a regular cube (48 structural elements in 24 pairs). Constructing the geometry of a cube using Kepler and Fibonacci triangles requires 24 pairs of them. Figure 5 shows two variants (out of six similar to these two) of diagonal sections of a cube, each of which passes through its 4 vertices and the center of symmetry.

In the first variant of the diagonal section of the cube (the first type of cube plane (4 planes in total)) (Figure 5, top) two large and two smaller Kepler triangles are connected in one case by hypotenuses with 2 large Fibonacci triangles, and in the other by a hypotenuse with a large leg of the larger Fibonacci triangle. The junctions of the large legs and hypotenuses of the Kepler and Fibonacci triangles in this variant form the center of symmetry of the cube.

In the second variant of the diagonal section of the cube (the second type of the cube plane (2 planes in total)) The vertices of the cube are formed by connecting the legs of the 4 smaller Fibonacci triangles, the hypotenuses of which are common to the smaller legs of the 4 large Kepler triangles (Figure 5, bottom). The junctions of the large legs and hypotenuses of the Kepler triangles in this variant form the center of symmetry of the cube.

The total number of structural elements of the cube, their size (the initial size of the shape with a sphere with a radius of 3 units inscribed in it) and their area are shown in Table 4.

Table 4 – Structural elements of two families of a regular polygon "cube"

Figure	The area of all structural elements of the shape, S	Structural elements 1st family						Number of elements	Structural elements 2nd family						Number of elements
		The smaller Triangle Kepler, conventional units *			The larger triangle Kepler, conventional units *				The smaller Triangle Fibonacci, conventional units *			The larger triangle Fibonacci, conventional units *			
		L_1	L_2	S	L_1	L_2	S	∑	L_1	L_2	S	L_1	L_2	S	∑
Cube	∑195,528	2,37	3,0146	3,5723	2,8	3,58	5,012	8/16	1,45	2,34	1,6965	2,37	3,86	4,5741	8/16

* Note: relative to the initial size of the figure with a sphere with a radius of 3 conventional units inscribed (in this figure)

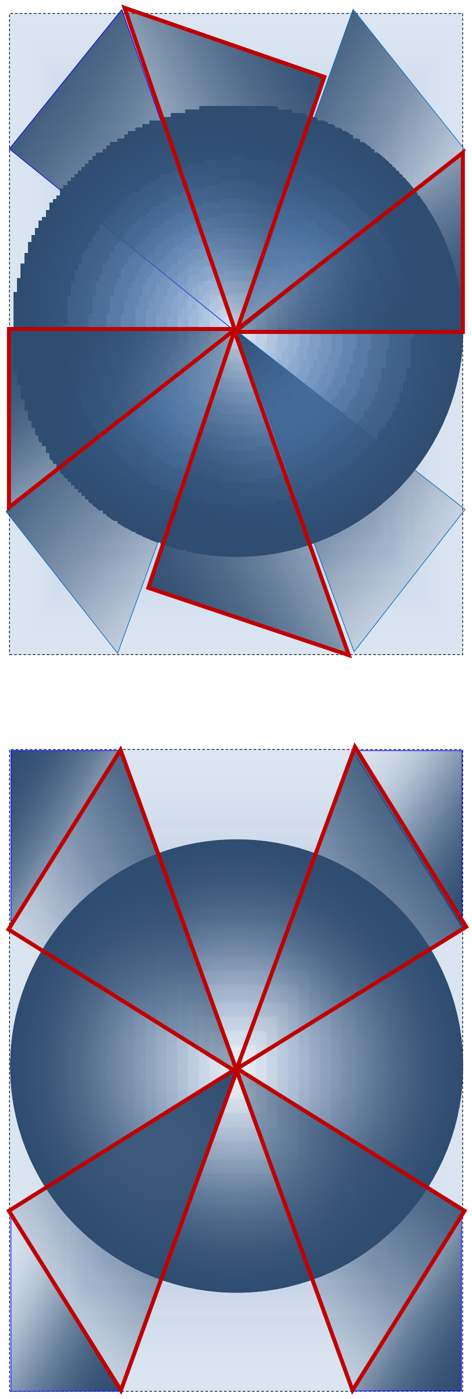

Figure 5 is a diagonal section of a cube passing through its 4 vertices and the center of symmetry. Above is the first type of cube plane (4 planes in total), below is the second type of cube plane (2 planes in total). Kepler triangles are indicated in red, Fibonacci triangles in blue.

To visually represent the geometry of a cube with Kepler and Fibonacci triangles, we have presented a drawing of two diagonal sections of a cube of the second type (Figure 5, bottom) in a three-dimensional plane (Figure 6). We considered it unnecessary to represent all six sections of the cube in one drawing, as such an image would be excessively difficult to perceive.

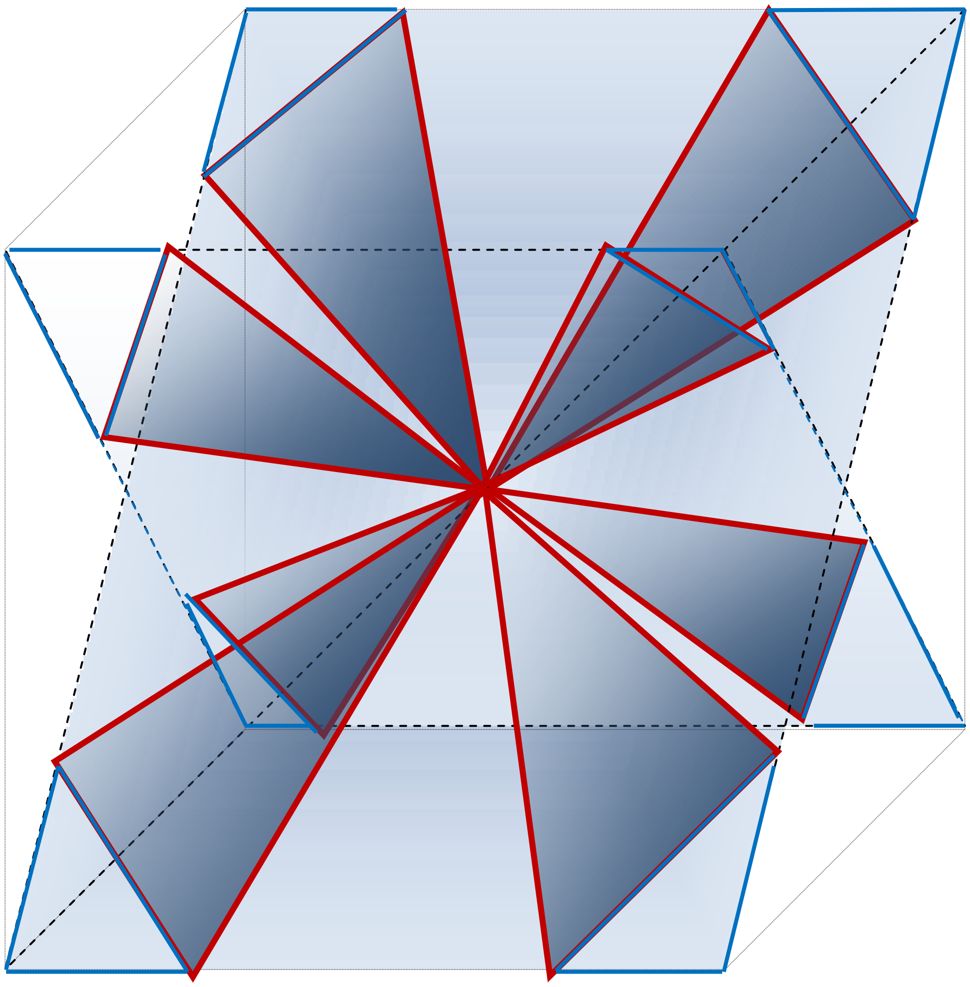

Figure 6 – Two diagonal sections of a cube through its vertices in a three-dimensional plane. Top-front-side view. Kepler triangles are indicated in red, Fibonacci triangles in blue.

Construction of a regular icosahedron (60 structural elements in 30 pairs). It is possible to construct an icosahedron from thirty pairs of Kepler and Fibonacci triangles. For clarity, when constructing the icosahedron, we used its cross-section passing through its center of symmetry and four opposite vertices (two edges) [1] (Figures 7, 8). The edges of the figure are formed by smaller catheters of the Kepler and Fibonacci triangles. The vertices of the icosahedron form the junctions of the smaller catheter and the hypotenuse of the two Keplerian and Fibonacci triangles (Figures 7, 8). The center of symmetry of the icosahedron is located at the junction of the hypotenuses with the large catheters of two types of triangles. The icosahedron contains 30 edges, therefore, the sections we are considering in it are 15. Each section contains 2 pairs of Kepler and Fibonacci triangles. The total number of structural elements of the icosahedron is thus 60 (Table 5).

Table 5 – Structural elements of two families of the regular polygon "icosahedron"

Figure	The area of all structural elements of the shape, S	Structural elements 1st family						Number of elements	Structural elements 2nd family						Number of elements
		The smaller Triangle Kepler, conventional units *			The larger triangle Kepler, conventional units *				The smaller Triangle Fibonacci, conventional units *			The larger triangle Fibonacci, conventional units *			
		L_1	L_2	S	L_1	L_2	S	∑	L_1	L_2	S	L_1	L_2	S	∑
The Icosahedron	∑196,47	2,3	2,94	3,381	–	–	–	30	1,98	3,2	3,168	–	–	–	30

* Note: relative to the initial size of the figure with a sphere with a radius of 3 conventional units inscribed (in this figure)

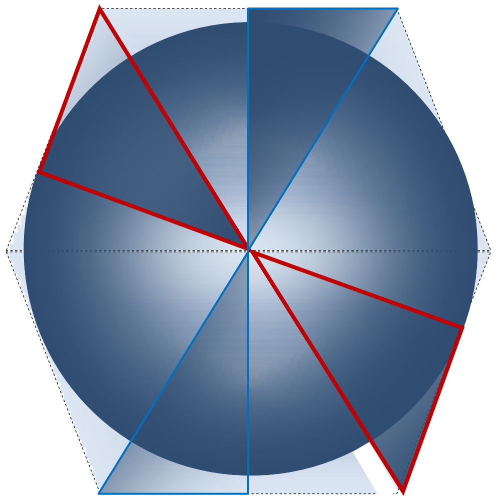

Figure 7 is a cross section of an icosahedron passing through its center of symmetry and four opposite vertices (two edges) [1]. Kepler triangles are indicated in red, Fibonacci triangles in blue.

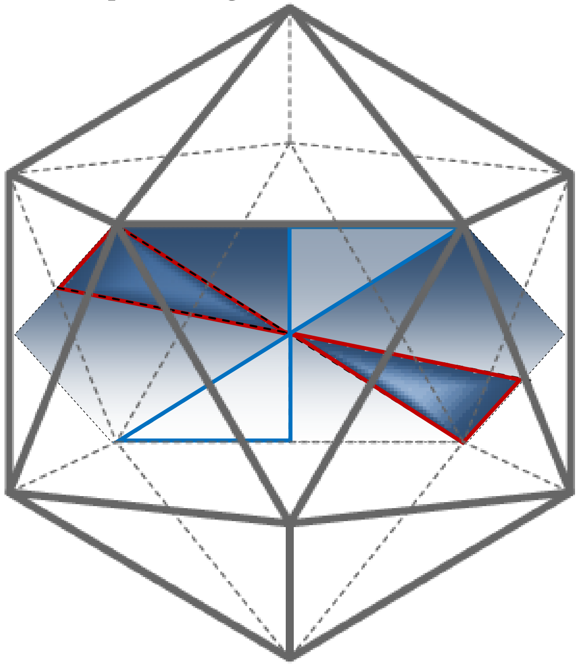

Figure 8 is a section of an icosahedron in a three–dimensional plane. The top view is from the front. Kepler triangles are indicated in red, Fibonacci triangles in blue.

Construction of a regular dodecahedron (240 structural elements in 120 pairs). The dual icosahedron dodecahedron can be reproduced from 240 triangles – 60 pairs of Kepler and Fibonacci triangles (Table 6). Figures 9 and 10 show a section of a dodecahedron passing through its center of symmetry and four opposite vertices [1]. The section under consideration allows you to place in it four pairs of large Kepler and Fibonacci triangles with the center of symmetry in the middle of the geometric body (4*15=60). At the same time, the pairs of large Kepler and Fibonacci triangles have in common their large legs. With their smaller catheters, they form a section line of the dodecahedron (its edges). The smaller Kepler and Fibonacci triangles form the faces of the dodecahedron: each of the 12 faces of the figure contains 5 pairs of smaller Kepler and Fibonacci triangles (12*5=60). The triangles of the pairs touch the large catheters, and the smaller catheters form the edges of the figure (Figure 9).

Table 6 – Structural elements of two families of the regular polygon "dodecahedron"

Figure	The area of all structural elements of the shape, S	Structural elements 1st family						Number of elements	Structural elements 2nd family						Number of elements
		The smaller Triangle Kepler, conventional units *			The larger triangle Kepler, conventional units *				The smaller Triangle Fibonacci, conventional units *			The larger triangle Fibonacci, conventional units *			
		L_1	L_2	S	L_1	L_2	S	∑	L_1	L_2	S	L_1	L_2	S	∑
Dodecahedron	∑488,74	1,3522	1,72	1,16289	2,3113	2,94	3,3976	60/60	1,063	1,72	0,91418	1,817	2,94	2,671	60/60

* Note: relative to the initial size of the figure with a sphere with a radius of 3 conventional units inscribed (in this figure)

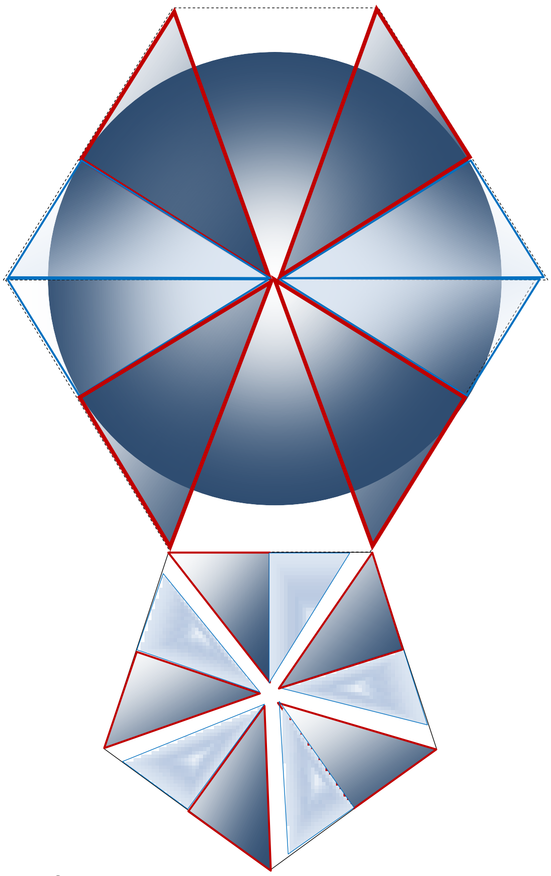

Figure 9 is a cross–section of a dodecahedron passing through its center of symmetry and four opposite vertices (two edges) [1]. Kepler triangles are indicated in red, Fibonacci triangles in blue.

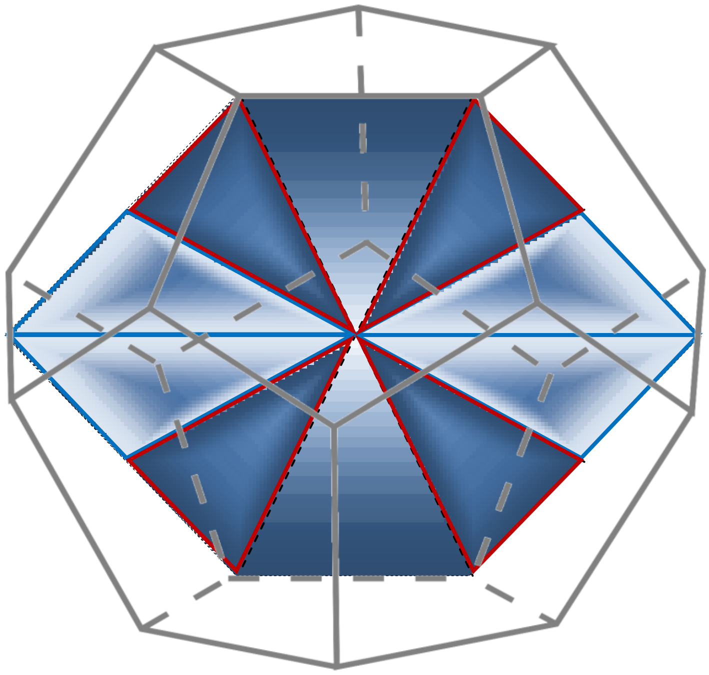

Figure 10 is a section of a dodecahedron in a three–dimensional plane. The top view is from the front. Kepler triangles are indicated in red, Fibonacci triangles in blue.

The experience of constructing Platonic solids (their geometric structure from structural elements) indicates the possibility of creating these geometric shapes using structural elements – Kepler and Fibonacci triangles – provided that the area of these elements remains unchanged. The number of elements (or pairs of elements) that make up the structure of the "tetrahedron", "octahedron", "cube" increases, thus, by two times, and the "icosahedron" – by five times in relation to the number of elements of the tetrahedron, while the indicator "the area of all structural elements of the figure" remains unchanged (Table 7).

Table 7 – Summary table: structural elements of two families of geometric objects – Platonic solids

Figure	The area of all structural elements of the shape, S*	Number of pairs	Number of elements
		∑	∑
The tetrahedron	∑194,315	6	12
The octahedron	∑195,704	12	24
Cube	∑195,528	24	48
The Icosahedron	∑196,47	30	60
Dodecahedron	∑488,74	120	240

* Note: relative to the initial size of the figure with a sphere with a radius of 3 conventional units inscribed (in this figure)

The following patterns were found in the construction of Platonic solids by means of structural elements:

– in sum, the area of all structural elements of any of the four Platonic solids – "tetrahedron", "octahedron", "cube", "icosahedron" – is related to the area (equal to its volume) of an inscribed sphere (with a radius of 3) in these geometric shapes as √3:1 (≈195.8902754094751 and 113.0973032392904);

– the area of the elements of each of the four Platonic solids – "tetrahedron", "octahedron", "cube", "icosahedron" – is very close to the value 195,8902754094751=√38373.

Thus: √3*√12791 (√12791=113,0973032392904 – the area value of the sphere) = 195,8902754094751=√38373. The values of the numbers √38373 and √12791 are correlated as 3:1 – this is the ratio of the area of the elements of any of the four Platonic solids (tetrahedron, octahedron, cube, icosahedron) to the area of a sphere with radius 3. Thus: the area of the elements of each of the four Platonic solids – tetrahedron, octahedron, cube, icosahedron – It is related to the area (or volume) of the sphere inscribed in these four bodies as √3:1.

As already shown, among the four Platonic solids – the irregular tetrahedron and octahedron, cube, and icosahedron – there is a certain numerical pattern: the ratio of pairs of Kepler and Fibonacci triangles in the geometric structure of these bodies determined by us (in order of their complexity from tetrahedron, octahedron, and cube): 6 pairs in a tetrahedron, 12 pairs in an octahedron, 24 pairs in a cube, 30 pairs in an icosahedron, 120 pairs in a dodecahedron. Thus, if we take as a unit of reference the number of pairs of dodecahedrons (120 pairs = 240 elements), then the number of these pairs will be a multiple of the number of elements of 20 tetrahedra, 10 octahedra, 5 cubes, 4 icosahedra. At the same time, the area of the elements of the tetrahedron, octahedron, cube, and icosahedron is the same and is equal to the value of √3 multiplied by the area of the inscribed circle.

From the area of the structural elements of two dodecahedra (S=√959325), you can get an area equal to the area of the structural elements of any 5 Platonic solids, for example, 5 tetrahedra or octahedra, cubes, icosahedra... and so on (S=√38373).

The assumption of the possibility of transitions (degenerations) of Platonic bodies into each other (with a certain constant value (parameter) and other variables) turned out to be practically feasible with a constant value (parameter) of the "area of structural elements", and as a variable – their multiple number in the "structure" of Platonic bodies.

3 Conclusion

The possibility of constructing Platonic solids from structural elements is shown – Kepler triangles (ratio of legs 1:√1.618..) and Fibonacci (ratio of legs 1:1.618...) – provided that the area of these elements remains unchanged. The number of elements making up the structure of the "tetrahedron", "octahedron", "cube" increases, thus, by two times, and the "icosahedron" – by five times in relation to the number of elements of the tetrahedron, while the indicator "area of all structural elements of the figure" (S=√38373) and radius (r=3) inscribed in the Platonic solids of the sphere.

The area of the structural elements of two dodecahedra (S=√959325) is equal to the area of the structural elements of any 5 Platonic solids, for example, 5 tetrahedra, octahedra, cubes, icosahedra (S=√38373). The possibility shown is in accordance with the text of Plato's work Timaeus, according to which Platonic bodies can "transform into each other...".

The assumption about the possibility of transitions (degenerations) of Platonic bodies into each other (with a certain constant value (parameter) and other variables) turned out to be practically feasible with a constant value (parameter) of the "area of structural elements", and as a variable – their multiple number in the "structure" of Platonic bodies.

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