The Relationship Between Pivot Vectors and Rotation Quaternions

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Abstract

The connection between the Pivot Vector and quaternion parameterizations of vehicle attitude transformation was investigated to enhance the understanding of each parameter set. Attitude transformations using quaternions involve special product rules for axial rotations in hypercomplex 4-dimensional space. Pivot Vectors involve slewing motion resulting in angular displacements in the 2-dimensional rotation plane. In spite of these differences, Pivot Vectors and quaternions share the same rule for combining rotational transformations and the signature half angle rotation parameter. The Pivot Vector Method defines an attitude transformation by the slewing motion of an axis extending from the center of a unit sphere to its surface. Two Pivot Vectors that also reside in the equatorial plane drive the axis along a portion of a great circle arc in the equatorial plane. A 180 degree rotation about the first Pivot Vector followed by a 180 degree rotation about the second Pivot Vector slews the axis by twice the angular separation between the Pivot Vectors. This explains why the angle between Pivot Vectors is one half the desired rotation angle. The slewing motion of the axis in the equatorial plane produces a rotation about the sphere's polar axis that applies to all points on the spherical surface, thereby, changing the longitude of each point on the surface while leaving the associated latitude value unchanged. This feature clearly shows that rotations are 2-dimensional. Two sequential transformations are combined by aligning the second Pivot Vector of the first transformation with the first Pivot Vector of the second transformation, even if the Pivot Vector pairs lie in different rotation planes. The linking of the Pivot Vector pairs is achieved because the two 180 degree rotations at the junction cancel, leaving the remaining Pivot Vectors to define the combined transformation. The linking of two Pivot Vector pairs into a single Pivot Vector pair clarifies the geometry of combining rotational transformations and leads to the composition rule for both Pivot Vectors and quaternions. The associated rotational quaternion can be easily derived, since its vector component is the cross product of the Pivot Vectors and its scalar component is the dot product of the Pivot Vectors. A Pivot Vector pair can be obtained from the associated quaternion once its clocking location in the equatorial plane is defined. The quaternion equation to rotate a vector is given a geometric interpretation using the associated Pivot Vectors.

Keywords: Pivot Vector, quaternion, attitude parameters, sequential transformations, direction cosine matrix

Nomenclature

a, b, c, d, e, f	Pivot Vectors
A, B, C, D	Pivot Vectors
DCM	direction cosine matrix
$\mathbf{e}_i$	rotation axis components
I, j, k	coordinate axis unit vectors
L	latitude
n, N	rotation axis
P	longitude
$\mathbf{q}_t$	total quaternion

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$\mathbf{q}$	vector component of quaternion
q0	scalar component of quaternion
$\mathbf{q}^{-1}$	inverse quaternion
$\mathbf{R}$	rotation axis
T, T_1, T_2	angular displacements
$\mathbf{U}$	attitude transformation
$\mathbf{V}$	Vector to rotate
$\mathbf{V_n}$	Vector component normal to rotation axis
$\mathbf{V_p}$	Vector component parallel to rotation axis
δ	rotation angle
σ	rotation angle
$\Delta\lambda$	latitude displacement
$\Delta\Omega$	longitude displacement
$\Delta\phi$	longitude
$\Delta\theta$	latitude
$\mathbf{\Psi}$	DCM for 180 deg. rotation
Θ	rotation angle

1. Introduction

This work focuses on the connection between Pivot Vectors, PVs (Patera, 2017) and rotational quaternions (Kuipers, 1999), (Rose, 2015), (Wyse-Gallifent, 2021) and does not repeat existing literature relating quaternions to traditional attitude parameterizations, such as, Euler Angles, direction cosine matrices, DCMs, axis-angle, etc. (Shuster, 1999). Therefore, much of the content in this work involves PVs, due to their limited reference material.

Although PVs were developed to clarify the geometry of combining rotations in a rotational sequence, they also have applications in attitude representation and attitude kinematics. A few examples are included below. The geometry of linking transformations was applied to Euler Angle sequences and enabled each of the 12 Euler Angle sequences to be reduced to a single pair of PVs (Patera, 2017). Using PVs and associated geometry, the quaternion composition rule was derived (Patera, 2017). PVs proved very useful in accurately computing the solid angles for spherical polygons and other related shapes of interest to the medical field (Patera, 2020a), (Patera, 2020b). The absolute rotation and rotation rate of a Foucault Pendulum and its mounting fixture were computed using PVs, which clarifies the associated non-intuitive kinematics of the pendulum (Patera, 2022).

Any attitude parameter set must define both the angle of rotation and the associated axis of rotation. The rotation quaternion has scalar component that defines the rotation angle and a vector component that defines the axis direction. A PV pair, which is equivalent to a quaternion, lies in the rotation plane and defines the rotation axis as the cross product of the PVs. The rotation angle, θ, is defined by the angular separation between the PVs, which is θ/2. This explains the one half angle that appears in both PVs and rotation quaternions. PVs are the building blocks of the rotational quaternion, since the PV dot product and cross product are the scalar and vector components of the associated rotational quaternion. Attitude transformations using rotational quaternions involve special product rules for axial rotations in hypercomplex 4-dimensional space. PVs use 180 degree rotations about each PV in sequence to achieve an attitude transformation. The DCMs for PVs and rotation quaternions are given in Section 2.

Section 3 provides a short review of the Pivot Vector Method, PVM, (Patera, 2017), since many readers may not be familiar with them. The PVM involves angular displacements in the 2-dimensional plane that have associated rotations about the respective polar axes. PVs can be used to rotate an arbitrary point on the surface of a unit sphere by a desired rotation angle. A feature of all PVs is that they can be easily combined, so two PV pairs can be combined into a single pair. The process can be repeated to combine any number of PV pairs. Since PVs reside in the rotational plane, the resulting rotation extends to three-dimensional space. If two pairs of PVs reside in the equatorial plane of a spherical surface, they can be easily combined using the PVM. In addition, since the PV pairs employ the same rotational axis, they commute.

Section 4 shows that PVM has the ability to combine rotations associated with PV pairs having different rotational planes on a spherical surface. Each PV pair is simply moved to the intersection of the rotational planes before combining into a single PV pair representing the combined rotation. The resulting PV pair defines its own rotation plane, so it can be combined with other rotations in a similar fashion. In this manner any number of rotations can be combined into a single rotation PV pair. The DCM for the final PV pair is easily computed as the product of the respective PV DCMs.

Section 5 shows that the rotational quaternion is defined in terms of PVs and PVs are defined in terms of rotational quaternions. Since there are an infinite number of PV pairs that define a given attitude transformation, additional information on the PV clocking in the equatorial plane is needed before PVs can be uniquely defined from the associated rotational quaternion. Gimbal lock and singularities do not occur when using rotational quaternions or PVM (Rose, 2015), (Kuipers, 1999). The quaternion representing the identity transformation is easily understood using PVM. The quaternion representation of a vector is derived using PVM.

Section 6 derives the composition rule for PVM by linking PV pairs at the intersection of the respective rotation planes (Patera, 2017). The quaternion composition rule, which is usually found using the quaternion product (Rose, 2015), (Wyse-Gallifent, 2021), (Kuipers, 1999), is derived from the PV composition rule. The half angle of the desired rotation appears in PVM and is inherited by the rotational quaternions.

In Section 7, the PV method for rotating a vector is derived and clearly shows that a rotation occurs in a 2-dimensional plane and only involves vector dot products. In Section 8, the rotation quaternion method for rotating a vector is presented (Rose, 2015), (Kuipers, 1999). It relies on an equation involving two quaternion products that is conceptually challenging due to the four-dimensional quaternion parameterization. The equation is validated by using the scalar vector form of the rotational quaternion (Rose, 2015), (Wyse-Gallifent, 2021), (Kuipers, 1999).

Section 9 interprets the quaternion rotation equation using the associated Pivot Vector representation for each quaternion in the equation. The geometry of each PV pair is provided to clarify the process of linking the PV pairs. The translation of the quaternion rotation equation into the associated Pivot Vectors enhances the conceptual understanding and reveals the underlying geometry hidden in the equation. Section 9 provides a conclusion that summarizes the work.

2. Attitude transformation using Pivot Vectors

The attitude transformation due to Pivot Vectors **a**, **b** can be found by multiplying their associated DCMs. The DCM, for a rotation of θ degrees about the unit vector axis, **e**, is given by eq. (1), where C = cos(θ) – 1, S = sin(θ). Eq. (1) is the axis-angle representation of a rotation of θ about axis **e**. For a single Pivot Vector rotation of 180 degrees, C = -2, S = 0 and eq. (1) reduces to the symmetric matrix in eq. (2).

$$\mathbf{U}(\mathbf{e},\boldsymbol{\theta}) = \begin{pmatrix} 1+({e_z}^2+{e_y}^2)C & -(e_x e_y C + e_z S) & e_y S - e_x e_z C \\ e_z S - e_x e_y C & 1+({e_x}^2+{e_z}^2)C & -(e_z e_y C + e_x S) \\ -(e_z e_x C + e_y S) & e_x S - e_y e_z C & 1+({e_x}^2+{e_y}^2)C \end{pmatrix} \tag{1}$$

$$\boldsymbol{\psi}(\mathbf{e}) = \mathbf{U}(\mathbf{e},180) = \begin{pmatrix} 2e_x^2-1 & 2e_x e_y & 2e_x e_z \\ 2e_x e_y & 2e_y^2-1 & 2e_z e_y \\ 2e_x e_z & 2e_z e_y & 2e_z^2-1 \end{pmatrix} \tag{2}$$

Using eq. (2), one finds the DCM for the two sequential 180 degree rotations about **a** and **b**, which are separated by θ/2, as **U**(**e**, θ) in eq. (3). The transformation results in a rotation of θ degrees about the axis **e**, which is normal to the rotation plane defined by **a** and **b**. Note that the transformation in eq. (3) involves only the components of the associated rotation axes, **a** and **b**, as indicated by the subscripts in eq. (2), and does not involve trigonometric functions. Since each matrix in eq. (3) is symmetric, computational effort is reduced.

$$\mathbf{U}(\mathbf{e},\theta) = \boldsymbol{\psi}(\mathbf{a})\ \boldsymbol{\psi}(\mathbf{b}) \tag{3}$$

The DCM in eq. (3) can be used to rotate any vector **V**, as shown in eq. (4).

$$\mathbf{V}' = \mathbf{U}(\mathbf{e},\theta)\ \mathbf{V} = \boldsymbol{\psi}(\mathbf{a})\ \boldsymbol{\psi}(\mathbf{b})\ \mathbf{V} = [\boldsymbol{\psi}(\mathbf{a})\ \boldsymbol{\psi}(\mathbf{b})]\ \mathbf{V} \tag{4}$$

For completeness the corresponding transformation matrix for quaternions is given in eq. (5), where the four quaternion parameters are defined in eqs. (6) - (10).

$$\mathbf{U}(\mathbf{q}) = \begin{bmatrix} q_1^2-q_2^2-q_3^2+q_0^2 & 2(q_1q_2-q_3q_0) & 2(q_1q_3+q_2q_0) \\ 2(q_1q_2+q_3q_0) & -q_1^2+q_2^2-q_3^2+q_0^2 & 2(q_2q_3-q_1q_0) \\ 2(q_1q_3-q_2q_0) & 2(q_2q_3+q_1q_0) & -q_1^2-q_2^2+q_3^2+q_0^2 \end{bmatrix} \tag{5}$$

$$\mathbf{Q} = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} \tag{6}$$

$$\mathrm{q}_0 = \cos\left(\frac{\theta}{2}\right) \tag{7}$$

$$\mathbf{q_1} = \mathbf{e_1}\sin\left(\frac{\theta}{2}\right) \tag{8}$$

$$\mathbf{q_2} = \mathbf{e_2}\sin\left(\frac{\theta}{2}\right) \tag{9}$$

$$\mathbf{q_3} = \mathbf{e_3}\sin\left(\frac{\theta}{2}\right) \tag{10}$$

3. Displacements on a spherical surface using Pivot Vectors

Pivot Vectors **A** and **B** that reside in the equatorial plane can be used for angular displacement on the surface of a unit sphere. The sequence of two 180 degree rotations about **A** and **B,** respectively, result in a rotation about the sphere's polar axis equal to twice the angular separation of **A** and **B**. Fig. 1 illustrates a 180 degree rotation about **A** that moves **B** to **B'** followed by a 180 degree rotation about **B'** that moves **A** to **A'**. as shown. The direction of rotation is from **B** to **A**. Thus, a displacement along the equatorial plane is equivalent to a rotation about the polar axis, which applies to every point on the surface of the sphere. The inverse rotation is given by rotating about **B** before **A**. The PV pair can be moved to any location in the equatorial plane while yielding the same transformation. Since a 180 degree rotation is equivalent to a -180 degree rotation, the direction of each PV can be reversed without changing the result but the order of the rotations must be maintained. This feature enables the separation angle between PVs to be equal to or less than 90 degrees, while still representing any desired rotational transformation.

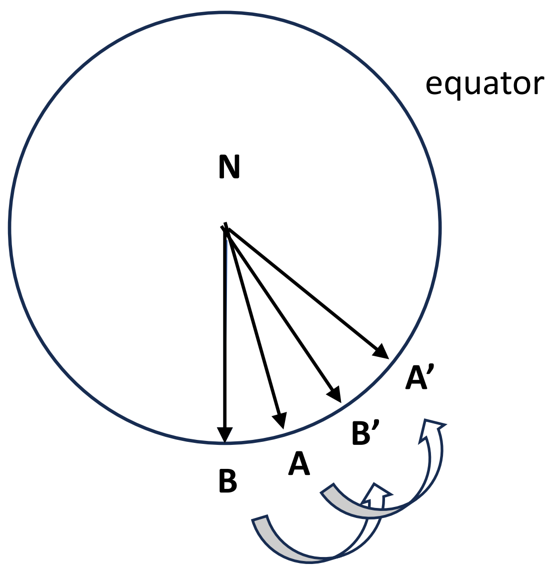

Fig. 1. Sequential rotation of 180 degrees about **A** and **B** result in a rotation about the polar axis by twice the separation angle between **A** and **B**.

If another PV pair, **C**, **D** is also located in the same equatorial plane as **A**, **B**, the associated rotations can be combined by moving the pairs such that **B** aligns with **C**. The two 180 degree rotations at **B** cancel, which results in a combined displacement defined by PV pair **A**, **D**.

Consider two PVs **A** and **B** normal to a spherical surface that extend from the center of a unit sphere to its surface. Let σ be the angular separation between points **A** and **B**, which are separated by a geodesic arc of length, σ, since the radius of the sphere is one. An axis, **P**, is also located on the great circle arc that contains **A** and **B**. The great circle arc can be considered an equatorial plane of the spherical surface with the polar axis normal to the equatorial plane. Position along the equatorial plane is defined by the angular parameter, longitude similar to that used for Earth coordinates. The angular distance, **Δϕ**, between **P** and **A** is given in eq. (11).

$$\boldsymbol{\Delta\phi} = \mathbf{P} - \mathbf{A} \tag{11}$$

A 180 degree rotation about **A** relocates **P** to **P1**, as shown in eq. (12), where eq. (11) has been used.

$$\mathbf{P1} = \mathbf{A} - \boldsymbol{\Delta\Phi} = \mathbf{2A} - \mathbf{P} \tag{12}$$

In a similar manner **B** relocates to **B1**, as shown in eq. (13).

$$\mathbf{B1} = \mathbf{A} - \Delta\mathbf{B} = \mathbf{2A} - \mathbf{B} \tag{13}$$

The distance between **P1** and **B1** is **ΔB1**, as shown in eq. (14).

$$\boldsymbol{\Delta}\mathbf{B1} = \mathbf{P1} - \mathbf{B1} \tag{14}$$

A 180 degree rotation about **B1** relocates **P1** to **P2** as shown in eq. (15).

$$\mathbf{P2} = \mathbf{B1} - \boldsymbol{\Delta}\mathbf{B1} = \mathbf{2B1} - \mathbf{P1} = \mathbf{2}(\mathbf{A} - \mathbf{B}) + \mathbf{P} \tag{15}$$

The net displacement, **T1**, after the two sequential 180 degree rotation is **P2** - **P**, as shown in eq. (16).

$$\mathbf{T1} = \mathbf{P2} - \mathbf{P} = \mathbf{2}\ (\mathbf{A} - \mathbf{B}) \tag{16}$$

The angular displacement value shown in eq. (16) is independent of the location of **P** along the equator. It only depends upon the relative angular separation between points **A** and **B**. Therefore, points **A** and **B** can be clocked to any location along the equator, as long as, their separation distance remains the same. The angular displacement, **T1**, resulting from the sequential rotations of 180 degrees about **A** and **B** is equivalent to a rotation about the polar axis normal to the equatorial plane containing **A** and **B** and is independent of the clocking of the **A, B** pair. Since the angular separation between **A** and **B** is σ, the associated rotation for the displacement in eq. (16) is given in eq. (17). This is how Pivot Vectors generate rotation about an axis normal to the plane containing **A** and **B**.

$$\mathbf{T1} = \mathbf{P2} - \mathbf{P} = \mathbf{2}\ (\mathbf{A} - \mathbf{B}) = 2\sigma \tag{17}$$

Another angular displacement, **T2**, associated with PVs **C** and **D,** which also are on the equatorial geodesic and are separated by angle, δ, can be combined with the displacement of **A** and **B**, as given in eq. (18).

$$\mathbf{T} = \mathbf{T1} + \mathbf{T2} = \mathbf{2}(\mathbf{A} - \mathbf{B}) + \mathbf{2}(\mathbf{C} - \mathbf{D}) \tag{18}$$

Since only the relative position of **A** with respect to **B** and the relative position of **C** with respect to **D** determine the associated angular displacements, each pair of points can be moved about in the equatorial plane while yielding the same angular displacements. As a result, one can move the **C**, **D** pair such that **C** aligns with **B**, and eq. (18) simplifies to eq. (19).

$$\mathbf{T} = \mathbf{2}(\mathbf{A} - \mathbf{B}) + \mathbf{2}(\mathbf{C} - \mathbf{D}) = \mathbf{2}(\mathbf{A} - \mathbf{D}) = 2(\sigma + \delta) \tag{19}$$

The 180 degree rotations about **B** and **C** results in a 360 degree rotation or no rotation at all. This feature of PVs enables them to link transformations, thereby, simplifying the results for the angular displacement along a geodesic on a spherical surface. The inverse transformation is obtained by reversing the order of the rotations in eq. (19) to obtain eq. (20).

$$\mathbf{T} = \mathbf{2}(\mathbf{D} - \mathbf{A}) = -2(\sigma + \delta) \tag{20}$$

The results derived above can be generalized for any point on the spherical surface, including those points that are out of the equatorial plane. A general point on the spherical surface can be defined by its longitude, **P**, and latitude, **L**. Vectors **A** and **B** remain on the equatorial geodesic arc. The angular distance between **P** and **A** now has two components, longitude, **Δϕ**, and latitude, **Δθ**, as given in eq. (21) and eq.(22). Notice that both **A** and **B** have zero values of latitude, since they are in the equatorial plane.

$$\boldsymbol{\Delta\phi} = \mathbf{P} - \mathbf{A} \tag{21}$$

$$\boldsymbol{\Delta\theta} = \mathbf{L} \tag{22}$$

A 180 degree rotation about **A** relocates **P** to **P1**, as shown in eq. (23), where eq. (21) has been used.

$$\mathbf{P1} = \mathbf{A} - \boldsymbol{\Delta\Phi} = \mathbf{2A} - \mathbf{P} \tag{23}$$

In a similar manner, a 180 degree rotation about **A** relocates the latitude value of **P1** to a negative value, as shown in eq. (24).

$$\mathbf{L1} = -\boldsymbol{\Delta\theta} = -\mathbf{L} \tag{24}$$

The rotation about **A** also relocates **B** to **B1**, as shown in eq. (25).

$$\mathbf{B1} = \mathbf{A} - \Delta\mathbf{B} = \mathbf{2A} - \mathbf{B} \tag{25}$$

The latitude value of **B** remains zero, since it is on the equatorial plane. The distance between **P1** and **B1** is **ΔB1**, as shown in eq. (26).

$$\boldsymbol{\Delta}\mathbf{B1} = \mathbf{P1} - \mathbf{B1} \tag{26}$$

A 180 degree rotation about **B1** relocates **P1** to **P2** as shown in eq. (27).

$$\mathbf{P2} = \mathbf{B1} - \boldsymbol{\Delta}\mathbf{B1} = \mathbf{2B1} - \mathbf{P1} = \mathbf{2}(\mathbf{A} - \mathbf{B}) + \mathbf{P} \tag{27}$$

The 180 degree rotation about **B1** also restores the latitude value of **-L** back to **L.** The net longitudinal displacement, **ΔΩ**, after the two sequential 180 degree rotation is **P2** - **P**, as shown in eq. (28). The net latitudinal displacement is zero, as given in eq. (29).

$$\boldsymbol{\Delta\Omega} = \mathbf{P2} - \mathbf{P} = \mathbf{2}\ (\mathbf{A} - \mathbf{B}) \tag{28}$$

$$\boldsymbol{\Delta\lambda} = \mathbf{L} - \mathbf{L} = 0 \tag{29}$$

In addition to the longitudinal displacement in eq. (28), there is an associated rotation about the polar axis of magnitude 2 σ that applies to any arbitrary axis **P** in the equatorial plane. If one desires a rotation of σ, then the angular separation between **A** and **B** must be σ/2, which clarifies the origin of the half angle that appears in both PV and quaternion representations. The preceding analysis shows how PVs generate a displacement along the equatorial plane on a unit sphere and a rotation about the associated polar axis that applies to any point on the spherical surface. Since the spherical surface is fixed with respect to the associated rigid body, the rotation about the polar axis applies to the rigid body, as well.

4. Combining displacements on a spherical surface

The previous section showed how two Pivot Vectors located in the equatorial plane and separated by angle σ can be used to rotate any point on a spherical surface about the polar axis by 2 σ. The rotation about the polar axis rotates all points located on the surface of the sphere not just those located in the equatorial plane.

One can combine the rotation due to Pivot Vectors **A** and **B** with the rotation generated by arbitrary Pivot Vectors **C** and **D** that are not in the equatorial plane, as shown in Fig.2. The Pivot Vectors **C** and **D** are on their own geodesic that differs from that of **A** and **B**. Each pair of Pivot Vectors can be clocked along their respective great circle arcs without changing their associated rotations. There is no need to change the variable name due to the clocking transformation, since each PV pair represents its respective transformation no matter where it is located in its rotational plane.

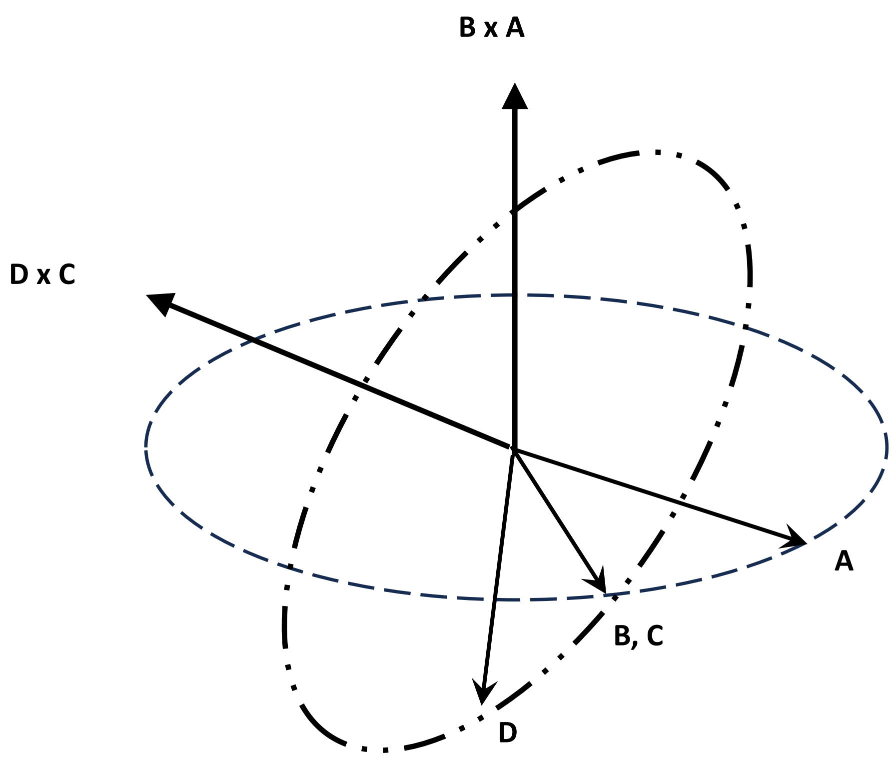

Fig. 2. The **A, B** rotation combines with the **C, D** rotation by aligning **C** with **B** and results in the combined rotation defined by the **A, D** pair.

One can clock each pair so that Pivot Vector **B** aligns with Pivot Vector **C** at the intersection, **N**, of the great circle arcs, given by eq. (30). Fig. 2 illustrates the intersecting rotational planes that enable the rotations about **B** and **C** to cancel, leaving the Pivot Vector pair **A** and **D** to define the combined transformation.

$$\mathbf{N} = \frac{(\mathbf{B}\times\mathbf{A})\times(\mathbf{D}\times\mathbf{C})}{[(\mathbf{B}\times\mathbf{A})\times(\mathbf{D}\times\mathbf{C})]} = \mathbf{B} = \mathbf{C} \tag{30}$$

Using eq. (30), one can find the clocked **A** and **D** using eq. (31) and eq. (32), since the location of **B** and **C** are known to be **N**. **A**, **D** define a new equatorial plane and associated polar axis, as shown in eq. (31) and eq. (32), where the triple cross product identity has been used.

$$\mathbf{A} = (\mathbf{A}\cdot\mathbf{B})\,\mathbf{N} + (\mathbf{B}\times\mathbf{A})\times\mathbf{N} = (\mathbf{A}*\mathbf{B})\,\mathbf{N} + (\mathbf{N}\cdot\mathbf{B})\,\mathbf{A} - (\mathbf{N}\cdot\mathbf{A})\,\mathbf{B} \tag{31}$$

$$\mathbf{D} = (\mathbf{C}\cdot\mathbf{D})\,\mathbf{N} - (\mathbf{D}\times\mathbf{C})\times\mathbf{N} = (\mathbf{C}\cdot\mathbf{D})\,\mathbf{N} + (\mathbf{N}\cdot\mathbf{C})\,\mathbf{D} - (\mathbf{N}\cdot\mathbf{D})\,\mathbf{C} \tag{32}$$

The total rotation can be generated by the **A**, **D** pair, as given in eq. (33). The two 180 degree rotations about Pivot Vector **N** cancel each other, which reduces four Pivot Vectors to just two Pivot Vectors, shown in eq. (33).

$$\mathbf{U} = \boldsymbol{\psi}(\mathbf{A})\boldsymbol{\psi}(\mathbf{B})\boldsymbol{\psi}(\mathbf{C})\boldsymbol{\psi}(\mathbf{D}) = \boldsymbol{\psi}(\mathbf{A})\boldsymbol{\psi}(\mathbf{N})\boldsymbol{\psi}(\mathbf{N})\boldsymbol{\psi}(\mathbf{D}) = \boldsymbol{\psi}(\mathbf{A})\boldsymbol{\psi}(\mathbf{D}) \tag{33}$$

The result derived in eq. (33) shows that any two arbitrary rotations can be combined into a single rotation defined by just two PVs. The associated composition rule is derived in Section 6. The process can be applied to any number of sequential rotations spanning the surface of a unit sphere. The resulting single PV pair represents the combination of all the rotations. The PV pair defines the angular displacement in the rotational plane and the associated rotation about the polar axis normal to the rotational plane.

5. Relating Pivot Vectors to rotational quaternions

Pivot Vectors are closely related to rotational quaternions, since the scalar and vector components of the quaternion are defined by the dot and cross products of the associated Pivot Vectors, respectively. A single rotational quaternion, $\mathbf{q}_t$, that represents a rotation of θ degrees about the rotation axis, **n** ,is defined in eq. (34) and contains the same information as the PV pair **a, b**. It is clear from eq. (34) that the angular separation between **a** and **b** is θ/2 and that they lie in the rotation plane.

$$\boldsymbol{q_t} = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\mathbf{n} = \mathrm{q}0 + \mathbf{q} = \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\times\mathbf{a} \tag{34}$$

Eq. (34) shows how a quaternion can be defined by a Pivot Vector pair and how it inherited the half angle from the Pivot Vector formulation. Note that a single transformation has two rotational quaternion solutions, given by $\mathbf{q}_t$ and $-\mathbf{q}_t$, that may have to be resolved for some applications. A single transformation has PV pair **a** and **b** separated by θ/2 that can be clock anywhere around the rotation plane, but the order and separation angle of the pair must be the same.

Since PVs can be clocked to any location in the rotation plane, they are not unique and additional information is needed before a quaternion can be used to define a specific PV pair. If one defines the location of PV **a**, eq. (35) can be used to obtain PV **b**. If the clocking location of PV **b** is known, PV **a** can be obtained using eq. (36). Thus, a quaternion can be used to define a PV pair as long as the clocking information is provided.

$$\mathbf{b} = \mathrm{q}0\,\mathbf{a} + \mathbf{a}\times\mathbf{q} \tag{35}$$

$$\mathbf{a} = \mathrm{q}0\,\mathbf{b} + \mathbf{q}\times\mathbf{b} \tag{36}$$

The scalar quantity, q0, in the quaternion formulation has an important role in PVM by contributing to **a** and **b,** as shown in eqs. (35) and (36).

A given transformation is represented by a single PV pair, since changing the sign of one or both of the PVs yields the same transformation. In addition, changing the 180 degree rotation about either PV to a -180 degree rotation results in the same transformation. Thus, if the PV dot product is negative one can change the sign of one of the PVs to ensure that the dot product remains positive and the angle between them, θ/2, is equal to or less than 90 degrees. Notice that changing the signs of q0 and **q** in eqs. (35) and (36) is equivalent to changing the signs of **a** and **b**, which leaves the transformation unchanged in PVM.

The identity transformation can be found when θ/2 equals 0 and **a** equals **b**, with the associated quaternion obtained from eq. (34) and is given by eq. (37).

$$\mathbf{q_t} = (1, \quad 0, \quad 0, \quad 0) \tag{37}$$

A pure vector, **V**, can be found when θ/2 equals 90 degrees and **b**, **a**, **v** forms an orthogonal triad. The related quaternion is found from eq. (34) and is given by eq. (38), where **V** is normalized to unity.

$$\mathbf{q_t} = (0, \quad \mathbf{V}_X, \quad \mathbf{V_Y}, \quad \mathbf{V_{Z,}}) \tag{38}$$

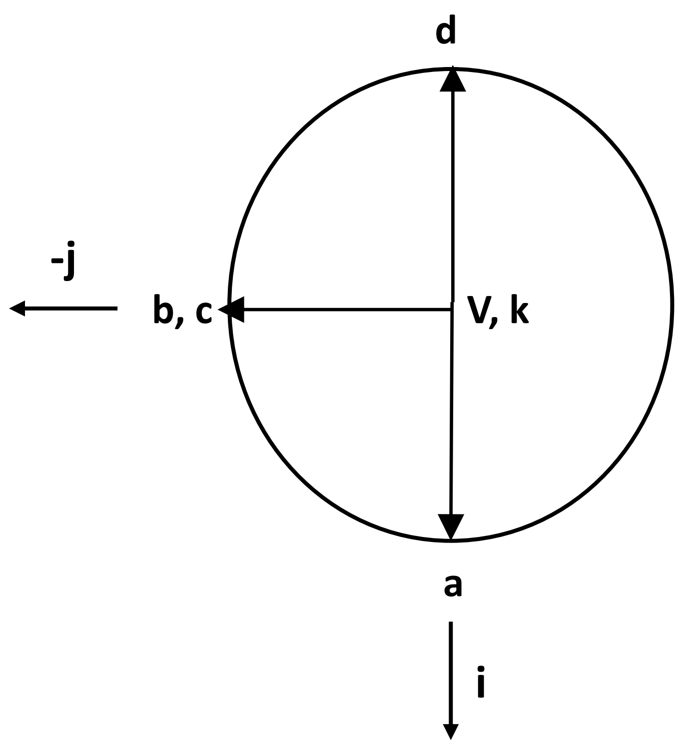

Fig. 3. The product of two pure vectors is found by linking the PV pairs **a**, **b** and **c**, **d** to obtain pair **a**, **d**.

The product of two quaternions representing pure unit vectors, **V**, can be understood using Fig. 3 with **a** aligned with the x-axis, **b** and **c** are aligned with the negative y-axis, **d** is aligned with the negative x-axis and **V** is aligned with the z-axis. PVs **b, a, V** form an orthogonal triad, as well as PVs **d, c, V**. One aligns **b** and **c** to combine the transformations, as shown in eq. (39). The combined transformation is governed by the PV pair **d**, **a**, and its associated quaternion shown in scalar vector form in eq. (40).

$$\mathbf{U} = \boldsymbol{\psi}(\mathbf{a})\,\boldsymbol{\psi}(\mathbf{b})\,\boldsymbol{\psi}(\mathbf{c})\,\boldsymbol{\psi}(\mathbf{d}) = \boldsymbol{\psi}(\mathbf{a})\,\boldsymbol{\psi}(\mathbf{d}) \tag{39}$$

$$\mathbf{q_t} = (\mathbf{a}\cdot\mathbf{d}, \quad \mathbf{d}\times\mathbf{a}) \tag{40}$$

Since **a** = 1 **i** and **d** = -1 **i**, as is apparent from Fig. 3, eq. (40) reduces to eq(41).

$$\mathbf{q_t} = (-1, \quad 0, \quad 0, \quad 0) \tag{41}$$

One can multiply **V** by a scalar to increase its magnitude to any desired value. The result obtained in eq. (41) seems non-intuitive, but Fig. 3 clarifies why the negative one appears in the result. Note that the PVs **a** and **d** result in the identity transformation in PVM, since negative **d** can be replaced with positive **d** without changing the results.

The PVM inverse is obtained by reversing the order of **a** and **b**, which changes the quaternion to its inverse, as shown in eq. (42).

$$q_t^{-1} = \mathbf{b}\cdot\mathbf{a} + \mathbf{a}\times\mathbf{b} = \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\times\mathbf{a} = \cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\mathbf{n} = \mathrm{q}0 - \mathbf{q} \tag{42}$$

For both quaternions and PVs, the axis and rotation angle can be found using eq. (43) and eq. (44) to obtain the result in the axis-angle representation. Note that **a** dot **b** is always positive in PVM, so θ is equal to or less than 90 degrees, whereas, two solutions, **+q** and **-q**, are permitted in the quaternion formulation.

$$\theta = 2\cos^{-1}(\mathbf{a}\cdot\mathbf{b}) = 2\cos^{-1}(\mathrm{q}0) \tag{43}$$

$$\mathbf{n} = \frac{\mathbf{b}\times\mathbf{a}}{|\mathbf{b}\times\mathbf{a}|} = \frac{\mathbf{q}}{|\mathbf{q}|} \tag{44}$$

6. Deriving the composition rule for PVs and rotational quaternions

Pivot Vectors and rotational quaternions share the same rule for combining rotations, (Patera, 2017). Consider combining a rotation of θ about axis **U** with a rotation of ϕ about axis **V**, where **U** and **V** are unit vectors. The associated PV pairs for rotation axes **U** and **V** are (**a, b**) and (**c, d**), respectively, as shown in eq (45) and eq. (46).

$$\sin\left(\frac{\theta}{2}\right)\mathbf{U} = \mathbf{q_U} = \mathbf{b}\times\mathbf{a} \tag{45}$$

$$\sin\left(\frac{\Phi}{2}\right)\mathbf{V} = \mathbf{q_V} = \mathbf{d}\times\mathbf{c} \tag{46}$$

In order to combine the rotations, we construct the Pivot Vectors at the intersection of the rotation planes, which defines vectors **b** and **c**, as given in eq. (47).

$$\mathbf{N} = \frac{\mathbf{U}\times\mathbf{V}}{|\mathbf{U}\times\mathbf{V}|} = \mathbf{b} = \mathbf{c} \tag{47}$$

Since **b** = **N**, one can construct the **a** using eq. (36), as shown in eq. (48).

$$\mathbf{a} = \mathrm{q}0\,\mathbf{b} + \mathbf{q}\times\mathbf{b} = \mathbf{N}\cos\left(\frac{\theta}{2}\right) + (\mathbf{U}\times\mathbf{N})\sin\left(\frac{\theta}{2}\right) \tag{48}$$

The first rotation about axis **A** is defined by the **PV** pair (**a, b**) and its **DCM** is given in eq. (49).

$$\mathbf{U}(\mathbf{a},\mathbf{N}) = \boldsymbol{\psi}(\boldsymbol{a})\,\boldsymbol{\psi}(\mathbf{N}) = \boldsymbol{\psi}(\mathbf{a})\,\boldsymbol{\psi}(\mathbf{b}) \tag{49}$$

In the same manner, one can construct **d** for the second rotation about axis **V** using eq. (35), as shown in eq. (50).

$$\mathbf{d} = \mathrm{q}0\,\mathbf{c} + \mathbf{c}\times\mathbf{q} = \mathbf{N}\cos(\phi/2) + (\mathbf{N}\times\mathbf{V})\sin(\phi/2) \tag{50}$$

Therefore, the second rotation is defined by the **PV** pair (**c, d**) and its **DCM** is given by eq. (51).

$$\mathbf{U}(\mathbf{N},\mathbf{d}) = \boldsymbol{\psi}(\mathbf{N})\,\boldsymbol{\psi}(\mathbf{d}) = \boldsymbol{\psi}(\mathbf{c})\,\boldsymbol{\psi}(\mathbf{d}) \tag{51}$$

The rotations can now be combined, since the two rotations of 180 degrees about **N** cancel and result in the **PV** pair (**a, d**) representing the combined rotation. The final **DCM** is shown in eq. (52).

$$\boldsymbol{\psi}(\mathbf{a})\,\boldsymbol{\psi}(\mathbf{N})\,\boldsymbol{\psi}(\mathbf{N})\,\boldsymbol{\psi}(\mathbf{d}) = \boldsymbol{\psi}(\mathbf{a})\,\boldsymbol{\psi}(\mathbf{d}) \tag{52}$$

The PVM clarifies how quaternions combine. Based on eq. (34), the quaternions for the rotations about axis **U** and axis **V** are given in eqs. (53) and (54), where the scalar and vector portions of each quaternion are separated.

$$\mathbf{q_U} = \mathbf{b}\times\mathbf{a} = \mathbf{U}\ \sin(\theta/2), \qquad \mathrm{q}0_\mathrm{U} = \mathbf{a}\cdot\mathbf{b} = \cos(\theta/2) \tag{53}$$

$$\mathbf{q_V} = \mathbf{d}\times\mathbf{c} = \mathbf{V}\ \sin(\phi/2), \qquad \mathrm{q}0_\mathrm{V} = \mathbf{d}\cdot\mathbf{c} = \cos(\phi/2) \tag{54}$$

The vector portion of the combined quaternion, **q**, is obtained as the cross product of **d** with **a,** and the scalar portion is the dot product of **a** and **d**, as shown in Eq. (55).

$$\mathbf{q} = \mathbf{d}\times\mathbf{a} = \sin(\theta/2)\cos(\phi/2)\,\mathbf{U} + \sin(\phi/2)\cos(\theta/2)\,\mathbf{V} + \sin(\theta/2)\sin(\phi/2)\,(\mathbf{U}\times\mathbf{V}) \tag{55}$$

$$\mathrm{q}0 = \mathbf{a}\cdot\mathbf{d} = \cos(\theta/2)\cos(\phi/2) - (\mathbf{U}\cdot\mathbf{V})\sin(\theta/2)\sin(\phi/2)$$

Using Eqs. (53) & (54) in Eq. (55), results in the expected equation for the combination of two rotations using quaternions, as shown in eq. (56). Thus, the quaternion composition rule in eq. (56) was derived from the PV composition rule. Note that the quaternion composition equation in eq. (56) is normally derived by quaternion multiplication, as shown in eq. (57), (Kuipers, 1999).

$$\mathbf{q} = \mathrm{q}0_\mathrm{B}\,\mathbf{q_U} + \mathrm{q}0_\mathrm{A}\,\mathbf{q_V} + (\mathbf{q_U}\times\mathbf{q_V})$$

$$\mathrm{q}0 = \mathrm{q}0_\mathrm{U}\,\mathrm{q}0_\mathrm{V} - \mathbf{q_U}\cdot\mathbf{q_V} \tag{56}$$

$$\mathbf{q_t} = \mathrm{q}0 + \mathbf{q} = \mathbf{q_V}\,\mathbf{q_U} \tag{57}$$

7. Rotating a vector with Pivot Vectors

Section 3 showed how an arbitrary vector can be rotated by a PV pair. The PV pair defines an equatorial plane and an arbitrary vector has an associated latitude and longitude. After rotation by the PV pair, the longitude of the vector was increased by twice the angular separation between **A** and **B**, while its latitude remained fixed. A general equation can be derived that doesn't refer to latitude of longitude by using a method similar to that of Section 3.

One can rotate a vector, **V**, by θ degrees about rotation axis **R** using the PV pair **a**, **b**, separated by θ/2 degrees assuming **V** is not aligned with **R**. If **V** is aligned with **R**, then **V** does not change due to the rotation. Let **b** be a PV normal to the **V, R** plane, as defined by eq. (58).

$$\mathbf{b} = \frac{\mathbf{R} \times \mathbf{V}}{|\mathbf{R} \times \mathbf{V}|} \tag{58}$$

Pivot Vector **a** can be constructed from **R** and **b**, as shown in eq. (59), based on eq. (36).

$$\mathbf{a} = \cos\left(\frac{\theta}{2}\right)\mathbf{b} + \sin\left(\frac{\theta}{2}\right)\left(\frac{\mathbf{R} \times \mathbf{b}}{|\mathbf{R} \times \mathbf{b}|}\right) \tag{59}$$

The PV pair, **a**, **b** can rotate an arbitrary vector, **V**. The component of **V** normal to vector **a** is given by subtracting **V**'s parallel component from **V**, as shown in eq. (60), where **h** is the normal component.

$$\mathbf{h} = \mathbf{V} - (\mathbf{V} \cdot \mathbf{a})\,\mathbf{a} \tag{60}$$

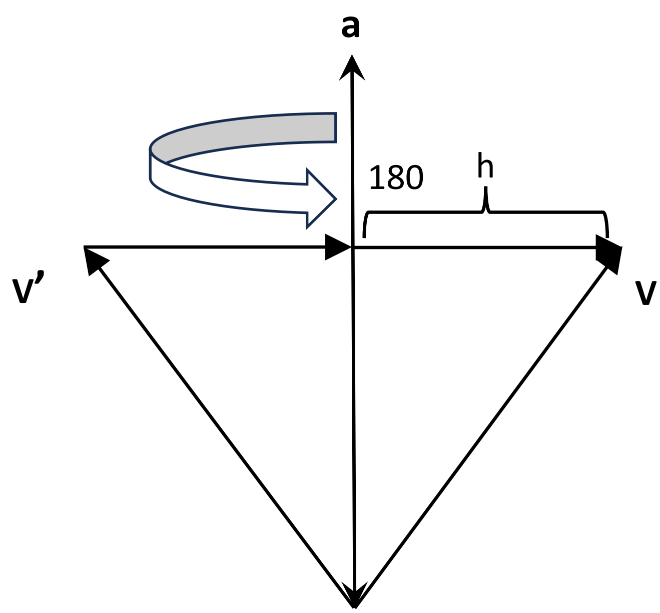

Fig. 4. Rotating **V** about a by 180 degrees is equivalent to subtracting 2**h** from **V**.

The rotation of **V** about **a** by 180 degrees is equivalent to subtracting 2 **h** from **V** and using eq. (60), as given in eq. (61). Fig. 4 illustrates how a 180 degree rotation of **V** about a can be accomplished by subtracting 2h from **V**

$$\mathbf{V}' = \mathbf{V} - 2\,\mathbf{h} = 2\,(\mathbf{V} \cdot \mathbf{a})\,\mathbf{a} - \mathbf{V} \tag{61}$$

The same 180 degree rotation about **a** relocates **b** to **b'**, as shown in eq. (62).

$$\mathbf{b}' = 2\,(\mathbf{b} \cdot \mathbf{a})\,\mathbf{a} - \mathbf{b} \tag{62}$$

Now **V'** is rotated about **b'** by 180 degrees using eq. (61) with **b'** replacing **a**, as given in eq. (63).

$$\mathbf{V}'' = 2(\mathbf{V}' \cdot \mathbf{b}')\,\mathbf{b}' - \mathbf{V}' \tag{63}$$

Using eq. (61) and eq. (62) in eq. (63), one obtains eq. (64) after simplification.

$$\mathbf{V}'' = [4\,(\mathbf{V}\cdot\ \mathbf{b})(\mathbf{b}\cdot\mathbf{a}) - 2(\mathbf{V}\cdot\mathbf{a})]\,\mathbf{a} - 2(\mathbf{V}\cdot\mathbf{b})\,\mathbf{b} + \mathbf{V} \tag{64}$$

Since the vector, **V**, is arbitrary, the rotation generated by a rotation of θ about axis **R** is given by eq. (64) with **a** and **b** defined in eq. (58) and eq. (59). Eq. (64) shows that the only change to **V** occurs in the plane containing **a** and **b**, which indicates that the rotation occurs in a 2-dimensional plane. The component of **V** normal to the plane containing **a** and **b** is not changed by the rotation.

8. Rotating a vector with quaternions

The equation to rotate a vector using quaternions is created by converting the vector into a quaternion, (Kuipers, 1999), with a scalar portion of zero, as shown in eq. (65). This occurs when the associated PVs, **a** and **b** are orthogonal, so that **a** dot **b** is zero. The quaternion equation to rotate a vector is given by eq. (66), where the combine quaternion on the right hand side of eq. (66) has only a vector component with its scalar portion equal to zero. The quaternion and inverse quaternion are given in eqs. (67) and (68).

$$\mathbf{V} = 0 + \mathbf{V} \tag{65}$$

$$\mathbf{V}' = \mathbf{qVq^{-1}} \tag{66}$$

$$\mathbf{q} = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\mathbf{n} \tag{67}$$

$$\boldsymbol{q}^{\mathbf{-1}} = \cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\mathbf{n} \tag{68}$$

The first quaternion multiplication in eq. (66) is found using the quaternion composition rule and yields eq. (69).

$$\mathbf{qV} = -\sin\left(\frac{\theta}{2}\right)(\mathbf{n}\cdot\mathbf{V}) + \cos\left(\frac{\theta}{2}\right)\mathbf{V} + \sin\left(\frac{\theta}{2}\right)(\mathbf{n}\times\mathbf{V}) \tag{69}$$

Multiplying eq. (69) and $\mathbf{q}^{-1}$ yields the final quaternion, which is the rotated vector, as given in eq. (70).

$$\mathbf{qVq^{-1}} = \cos^2\left(\frac{\theta}{2}\right)\mathbf{V} + 2\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)(\mathbf{n}\times\mathbf{V}) + \sin^2\left(\frac{\theta}{2}\right)[(\mathbf{n}\cdot\mathbf{V})\mathbf{n} - (\mathbf{n}\times\mathbf{V})\times\mathbf{n}] \tag{70}$$

The unrotated vector can be decomposed into a component parallel to the rotation axis and a component orthogonal to the rotation axis, as shown in eq. (71).

$$\mathbf{V} = (\mathbf{n}\cdot\mathbf{V})\mathbf{n} + (\mathbf{n}\times\mathbf{V})\times\mathbf{n} \tag{71}$$

Using **V** from eq. (71) in the first term on the right hand side of eq. (70), yields eq. (72) after the double angle formulas for sine and cosine were used.

$$\mathbf{V}' = (\mathbf{n}\cdot\mathbf{V})\mathbf{n} + \cos(\theta)[(\mathbf{n}\times\mathbf{V})\times\mathbf{n}] + \sin(\theta)(\mathbf{n}\times\mathbf{V}) \tag{72}$$

V’ in eq. (72) is the vector after being rotated about axis **n** by an angle θ, where the three terms are orthogonal to each other. The first term in eq. (72) is the component of **V** parallel to the rotation axis and remains unchanged by the rotation. The component of **V** normal to the rotation axis is changed by the rotation and results in the last two terms in eq. (72). If θ = 0, then eq. (72) reduces to eq. (71) and **V’** = **V,** as it should.

9. Explanation of the quaternion rotation equation

The quaternion equation to rotate a vector, given in eq. (73), can be analyzed and understood using the associated PV parameters, where each quaternion is replaced with its equivalent PV pair.

$$\mathbf{V}' = \mathbf{q}\mathbf{V}\mathbf{q}^{-1} \tag{73}$$

To improve the visualization of the geometry, **V** can be divided into a component parallel to the rotation axis and a component normal to the rotation axis, as shown in eq. (74), where each component is normalized to unity and **V** has magnitude of $\sqrt{2}$.

$$\mathbf{V} = \mathbf{V_p} + \mathbf{V_n} \tag{74}$$

One can consider the effect of the rotation on each component separately and add the components by aligning and cancelling the linking Pivot Vectors while leaving a single PV pair to define the transformation. We assign PV pairs to each quaternion in eq. (73) as shown in eq. (75). Then we align **b** with **c,** and align **d** with **e**, so that the final transformation is the single PV pair **a**, **f.**

$$\mathbf{q} = \mathbf{a}, \mathbf{b} \qquad \mathbf{V_p} = \mathbf{c}, \mathbf{d} \qquad \mathbf{q}^{-1} = \mathbf{e}, \mathbf{f} \tag{75}$$

We consider $\mathbf{V_p}$ first and let **c** and **d** be the two Pivot Vectors defining $\mathbf{V_p}$ as shown in eq. (76), where **c** is aligned with the x-axis, **d** is aligned with the negative y-axis and $\mathbf{V_p}$ is aligned with the z-axis.

$$\mathbf{V_p} = \mathbf{c} \cdot \mathbf{d} + \mathbf{d} \times \mathbf{c} = \mathbf{d} \times \mathbf{c} \tag{76}$$

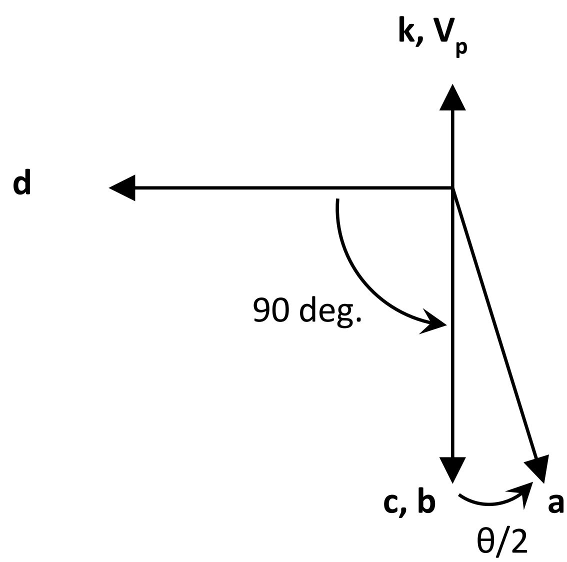

Fig. 5. PVs **b** and **c**, cancel, which leaves PV pair **a, d** representing the product of quaternions **q** and $\mathbf{V_p}$.

Note that $\mathbf{V_p}$ has no scalar component because **c** and **d** are orthogonal unit vectors and $\mathbf{c} \cdot \mathbf{d}$ is zero. Let the rotation axis be along the z-axis of the orthogonal coordinate frame, as shown in Fig. 5. All the vectors in Fig. 5 lie in the equatorial plane except **k** and $\mathbf{V_p}$. The Pivot Vectors defining the rotation of θ degrees about the z-axis are shown in eq. (77), where both **b** and **c** are aligned with the x-axis for convenience.

$$\mathbf{a} = cos\left(\frac{\theta}{2}\right)\mathbf{i} + sin\left(\frac{\theta}{2}\right)\mathbf{j}\ , \quad \mathbf{b} = \mathbf{i}\ , \qquad \mathbf{c} = \mathbf{i} \tag{77}$$

Since the associated rotations for **q** and $\mathbf{q}^{-1}$ as well as **V** align with the z-axis, the transformations in eq. (73) commute. Therefore, the order of the transformations in eq. (73) can be modified to simplify the analysis, as shown in eq. (78). Note that **q** cancels its inverse $\mathbf{q}^{-1}$. Eq. (78) indicates that the rotation does not change the component of **V** parallel to the rotation axis, which is the expected result.

$$\mathbf{V}_\mathbf{P}' = \mathbf{q}\mathbf{V}\mathbf{q}^{-1} = \mathbf{q}\mathbf{q}^{-1}\mathbf{V}_\mathbf{P} = \mathbf{V}_\mathbf{P}\mathbf{q}\mathbf{q}^{-1} = \mathbf{V}_\mathbf{P} \tag{78}$$

Although eq. (78) proves that the component of a vector parallel to the rotation axis remains unchanged by the rotation, it is instructive to go through the individual transformations in their original order in eq. (73). One finds $\mathbf{qV_p}$ by aligning Pivot Vector **c** with Pivot Vector **b** and noting that $\mathbf{V_p}$ lies along the z-axis to be parallel with the rotation axis. Therefore, **d** and **c** take on values shown in eq. (79). The combined transformation, $\mathbf{qV_P}$ is given by the PV pair **a, d.**

$$\mathbf{c} = \mathbf{b} = \mathbf{i}\ , \qquad \mathbf{d}\ =\ -\mathbf{j}, \qquad \mathbf{V_p} = \mathbf{k} \tag{79}$$

Let the PV pair **e**, **f,** define the inverse quaternion, $\mathbf{q}^{-1}$, and align **e** with **d** to combine the transformations, as shown in eq. (80). Note that the PV pair **e, f** is equivalent to $\mathbf{q}^{-1}$, as shown in Fig. 6.

$$\mathbf{e} = \mathbf{d} = -\mathbf{j}\ , \qquad \mathbf{f} = sin\left(\frac{\theta}{2}\right)\mathbf{i} - cos\left(\frac{\theta}{2}\right)\mathbf{j} \tag{80}$$

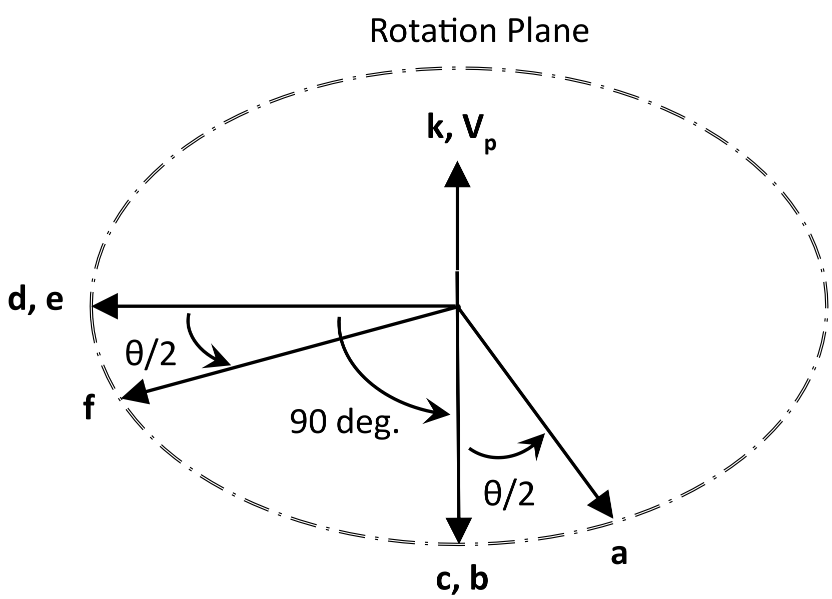

Fig. 6. PVs **b** and **c** cancel, also **d** cancels **e,** which leaves PV pair **a, f** to define the rotated parallel vector, $\mathbf{V_p}$.

The final result of the rotation on the parallel component is given in eq. (81). Note that the angle between **f** and **a** is 90 degrees, so there is no scalar component of the associated quaternion. Of course, $\mathbf{V_P}$**,** which was assumed to be a unit vector, can be scaled to any desired magnitude.

$$V'_p = \mathbf{q}V_p\mathbf{q}^{-1} = \mathbf{f} \times \mathbf{a} = \begin{pmatrix} 0 \\ 0 \\ \cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{81}$$

The result of eq. (81) indicates that the parallel component of **V** does not change due to the rotation, which is the expected result.

The effect of the rotation on the normal component can be evaluated in a similar fashion. Pivot Vectors for the rotation remain the same as in eq. (77), but now $\mathbf{V_n}$ must lie in the rotation plane. We assign new PV pairs to each quaternion in eq. (73) as shown in eq. (82). Next, we align **b** with **c,** and **d** with **e**, so that the final transformation is the single PV pair **a**, **f.**

$$\mathbf{q} = \mathbf{a}, \mathbf{b} \qquad \mathbf{V_n} = \mathbf{c}, \mathbf{d} \qquad \mathbf{q}^{-1} = \mathbf{e}, \mathbf{f} \tag{82}$$

Therefore, the PV pair **c**, **d** is given in eq. (83) with **c** being aligned with **b** to be combined with the rotation PV pair **a**, **b**. $\mathbf{V_n}$ is also shown for clarity, as shown in Fig. 7.

$$\mathbf{c} = \mathbf{b} = \mathbf{i} \quad , \qquad \mathbf{d} = \mathbf{k} \quad , \qquad \mathbf{V_n} = \mathbf{j} \tag{83}$$

Combining the rotations yields the associated quaternion in eq. (84).

$$\mathbf{qV_n} = \begin{bmatrix} \mathbf{a} \cdot \mathbf{d} & -sin\left(\frac{\theta}{2}\right) & cos\left(\frac{\theta}{2}\right) & 0 \end{bmatrix} = \begin{bmatrix} 0 & -sin\left(\frac{\theta}{2}\right) & cos\left(\frac{\theta}{2}\right) & 0 \end{bmatrix} \tag{84}$$

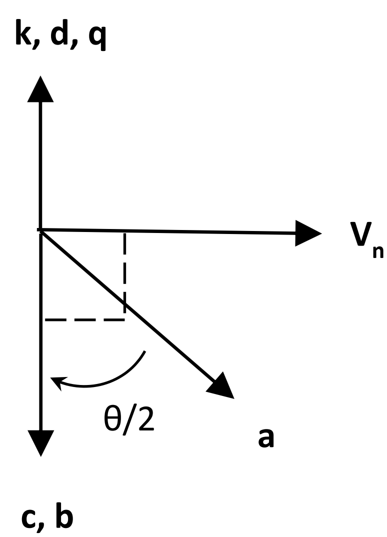

Fig. 7. PVs **c** cancels **b**, which leaves **a** and **d** to represent $\mathbf{qV_n}$**.**

Both **d** and **a** need to be clocked along their geodesic arc so that **d** moves into the rotation plane and **a** aligns with the negative z-axis, which is necessary to combine with $\mathbf{q}^{-1}$, as shown in eq. (85). Fig. 8 indicates the clocking motion of **d,** which leaves the transformation unchanged. The PV **a** is also clocked but not shown in Fig. 8, since it is in the negative **k** direction.

$$\mathbf{a} = -\mathbf{k} \quad , \qquad \mathbf{d} = \cos\left(\frac{\theta}{2}\right)\mathbf{i} + \sin\left(\frac{\theta}{2}\right)\mathbf{j} \tag{85}$$

The PV pair **e**, **f,** which is equal to $\mathbf{q}^{-1}$, lies in the equatorial plane but must be clocked so that **e** aligns with **d**, as shown in eq. (86). Notice that **f** is located θ/2 degrees from **e** to define $\mathbf{q}^{-1}$. The location of **f** involves the sum of two half angles, which becomes the full angle θ, as shown in eq. (86).

$$\mathbf{e} = \mathbf{d} = \cos\left(\frac{\theta}{2}\right)\mathbf{i} + \sin\left(\frac{\theta}{2}\right)\mathbf{j} \qquad , \qquad \mathbf{f} = \cos(\theta)\,\mathbf{i} + \sin(\theta)\,\mathbf{j} \tag{86}$$

The final result is equivalent to the PV pair **a, f** with associated quaternion given in eq. (87), where the scalar component of the quaternion is zero and the magnitude of the resulting vector is unity.

$$\mathbf{V}_\mathbf{n}' = \mathbf{q}\mathbf{V}_\mathbf{n}\boldsymbol{q}^{-1} = [\mathbf{a}\cdot\mathbf{f} \quad -\sin(\theta) \quad \cos(\theta) \quad 0] = [0 \quad -\sin(\theta) \quad \cos(\theta) \quad 0] \tag{87}$$

Since $\mathbf{V_n}$ was originally located along the y-axis from eq. (83), $\mathbf{V_n}'$ in eq. (87) is rotated by θ degrees in the counterclockwise direction about the z-axis, as expected.

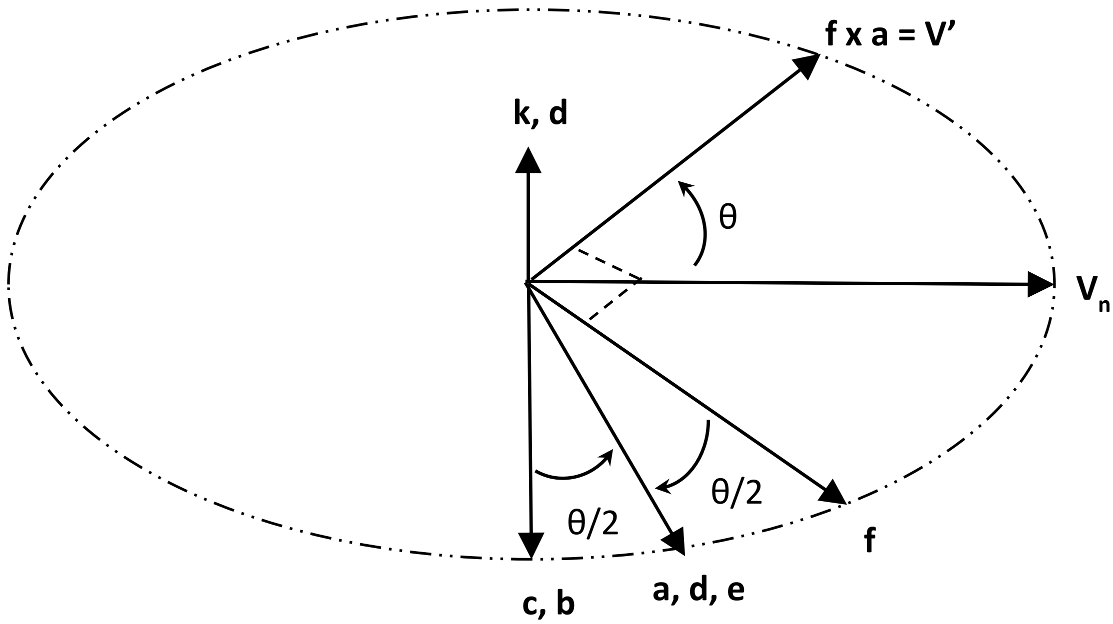

Fig. 8. PVs **c** cancels **b** and after clocking, **d** cancels **e**, which leaves **f** and the clocked **a** to represent the rotation of the normal component of **V'**.

Fig. 8 summarizes how the rotation quaternions rotate a vector. First a rotation involving **q** produces a rotation of θ/2, then **c** cancels **b** leaving PV pair **a** and **d** to define the transformation. Next, a clocking transformation is used to bring **d** into the rotation plane for alignment with the $\mathbf{q}^{-1}$ rotation which adds another θ/2 rotation. The two half angle rotation combine to yield the full θ rotation.

Now the parallel and normal components are combined to obtain the rotated **V',** which is clearly a rotation about the z-axis by θ degrees, as shown in eq. (88). Note that the magnitude of both **V** and **V'** is $\sqrt{2}$, since both $\mathbf{V_p}$ and $\mathbf{V_n}$ were defined to be unity. Of course, the vector can be scaled to any length without changing the rotation result given in eq. (88).

$$\mathbf{V}' = \mathbf{q}\mathbf{V}\mathbf{q}^{-1} = \mathbf{q}(\mathbf{V_p} + \mathbf{V_n})\mathbf{q}^{-1} = \mathbf{V_p} - \sin(\theta)\,\mathbf{i} + \cos(\theta)\,\mathbf{j} \tag{88}$$

Normally, the rotation quaternion method uses the quaternion product rule to evaluate eq. (88). In translating eq. (73) into the PVM, the associated PV pairs are clocked in their respective rotation planes and linked together to obtain the final single rotation PV pair. PVM treats the transformations as angular displacements rather than quaternion vector products. The two methods are complimentary, with PVM providing the geometry of each intermediate transformation in eq. (73). Of course, more efficient ways to rotate a vector with PVM are found in Section 2 and in Section 7.

10. Conclusion

The close relationship between Pivot Vector and rotational quaternion parameterization of attitude was investigated. Although quaternions involve vector products in hypercomplex 4-dimensional space and Pivot Vectors involve angular displacements in a 2-dimensional plane, many similarities were revealed. The attitude transformation represented by a single rotational quaternion is equivalent to the transformation of a single Pivot Vector pair. The scalar and vector components of a quaternion are the dot product and cross products of the associated Pivot Vectors. Since a Pivot Vector can be clocked anywhere in its rotation plane and still yield the same transformation, clocking information must be provided before a quaternion can define a unique Pivot Vector pair. Two quaternions representing two separate transformations are combined by vector multiplication to obtain a single quaternion representing the combined transformation. In the PV formulation, two PV pairs are linked at the junction of the respective rotational planes, which results in a single PV pair representing the combined attitude transformation. In spite of the differences, the resulting rule for combining two attitude transformations into one transformation is the same for both methods. In addition, the signature half angle parameter appears in both formulations. One can rotate a vector using a PV pair with an equation involving simple dot products. Rotating a vector using quaternions involves creating a quaternion from the vector and performing two quaternion vector products that appear in an equation that lacks a clear foundation. A geometrical interpretation of the quaternion vector rotation equation was provided using PVs to reveal the associated rotational displacements. The mathematical connection between Pivot Vectors and rotational quaternions that was presented in this work enhances the understanding of each parameter set and expands the set of tools available to the analyst.

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