Can Gravity Be Considered a Force?*

Scott R. Wilson, scott.wilson@octoengineering.ca

974 McKinnon Rd, Williams Lake, BC, Canada, V2G 2L3

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Abstract

We present Einstein's Field Equation with an explicit gravitational stress-energy term, in a flat space-time background. This treatment explicitly treats gravity as a force. Solutions appear easier to calculate, since in regions where the gravitational stress-energy tensor is zero the regular matter appears to be linearly related to the gravitational tensor potential. It has the added benefit that the flat space-time background allows the theory to be more easily integrated with other theories. As an application, we calculate a relation for the gravitational stress-energy in the Schwarzschild metric.

keywords: General Relativity, Gravity

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1 Introduction

Einstein's theory of General Relativity is interpreted as a theory that does away with the force of gravity, and instead interprets gravitational effects as the result of curved space-time. It may be asked if there is a consistent way to introduce gravitational force into the theory. We answer this question in the affirmative and explicitly construct equations to represent this viewpoint.

In a metric theory of gravitation, matter follows geodesics of a gravitational metric ($g_{\mu\nu}$) which is induced by matter ($T_{\mu\nu}$), and in which $T_{\mu\nu}$ has vanishing covariant divergence with respect to $g_{\mu\nu}$. Einstein's theory is a metric theory of gravity. Linear metric theories of gravity have been considered in the past, with a second flat space-time metric ($\eta_{\mu\nu}$) in which the gravitational metric ($g_{\mu\nu}$) is linearly proportional to $T_{\mu\nu}$ (see Will [1] for a discussion of these). These theories have to deal with an apparent inconsistency, in which the field equation demands a vanishing flat space-time divergence of $T_{\mu\nu}$, which is inconsistent with the vanishing covariant divergence. Approaches to deal with this include adding additional fields or adding a gravitational stress-energy term that itself is a source of gravity [1]. We have decided to include a gravitational stress-energy term, constructed so that the resulting theory is mathematically indistinguishable from Einstein's theory. The difference between the theory presented here and the one presented by Einstein is a difference of interpretation only. The advantage of this interpretation is the subject of this paper.

1.1 Flat Background Metric

We base our theory on an absolute background of flat space-time, given by the Minkowski metric $\eta_{\mu\nu}$. If using curved coordinates, we will refer to this metric as $N_{\mu\nu}$, with the knowledge that a coordinate transformation will take this metric back to the Minkowski metric. For the present article, we will use spherical coordinates (r, θ, ϕ) with radial distance

r, polar angle θ and azimuthal angle ϕ. With these coordinates the flat spacetime metric becomes:

$$N_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 sin^2\theta \end{bmatrix} \tag{1}$$

Differentiation using this metric will be denoted with a colon (:) so that $g_{\mu\nu:\alpha}$ indicates differentiation of $g_{\mu\nu}$ using the Christoffel symbols associated with $N_{\mu\nu}$, which is noted as different from covariant differentiation denoted with a semi-colon (;). The Christoffel symbols associated with $N_{\mu\nu}$ are presented in the Appendix so that the reader can use them to verify calculations. Also note that units are chosen so that the speed of light $c = 1$ for the rest of the document, so c is ignored in all formulas.

1.2 The Proposed Field Equation

We do not include the Cosmological Constant term in the current paper, but we assume that it could be added into the formulae with suitable adjustments. The proposed field equation, including gravitational stress-energy ($t_{\sigma\nu}$) is:

$$(T_{\sigma\nu} + t_{\sigma\nu})8\pi G = A_{\sigma\nu} - \frac{1}{2}AN_{\sigma\nu} \tag{2}$$

$$A_{\rho\sigma\mu\nu} = \frac{1}{2}(\Omega_{\rho\nu:\sigma\mu} - \Omega_{\rho\mu:\sigma\nu} + \Omega_{\sigma\mu:\rho\nu} - \Omega_{\sigma\nu:\rho\mu}) \tag{3}$$

$$A_{\sigma\nu} = N^{\rho\mu}A_{\rho\sigma\mu\nu} \tag{4}$$

$$A = N^{\sigma\nu}N^{\rho\mu}A_{\rho\sigma\mu\nu} \tag{5}$$

It is well known that the rhs of equation 2 has zero flat space-time divergence (See the Appendix for the demonstration.). Equation 2 is considered a statement of the theory, motivated by the similarity to Einstein's Field Equation. The fact that the rhs of the proposed field equation has zero divergence with respect to the flat background is a good reason to believe in the form presented, since the inclusion of gravitational stress-energy allows us to account for all stress-energy, ensuring it is conserved in total.

1.3 Gravitational Stress-Energy

The definition of gravitational stress-energy is made to conform to Einstein's Equation as follows:

$$8\pi G t_{\sigma\nu} = A_{\sigma\nu} - \frac{1}{2} A N_{\sigma\nu} - G_{\sigma\nu} \tag{6}$$

Note that with this definition, equation 2 manifestly complies with Einstein's equation, with $G_{\sigma\nu}$ being the Einstein tensor. See the Appendix for a derivation involving the gravitational stress-energy for the Schwarzschild metric, with the result for a spherically symmetric body:

The radial stress (gravitational + non-gravitational) is like a negative pressure equal to T. The other stresses (gravitational + non-gravitational) appear to be containment stresses that balance the radial tension, similar to those created in a pressure vessel containing a pressurized gas, but reversed

2 Discussion

The gravitational metric $g_{\mu\nu}$ represents the gravitational potential. This is the metric that is induced by $T_{\mu\nu}$ in Einstein's theory. We do not use it to lower indices, raise indices or contract indices. It's sole purpose is to act on matter to influence trajectories by way of the geodesic equation. It will give the illusion of a curved space-time, but we interpret the illusion to being gravity's effect on matter as a force, not an effect on space-time itself. The proposed field equation is linearly related to matter where there is no gravitational stress-energy (ie. where $t_{\sigma\nu} = 0$), which appears to be the case in the vacuum, in particular for the Schwarzschild exterior solution.

2.1 Gravitation Metric For the Schwarzschild Exterior solution

We start with a proposed metric for a static spherically symmetric point mass:

$$g_{\mu\nu} = \begin{bmatrix} 1 - \frac{2GM}{r} & \frac{2GM}{r} & 0 & 0 \\ \frac{2GM}{r} & -1 - \frac{2GM}{r} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 sin^2\theta \end{bmatrix} \tag{7}$$

[1]

This particular form of the Schwarzschild exterior solution was selected because it appears that we can linearly add potentials together from separate mass sources. A linear theory of matter is valid wherever the gravitational stress-energy is zero. In the vacuum, the gravitational stress-energy is expected to be zero when the Cosmological Constant is not considered. It is no surprise that a linear theory (in regards to matter) isn't apparently taken seriously, since a compatibility condition must be satisfied to ensure $t_{\mu\nu} = 0$. Simultaneously $A_{\sigma\nu}$ and the Ricci curvature tensor ($R_{\sigma\nu}$) must be zero in the absence of matter and gravitational stress-energy. It is easy to guarantee $A_{\sigma\nu} = 0$ by linearly adding potentials of the correct form, but is not clear that the Ricci tensor is zero whenever $A_{\sigma\nu} = 0$; however, we assume that this is the case, as is argued in the Appendix (See the Curvature Tensor). We believe that we can go beyond this and state that the condition of $t_{\mu\nu} = 0$ is actually not required for a linear combination of potentials when dealing with the combined term $T_{\mu\nu} + t_{\mu\nu}$. See the Schwarzschild solution in the Appendix for a demonstration of this.

[1] This metric can be generated from the usual Schwarzschild metric diag$(1 - \frac{2GM}{r}, -1/(1 - \frac{2GM}{r}), -r^2, -r^2 sin^2\theta)$ using the Eddington-Finkelstein coordinate transformation [2,3] $dt \mapsto dt + \frac{2GM}{r-2GM} dr$.

When introducing the stress-energy tensor into the theory, the effective mass is equal to the contracted tensor T integrated over the volume containing it. Note that we may typically equate T with $T_{00} + t_{00}$ because outside (external to) the source particle, the interior stresses cancel out and have no effect (see [4]).[2] We may state that potentials may be added, with each one having the form of:

$$\Omega_{\mu\nu} = \begin{bmatrix} -\frac{2GM}{r} & \frac{2GM}{r} & 0 & 0 \\ \frac{2GM}{r} & -\frac{2GM}{r} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

The mass M may be formed from an integration of T, so that $dM = TdV$. See the Appendix for the application to the Schwarzschild metric.

3 Conclusions

We have presented Einstein's Field Equation with an explicit gravitational stress-energy term, in a flat space-time background. This treatment explicitly treats gravity as a force, thus answering in the affirmative whether gravity may be considered a force. The benefits include:

1) The individual potentials for bodies may be linearly combined when considering the combined term $T_{\mu\nu}+t_{\mu\nu}$ as the source, using the potential in the form presented in equation 8.

2) It explicitly identifies gravitational stress-energy.

[2] For an example of how the stresses cancel in a compound object consider a very thin spherical shell of thickness t and radius r containing pressure P. It has wall tension of Pr/2t. The wall tension is summed over 2 principle directions in the wall and integrated over the wall volume of $4\pi r^2 t$ to obtain 3PV, where V is the volume contained. This exactly cancels the integrated volume of the 3 directions of pressure within the shell.

3) The background is flat space-time, so it is easier to combine with other theories.

4) Providing a definition of the gravitational stress-energy allows it to be studied like any other physical object. For example, for the Schwarzschild solution, we can say that the radial stress (gravitational + non-gravitational) is like a negative pressure equal to T. The other stresses (gravitational + non-gravitational) appear to be containment stresses that balance this radial tension, similar to those created in a pressure vessel containing a pressurized gas. As an example, we may imagine the mass being converted into photons in an explosion. The total gravitational and non-gravitational stress, combined with the momentum-momentum components of the photons, must be conserved in order to maintain overall conservation of the stress-energy.

4 References

[1]Will, C. M.(1973) Relativistic Gravity In the Solar System. 111. Experimental Disproof of a Class of Linear Theories of Gravitation, Astrophysical Journal, Vol. 185, pp. 31-42

[2]Eddington, A.S. (Feb 1924). "A Comparison of Whitehead's and Einstein's Formulæ". Nature. 113 (2832): 192.

[3]Finkelstein, David (1958). "Past-Future Asymmetry of the Gravitational Field of a Point Particle". Phys. Rev. 110 (4): 965–967

[4]Misner, C. W., and Putnam, P. (1959). Active Gravitational Mass. Physical Review, 116(4), 1045–1046.

5 Appendices

5.1 Christoffel Symbols for Spherical Coordinates

$$\Gamma^1_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & -rsin^2\theta \end{bmatrix} \tag{9}$$

$$\Gamma^2_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1/r & 0 \\ 0 & 1/r & 0 & 0 \\ 0 & 0 & 0 & -sin\theta cos\theta \end{bmatrix} \tag{10}$$

$$\Gamma^3_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/r \\ 0 & 0 & 0 & cos\theta/sin\theta \\ 0 & 1/r & cos\theta/sin\theta & 0 \end{bmatrix} \tag{11}$$

5.2 Flat Space-Time Divergence of the Field Equation

For this demonstration we use the Minkowski metric for the background metric. Note that colon (:) differentiation is commutative since the metric is flat, and with the Minkoski metric the derivatives become partial derivatives denoted with commas. We want to demonstrate that the following expression (rhs of equation 2) has zero divergence.

$$rhs = \eta^{\rho\mu}\frac{1}{2}(\Omega_{\rho\nu,\sigma\mu} - \Omega_{\rho\mu,\sigma\nu} + \Omega_{\sigma\mu,\rho\nu} - \Omega_{\sigma\nu,\rho\mu}) - \eta^{\lambda\tau}\eta^{\rho\mu}\frac{1}{4}(\Omega_{\rho\tau,\lambda\mu} - \Omega_{\rho\mu,\lambda\tau} + \Omega_{\lambda\mu,\rho\tau} - \Omega_{\lambda\tau,\rho\mu})\eta_{\sigma\nu} \tag{12}$$

We differentiate using the Minkowski metric in the index β and contract this with the index σ:

$$\begin{aligned}
&\eta^{\beta\sigma}\eta^{\rho\mu}\frac{1}{2}(\Omega_{\rho\nu,\sigma\mu\beta}-\Omega_{\rho\mu,\sigma\nu\beta}+\Omega_{\sigma\mu,\rho\nu\beta}-\Omega_{\sigma\nu,\rho\mu\beta})-\eta^{\beta\sigma}\eta^{\lambda\tau}\eta^{\rho\mu}\frac{1}{4}(\Omega_{\rho\tau,\lambda\mu\beta}-\Omega_{\rho\mu,\lambda\tau\beta}+\Omega_{\lambda\mu,\rho\tau\beta}-\Omega_{\lambda\tau,\rho\mu\beta})\eta_{\sigma\nu}\\
&=\eta^{\beta\sigma}\eta^{\rho\mu}\frac{1}{2}(\Omega_{\rho\nu,\sigma\mu\beta}-\Omega_{\rho\mu,\sigma\nu\beta}+\Omega_{\sigma\mu,\rho\nu\beta}-\Omega_{\sigma\nu,\rho\mu\beta})-\eta^{\beta\sigma}\eta^{\lambda\tau}\eta^{\rho\mu}\frac{1}{2}(\Omega_{\rho\tau,\lambda\mu\beta}-\Omega_{\rho\mu,\lambda\tau\beta})\eta_{\sigma\nu}\\
&=\eta^{\beta\sigma}\eta^{\rho\mu}\frac{1}{2}(\Omega_{\rho\nu,\sigma\mu\beta}-\Omega_{\rho\mu,\sigma\nu\beta}+\Omega_{\sigma\mu,\rho\nu\beta}-\Omega_{\sigma\nu,\rho\mu\beta})-\eta^{\lambda\tau}\eta^{\rho\mu}\frac{1}{2}(\Omega_{\rho\tau,\lambda\mu\nu}-\Omega_{\rho\mu,\lambda\tau\nu})\\
&=\eta^{\lambda\tau}\eta^{\rho\mu}\frac{1}{2}(\Omega_{\rho\nu,\tau\mu\lambda}-\Omega_{\rho\mu,\tau\nu\lambda}+\Omega_{\tau\mu,\rho\nu\lambda}-\Omega_{\tau\nu,\rho\mu\lambda})-\eta^{\lambda\tau}\eta^{\rho\mu}\frac{1}{2}(\Omega_{\rho\tau,\lambda\mu\nu}-\Omega_{\rho\mu,\lambda\tau\nu})=0
\end{aligned} \tag{13}$$

The above manipulations result from relabeling dummy indices.

5.3 The Schwarzschild Solution

We are using spherical coordinates for the background metric, and note that gravitational stress-energy is assumed present in the interior (in the presence of matter). Both the gravitational stress-energy and the regular matter are sources for the exterior potential, which is given in equation 8.

Note that the contracted potential Ω is equal to zero for the exterior solution, and this fact has been used to simplify the equations. The derivation takes advantage of the commutativity of derivatives for flat metrics. The rhs of the field equation is easily reduced to:

$$A_{\sigma\nu}-\frac{1}{2}AN_{\sigma\nu}=\frac{1}{2}N^{\rho\mu}(\Omega_{\rho\nu:\sigma}+\Omega_{\sigma\rho:\nu}-\Omega_{\sigma\nu:\rho})_{:\mu}-\frac{1}{2}N^{\rho\mu}N^{\alpha\theta}N_{\sigma\nu}(\Omega_{\rho\alpha:\theta})_{:\mu} \tag{14}$$

Introducing an intermediate variable $B_{\rho\sigma\nu}$, we have:

$$A_{\sigma\nu}-\frac{1}{2}AN_{\sigma\nu}=N^{\rho\mu}B_{\rho\sigma\nu:\mu} \tag{15}$$

We now apply the divergence theorem to the intermediate variable, noting that the integration area is over a sphere that is far enough removed from the particle that the particle is a point mass, since the calculation is not straightforward for a compound object. Because the radial coordinate is variable 1, which is normal to the area of integration, we force the first index to be 1, and after some simplification get:

$$B_{1\sigma\nu} = \frac{1}{2}(\Omega_{1\nu,\sigma} + \Omega_{\sigma 1,\nu} - \Omega_{\sigma\nu,1}) - \Omega_{11}\Gamma^1_{\sigma\nu} + \frac{1}{2r}N_{\sigma\nu}\Omega_{11} \tag{16}$$

The only nonzero results are:

$$\begin{aligned} B_{100} &= -\Omega_{00,1}N_{00} \\ B_{111} &= -\Omega_{00,1}N_{11} \\ B_{122} &= \frac{1}{2}\Omega_{00,1}N_{22} \\ B_{133} &= \frac{1}{2}\Omega_{00,1}N_{33} \end{aligned} \tag{17}$$

$$\begin{aligned} B_{100}\cdot\hat{r} &= \Omega_{00,1} \\ B_{111}\cdot\hat{r} &= \Omega_{00,1}N_{11} \\ B_{122}\cdot\hat{r} &= -\frac{1}{2}\Omega_{00,1}N_{22} \\ B_{133}\cdot\hat{r} &= -\frac{1}{2}\Omega_{00,1}N_{33} \end{aligned} \tag{18}$$

We may now apply the divergence theorem and list the nonzero results:

$$\oint_s B_{1\sigma\nu}\cdot\hat{r}ds = \oint_V N^{\rho\mu}B_{\rho\sigma\nu:\mu}dV = \oint_V (A_{\sigma\nu} - \frac{1}{2}AN_{\sigma\nu})dV \tag{19}$$

$$\begin{aligned} 8\pi G N_{00}\oint_V TdV &= \oint_V (A_{00} - \frac{1}{2}AN_{00})dV = 8\pi G\oint_V (T_{00}+t_{00})dV \\ 8\pi G N_{11}\oint_V TdV &= \oint_V (A_{11} - \frac{1}{2}AN_{11})dV = 8\pi G\oint_V (T_{11}+t_{11})dV \\ -4\pi G N_{22}\oint_V TdV &= \oint_V (A_{22} - \frac{1}{2}AN_{22})dV = 8\pi G\oint_V (T_{22}+t_{22})dV \\ -4\pi G N_{33}\oint_V TdV &= \oint_V (A_{33} - \frac{1}{2}AN_{33})dV = 8\pi G\oint_V (T_{33}+t_{33})dV \end{aligned} \tag{20}$$

We now see no reason to limit the solution to an integral over the particle. As long as we explicitly account for the $t_{\mu\nu}$ term, we still regard the the potential $\Omega_{\mu\nu}$ to be a linear function of T even in the interior and get, without knowing the details of the stresses within the particle:

$$\begin{aligned} T_{00} + t_{00} &= T \\ T_{11} + t_{11} &= -T \\ T_{22} + t_{22} &= \frac{Tr^2}{2} \\ T_{33} + t_{33} &= \frac{Tr^2 sin^2\theta}{2} \end{aligned} \tag{21}$$

The radial stress (gravitational + non-gravitational) is like a negative pressure. The other stresses (gravitational + non-gravitational) appear to be containment stresses that balance the radial tension, similar to those created in a pressure vessel containing a pressurized gas, but acting in reverse. We can also note that $(T_{\mu\nu} + t_{\mu\nu})N^{\mu\nu} = T$, thereby showing that the stress components do not contribute to the effective mass.

5.4 The Curvature Tensor

Note that the Christoffel symbols used in this subsection are those associated with the $g_{\mu\nu}$ metric. The curvature tensor may be written as:

$$R_{\rho\sigma\mu\nu} = A_{\rho\sigma\mu\nu} + g^{\lambda\eta}(\Gamma_{\eta\rho\nu}\Gamma_{\lambda\mu\sigma} - \Gamma_{\eta\rho\mu}\Gamma_{\lambda\nu\sigma}) \tag{22}$$

Note that we have explicitly written it in the Minkowski metric form of the flat space-time background to avoid including the Christoffel symbols of this metric. We use latin indices for normal coordinates. Therefore:

$$\Gamma_{\eta\rho\nu} = \frac{\partial x^i}{\partial x^{\rho\nu}} \frac{\partial x^\theta}{\partial x^i} g_{\theta\eta} = x^i_{,\rho\nu} x^\theta_{,i} g_{\theta\eta} \tag{23}$$

The curvature tensor may be rewritten as:

$$R_{\rho\sigma\mu\nu} = A_{\rho\sigma\mu\nu} + g_{\lambda\eta}(x^i_{,\rho\nu}x^j_{,\mu\sigma} - x^i_{,\rho\mu}x^j_{,\nu\sigma})x^\lambda_{,i}x^\eta_{,j} \tag{24}$$

Our current consideration is when $A_{\sigma\nu} = 0$; therefore, contracting with the Minkowski metric according to our definition of $A_{\sigma\nu}$ and setting $A_{\sigma\nu} = 0$:

$$\eta^{\rho\mu}R_{\rho\sigma\mu\nu} = \eta^{\rho\mu}g_{\lambda\eta}(x^i_{,\rho\nu}x^j_{,\mu\sigma} - x^i_{,\rho\mu}x^j_{,\nu\sigma})x^\lambda_{,i}x^\eta_{,j} \tag{25}$$

We want to know the value of $g^{\rho\mu}R_{\rho\sigma\mu\nu}$ when $A_{\sigma\nu} = 0$. Based on the result of the Schwarzschild metric we guess that it equals zero, so we also have:

$$g^{\rho\mu}A_{\rho\sigma\mu\nu} = -g^{\rho\mu}g_{\lambda\eta}(x^i_{,\rho\nu}x^j_{,\mu\sigma} - x^i_{,\rho\mu}x^j_{,\nu\sigma})x^\lambda_{,i}x^\eta_{,j} \tag{26}$$

Based on the apparent symmetry of the equations, we guess that in general $R_{\sigma\nu} = 0$ when $A_{\sigma\nu} = 0$. This would mean that the Einstein tensor is zero when $A_{\sigma\nu} = 0$, and thus in empty space we may assume that there is no gravitational stress-energy $t_{\sigma\nu}$.