Complex Space-Time

An Alternative View on the Problems of Special Relativity

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Dubitando ad veritatem pervenimus

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To Eliane Trevisan

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Introduction

The Special Theory of Relativity (SR) has aroused much controversy since the very beginning. Even today, although most physicists consider it a classic on a par with mechanics and electrodynamics, not all are entirely convinced by it. Since the Internet has become a free forum of expression, it is clear that despite the century that has passed since the theory was first developed, it is still not accepted indiscriminately. However, the mainstream science does not take these voices seriously, and scientific journals do not publish articles critical of SR. The current theory took a long time to develop, has a strong mathematical structure and has survived for 100 years, therefore it cannot be fundamentally wrong. A century is long enough time to see the SR from a distance. One can risk a claim that the SR is a prosthesis whose creators, in a very general formalism, have lost some details, without which the theory allows for erroneous conclusions euphemistically referred to as paradoxes. This monograph is the result of almost 40 years of work on creating a mathematical basis for an alternative theory which is, however, in line with Einstein's postulates. Finding a parallel path was possible thanks to the discovery by the author of a transformation invariant for the wave equation, previously unknown[1], because it does not belong to the Lorentz group.

Historical outline, or a few words about original sins

At the turn of the 19th and 20th centuries, an explosion of discoveries took place in physics. Hypotheses were made and new theories developed. Forty years earlier, James Clerk Maxwell had put the laws of electricity together, finding out that they were combined into a wave equation, the domain of which is time and space: real time and 3-dimensional Euclidean space, because at such time and in such space Coulomb, Faraday, Ampere, Hertz and many other outstanding experimenters conducted experiments and formulated the laws of physics. It was an era when mathematicians had already thoroughly studied mechanics and they knew that by analysing the laws of physics with mathematical methods, a lot of knowledge can still be obtained. Thus, they got down to electrodynamics. The basic problem was to define the universality of the newly discovered laws, and it quickly turned out that transformations that are invariant to mechanics change electrodynamics. The mathematicians concluded, therefore, that either the transformations were bad or the laws of electrodynamics did not deserve that ranking. Meanwhile, Albert Michelson and Eduard Morley, while searching for the ether, found that the speed of light in the vacuum was always the same. From this they concluded that ether didn't exist. For mathematicians, the constant speed of light in the vacuum meant that the wave equation should be invariant under the boost, because the speed of light is closely related to the solution of the homogeneous wave equation. This led them to a transformation that was later named after its most eminent discoverer, Hendrik Lorentz. The Lorentz transformation (LT), which is interpreted as a transfer of an observer between frames moving relative to each other at a speed comparable to light, does not change the form of the wave equation and thus preserves the speed of the electromagnetic wave. This resolved the contradictions that arose. The application of the Lorentz transformation in other areas of physics, in particular in mechanics, resulted in the creation of a completely new field of theoretical physics - *Special Theory of Relativity*.

The Special Theory of Relativity was one of these theories that changed the perception of the world. It showed to people a world that they had not known until recently - the world of particles moving at enormous speeds. The theoretical physicists had their hands full. The mathematics they had used to describe certain

[1] As it turned out later, the transformation was known to researchers of Clifford Algebras, but it was not tested for use in SR.

phenomena has become insufficient. It was necessary to extend the geometry created in antiquity with an additional dimension - time. Formulas devised by researchers have become more and more expanded. In order to write them down in a concise form, a tensor calculus was used. The concepts of vector and scalar, which were the basis of didactics, have disappeared. Tensor calculus combined them into one form and hid them behind meaningless indexes. The level of abstraction created a barrier to intuitive understanding of physical phenomena. Someone may ask: Was the direction taken by relativists 100 years ago correct? Is it not possible to create a different, more intuitive language to describe objects that move at the highest speeds? Popular publications show that the topic has not been dealt with. The presented interpretations are full of paradoxes and are contrary to common sense. The scientific mainstream has passed over this by stating that this is what modern physics is like and must be accepted as it is.

Although relativists have adopted tensor calculus as the basic language for the mathematical description of the theory, attempts are still being made to find a more intuitive tool based on imaginable geometric concepts. The most interesting direction is the constantly developing Geometric Algebra (GA), initiated by David Hestenes, based on concepts derived from Grassmann Algebra. William Baylis went in a similar direction, combining scalars with multivectors into complex paravectors and on them he built his Algebra of Physical Space (APS). Both theories apply a different formalism, but they are equivalent to each other and describe the problems of the current Special Theory of Relativity in a specific way. William Baylis showed that the Lorentz transformation can be represented by the relation $X' = \Lambda X \Lambda^*$, where Λ is an orthogonal paravector. We study the transformation $X' = \Lambda X$, which does not belong to the Lorentz group because it requires complex spacetime.

Elegant and friendly formalism is of great importance in didactics. The assessment of elegance depends on the individual sense of aesthetics, which is influenced not only by education but also by deeply subconscious intuition. If a person doesn't like something, it doesn't appeal to them. He may learn formulas by heart, but will never be convinced by them. An example of well-understood mathematical aesthetics are Maxwell's equations, which are commonly presented not in their original form or in tensor or matrix notation, but in Heaviside operator notation. Thanks to differential operators, they are the most elegant, i.e. simple, aesthetic, and most importantly – imaginable notation. Aesthetics, a seemingly irrational concept, is of great importance in mathematics. Richard Feynman, discussing Maxwell's equations in his *Lectures*, repeatedly admired their mathematical beauty. Symmetry has always been associated with the natural order and beauty of nature. It was probably such a sense of aesthetics that prompted Dirac to search for a magnetic monopole. He was not the only one to look for the elements missing to the symmetry of the equations in the theory of electromagnetism. It has come to the point that when browsing contemporary theoretical literature, one can get the impression that magnetic charges exist not only in literature but also in nature. Many recognized textbooks on the theory of electricity, such as *Classical Electrodynamics* by J.D. Jackson, have chapters on this topic. And although these theoretical considerations date back well over half a century, and research possibilities have changed significantly during this time, no magnetic charge has been detected. So far it is known that the magnetic field is generated by moving electric charges.

The scope of the issues raised

The Special Theory of Relativity originated from the study of electric field equations, therefore the content of this monograph largely applies to this branch of physics. The idea of this paper is to search for the possibility of building a theory alternative to the current SR. Its leitmotif is the analysis of the field equations of moving electric charges. The language used to describe the problems is the paravector calculus, which seems to be natural for space-time phenomena. The properties of paravectors are similar to vectors, which helps a lot in understanding problems and controlling calculations.

The presented reflections are consistent with the assumptions and the direction chosen by the authors of the current STR, and yet the results differ from the textbook knowledge. Why? The creators of STR, apart from the famous two Einstein's postulates, informally made one more assumption, which affected the direction in which the theory developed. This assumption is: *Space-time is a real structure* (in terms of real numbers), so they only found such transformations that work inside real space-time. A long time ago, the author found a different transformation that meets the postulates, but requires complex space-time. An investigation of the properties of this transformation, which is an equivalent to the Lorentz transformation, showed that it is much more elegant than LT. A question arises: Could space-time be more compound than we think? Today, after years of work, I can say with certainty that this is indeed the case, of which I will try to convince the reader.

In our study, we talk about physical and mathematical concepts. Simply put, physical concepts include those that can be measured. The others can only be classified as mathematical concepts. Of fundamental importance in the adopted reasoning is the separation of the concepts of time into time, which is the fourth dimension of the mathematical structure of space-time, which can be called *prototime*, and the proper time of a physical object, which is directed into the future and is discrete. For this reason, time intervals and space-time intervals will appear in the formulas. Discrete time is closer to the natural sense of time measured by the ticking of a clock. We will see that despite the use of complex transformations, the rectilinear motion of an inertial object in any frame is described by equations known as the Galilean transformation. The complex transformation as well as LT meets the correspondence principle, i.e. for non-relativistic approximations it evolves into the Galilean transformation. We will check the invariants of the electric field theory and special relativity, and we'll show the compatibility and differences of the constructed theory and of the classical one. In complex space-time, we will reduce the four Maxwell's equations to the one in which the imaginary part of the electric field is interpreted as a magnetic field, and the existence of a magnetic field excludes the existence of magnetic charges. Although the presented proposal of an alternative SR is consistent with the postulates of the classical theory, they are not equivalent. Based on the results of William Baylis we will show that the classical Lorentz transformation is one of several variants of complex transformations' combination.

The only information carrier is energy which is a fundamental physical concept and which is always real. It is for this reason that energy can undergoes the classical Lorentz transformation, but in our opinion it is not, as discussed in Chapter 10. We will make references to STR in Chapter 12 on the basis of the work of professor William Baylis of Windsor University, Canada. His *Algebra of Physical Space* is very well suited to describe space-time phenomena. The reader does not need to know it, and maybe it is even better if he does not, because then it will be easier for him to assimilate our formalism. Finally, we will outline the idea of the mathematical structure of space-time in which these considerations are possible.

Chapter 1 shows mathematically the origins of Lorentz transformation and explains why it is very important to know the transformations preserving the form of the electromagnetic wave equation. The method of checking the invariance of the wave equation under the classical Lorentz transformation is presented, together with some critical remarks on LT. Finally a linear complex transformation is presented, which also preserves the wave equation, and which, according to the author, is more general and mathematically much more elegant than Lorentz transformation.

Chapters 2 and 4 provide the mathematical basis for describing physical problems. Chapter 2 presents the language of the paravector algebra. The short Chapter 4 proves the identities with which it is possible to quickly change formulas containing space-time differentiation operators that underwent transformations described by paravectors. Chapter 3 prepares the reader for the problems of mathematical analysis involving differential equations containing space-time differentiation operators, showing how clear paravector notation of the laws of electric field theory looks. **A thorough reading of these chapters is necessary to understand the considerations presented in the further part of the monograph**.

Chapter 5 is devoted to explaining the concept of a phase interval, which extends the concept of time interval to the movement of objects in space-time. The analysis of the various examples of phase intervals in Chapters 5 and 6 will allow the reader to intuitively imagine relativistic phenomena in complex space-time and to understand the idea of space-time, which is not a space of points (affine space) but an interval space (vector space).

Equipped with the knowledge obtained so far, in Chapter 7 we will return to the electric field and we will see that Maxwell's equations in complex space-time require the removal of the current density (Ampere's law) from them, which was postulated during J.C.Maxwell's lifetime.

Chapter 8 hypothesizes on why we see the world as real despite the fact that the simplicity of the complex notation indicates that it is more compound. A transformation interpreted as a projection of phenomena from the complex space-time onto the real space-time of the observer, which we called realisation, is shown. This transformation is based on the hypothesis that energy is real in any frame of reference, and the only information carrier in nature is energy.

In Chapter 9 the basic issues of electric field theory after modifying Maxwell's equations are examined and it is shown there that despite this correction, the paravector equations still describe the electric field well.

In Chapter 10, we return to the geometric issues. By examining the relation between three objects, we find a relation corresponding to the metric, which allows us to specify the structure of complex space-time. We postulate that the metric is not a property of geometric space, but concerns only physical objects. We present the covariant equation of the motion of a charged particle in an electric field, and finally we show that complex space-time does not contradict the current relativistic theoretical mechanics or quantum mechanics.

Chapter 11 organizes the results obtained in the previous chapters to outline the structure of complex space-time.

Side-stories that the reader will surely come across, and whose explanation will facilitate understanding of the idea of complex space-time, are discussed in the appendices.

Assumptions valid throughout the monograph and notes that the reader should bear in mind when s/he has trouble locating considerations in the structure of space-time

To simplify the calculations, the following assumptions are made:

1. Physical issues are dealt with in a vacuum
2. The discussed problems concern inertial systems only
3. Prototime is continious, but a proper time of the physical object is a discrete quantity[2]
4. To simplify the formulas involving the elimination of the physical constants present in Maxwell's equations, it is assumed that the formulas are written in the natural system of Planck units, i.e.:
   - permittivity ϵ_0 and permeability μ_0 of free space are equal to 1.
   - velocities are dimensionless quantities and their values are relative to the speed of light. The modulus of the velocity of light vector is equal to 1, i.e. the change of formulas from natural units to SI units occurs by conversion:

Table 1: Symbol replacement table

	Natural units system	SI units system
velocity	v	v/c
time	t	ct
mass	m	mc^2
momentum	$\mathbf{p}$	$c\mathbf{p}$
charge density	ρ	$\rho\sqrt{\epsilon_0}$
current density	j	$j/\sqrt{\epsilon_0}$
scalar potential	φ	$\varphi/\sqrt{\epsilon_0}$
vector potential	$\mathbf{A}$	$\mathbf{A}/\sqrt{\mu_0}$
electric field intensity	$\mathbf{E}$	$\mathbf{E}\sqrt{\epsilon_0}$
magnetic field induction	$\mathbf{B}$	$\mathbf{B}/\sqrt{\mu_0}$
paravector of light speed	$(1, \mathbf{c})$, where $\lvert\mathbf{c}\rvert = 1$	$(c, \mathbf{c})$

[2] A brief explanation is given in Appendix 1

In cases where we will check what the obtained formula looks like for non-relativistic velocity (checking the correspondence rule), we will change the units system to SI.

5. Throughout the work, efforts were made to follow the rule that real quantities (numbers, vectors, paravectors) are marked with Roman letters, and complex ones with Greek letters. This does not apply to physical quantities marked with letters commonly used in the literature.

From the author

While developing this monograph, a plan I adopted that the mathematical structure of physical space-time would be shown at the end as a natural effect of the presented arguments. Perhaps the mathematical considerations would be simpler if I started with presenting the conditions that complex space-time meets, because it would be clear to the reader in which field he is located, but showing the structure of space-time at the beginning would give the impression that artificial assumptions have been made, which the reader would have the right not to trust. The discussions that I conducted electronically with Professor Zbigniew Oziewicz showed that preliminary remarks are necessary to emphasize that the considerations are not conducted in affine space-time (space of places) but in the space of motion, i.e. in the vector space. This is due to the assumption of the quantum nature of time. Although the real space of places (without time) is affine (points), the movement in this space does not take place continuously, which is conditioned by the advancing progress of time. This assumption is consistent with the methodology of time measurement and cannot be experimentally denied, to the contrary: it is impossible to prove that time runs continuously. Vector space-time makes it easy to assign imaginary components to real vectors, which cannot be done with the coordinates of points. The assumption that time is a discrete quantity is the same as saying that the basic concept is the **while**. Each longer distance of time is called an **interval**. The while is the shortest interval. An interval starts at one moment and ends at another, so like in geometry, an interval is an ordered pair of moments, but it is the intervals that define the moments, not the other way around. So it is better to define the moment like this: In the **moment** one interval ends and another begins. The reader should remember about this philosophical assumption while studying this monograph.

I hope that the reader will be interested in a different view on the electric field theory, on special relativity and, above all, on space-time. I would also like to draw the attention of the reader to a useful and simple mathematical apparatus designed to describe them. I think that the presented theory is already developed enough to inspire an inquisitive reader to work on its verification and further expansion. I ask the orthodox followers of the classical SR to treat the book as a curiosity or a mathematical game, while I would like to support the doubters and show them the directions of possible research. To both, I would like to remind that science is not a religion, and doubt is not a sin.

Please send your opinions, comments and all correspondence to: c4spacetime@gmail.com

Acknowledgements

My adventure with the Special Theory of Relativity began in the early 1980s. I guess it was a coincidence, but if I started looking for logic in these cases, I would certainly find a metaphysical explanation. It was a time when my plans for the future collapsed and trying to understand SR became a way of keeping my mind busy. It quickly turned out that my logical brain would not accept SR in either popular or scientific form, where tensor formalism was dominant. Then I started my own research, which directed me to the Institute of Theoretical Physics of the University of Wrocław to Dr. Bernard Jancewicz[3], who showed me the articles of William Baylis. From the very beginning, my efforts have been followed and supported spiritually by my school friend Jacek Lewiński. Certainly without his kind support, I would have quit my work on this theory long ago. In order to be able to present my results, it was necessary to master the TeX editor in which I was supported by my daughter Dorota, whose critical remarks were very accurate and gave the final shape to the second chapter "Algebra of Paravectors". Professor Zbigniew Oziewicz persuaded me to collect all my results together. The correspondence

[3]Professor Bernard Jancewicz died on May 16, 2021 as a result of an infection with the COVID-19 virus

with the Professor was very inspiring, but unfortunately, it was suddenly interrupted by the Professor's death as a result of COVID-19 virus infection. I cannot omit Dr. Ryszard Herbeć, without whose help I would not be able to finish my work. I would like to express my sincere thanks to all of them.

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Chapter 1

Invariance of the wave equation

In this chapter, the importance of transformations preserving the invariance of physics laws is presented. Invariance of the wave equation with respect to the Lorentz transformation is checked. A complex linear transformation is presented, which does not belong to the Lorentz group but which also preserves the invariance of the wave equations system.

Most of the laws of physics are described by differential equations. In the mathematical sense, the universality of the laws of physics means the equations representing these laws are invariant, regardless of the system in which they are written. The main issue is to find transformations that preserve the mathematical formula of the law under analysis. An electromagnetic wave observed from any frame of reference is always an electromagnetic wave, so it should always have a similar formula. The electromagnetic wave in a vacuum is described by a homogeneous wave equation.

$$\frac{\partial^2 \varphi(t,\mathbf{x})}{c^2 \partial t^2} - \nabla^2 \varphi(t,\mathbf{x}) = 0 \tag{1.1}$$

Since d'Alembert it is known that the most general solution to this equation has the following form

$$\varphi(t,\mathbf{x}) = f(\mathbf{x} - \mathbf{c}t) + g(\mathbf{x} + \mathbf{c}t) \tag{1.2}$$

where f and g are any double differentiable functions and the vector $\mathbf{c}$ is interpreted as the wave velocity. We can see that if we find a transformation of $(t,\mathbf{x}) \to (t',\mathbf{x}')$ that does not change the form of the wave equation, neither will it change the speed of the wave, so it will meet the demand for a constant speed of light.

Maxwell's equations transferred to potentials using Lorenz gauge condition make up a system of wave equations

$$\begin{aligned} \frac{\partial^2 \varphi(t,\mathbf{x})}{c^2 \partial t^2} - \nabla^2 \varphi(t,\mathbf{x}) &= \frac{\rho(t,\mathbf{x})}{\epsilon_0} \\ \frac{\partial^2 \mathbf{A}(t,\mathbf{x})}{c^2 \partial t^2} - \nabla^2 \mathbf{A}(t,\mathbf{x}) &= \frac{\mathbf{j}(t,\mathbf{x})}{\epsilon_0 c^2} \end{aligned} \tag{1.3}$$

One of the transformations that preserves invariance of this system of equations is the Lorentz transformation. While in a one-dimensional space where the Lorentz transformation takes the form of

$$t' = \frac{t - vx/c^2}{\sqrt{1-(v/c)^2}} \quad , \quad x' = \frac{x - vt}{\sqrt{1-(v/c)^2}} \tag{1.4}$$

the proof is obvious, in the three-dimensional space it seems that it was a bit stretched. One has the impression that the 1-dimensional transformation 'forcibly' fits into the 3-dimensional space, and the vector formula (Fock)

$$t' = \frac{t - \mathbf{v}\mathbf{x}/c^2}{\sqrt{1-(v/c)^2}}, \qquad \mathbf{x}' = \mathbf{x} - \mathbf{v}\left[\frac{\mathbf{x}\mathbf{v}}{v^2}\left(1 - \frac{1}{\sqrt{1-(v/c)^2}}\right) + \frac{t}{\sqrt{1-(v/c)^2}}\right] \tag{1.5}$$

is not very elegant.

Although most textbooks in the STR chapters talk about the invariance of the wave equation, or the covariance of the Maxwell's equations with respect to the Lorentz transformation, they rarely prove this, although it is essential. A proof made using tensor calculus can be found in W.A.Ugarow's textbook [19]. Since the manipulation of tensor indexes is not convincing enough for everyone, we will check it using the rules of well known differentiation calculus. Without losing generality, we assume that $c = 1$, which we get when we write the physical formulas in the natural units system. In this system, the one-dimensional Lorentz transformation takes the form of

$$t' = \frac{t - vx}{\sqrt{1-v^2}} \qquad , \qquad x' = \frac{x - vt}{\sqrt{1-v^2}} \tag{1.6}$$

1.1 Proof of invariance of the wave equation under the Lorentz transformation

1.1.1 One-dimensional space

We start from the simplest case, i.e. the wave equation in 1-dimensional space:

$$\frac{\partial^2 f(t,x)}{\partial t^2} - \frac{\partial^2 f(t,x)}{\partial x^2} = 0, \tag{1.7}$$

where f is a double-differentiable function. In the above equation, we will replace the unprimed arguments with primed ones, according to the equations (1.6) and we expect that by going to the primed frame we should get the equations in the form of:

$$\frac{\partial^2 f'}{\partial t'^2} - \frac{\partial^2 f'}{\partial x'^2} = 0, \tag{1.8}$$

where prime denotes a function with the same values as the f function, but on primed arguments $f' = f(t', x')$.

Proof.

By differentiating (1.6) over time, we get

$$\frac{\partial t'}{\partial t} = \frac{1}{\sqrt{1-v^2}} \qquad , \qquad \frac{\partial x'}{\partial t} = -\frac{v}{\sqrt{1-v^2}}$$

which, substituted into the formula for partial derivatives of a compound function

$$\frac{\partial \varphi'}{\partial t} = \frac{\partial \varphi'}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial \varphi'}{\partial x'}\frac{\partial x'}{\partial t}$$

gives

$$\frac{\partial f'}{\partial t} = \frac{\partial f'}{\partial t'}\frac{1}{\sqrt{1-v^2}} - \frac{\partial f'}{\partial x'}\frac{v}{\sqrt{1-v^2}}.$$

By differentiating again for t we get:

$$\begin{aligned}
\frac{\partial^2 f'}{\partial t^2} &= \frac{1}{\sqrt{1-v^2}}\frac{\partial}{\partial t}\left(\frac{\partial f'}{\partial t'} - v\frac{\partial f'}{\partial x'}\right) = \\
&= \frac{1}{\sqrt{1-v^2}}\left(\frac{\partial^2 f'}{\partial t'^2}\frac{\partial t'}{\partial t} + \frac{\partial^2 f'}{\partial t'\partial x'}\frac{\partial x'}{\partial t} - v\frac{\partial^2 f'}{\partial x'\partial t'}\frac{\partial t'}{\partial t} - v\frac{\partial^2 f'}{\partial x'^2}\frac{\partial x'}{\partial t}\right) = \\
&= \frac{1}{1-v^2}\left(\frac{\partial^2 f'}{\partial t'^2} - v\frac{\partial^2 f'}{\partial t'\partial x'} - v\frac{\partial^2 f'}{\partial x'\partial t'} + v^2\frac{\partial^2 f'}{\partial x'^2}\right) = \\
&= \frac{1}{1-v^2}\left(\frac{\partial^2 f'}{\partial t'^2} - 2v\frac{\partial^2 f'}{\partial t'\partial x'} + v^2\frac{\partial^2 f'}{\partial x'^2}\right)
\end{aligned}$$

By differentiating (1.6) for x we get:

$$\frac{\partial t'}{\partial x} = -\frac{v}{\sqrt{1-v^2}} \qquad , \qquad \frac{\partial x'}{\partial x} = \frac{1}{\sqrt{1-v^2}}$$

By acting in an analogous way, we get:

$$\frac{\partial \varphi'}{\partial x} = \frac{\partial \varphi'}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial \varphi'}{\partial t'}\frac{\partial t'}{\partial x} = \frac{1}{\sqrt{1-v^2}}\left(\frac{\partial \varphi'}{\partial x'} - v\frac{\partial \varphi'}{\partial t'}\right),$$

hence the second derivative is:

$$\begin{aligned}
\frac{\partial^2 f'}{\partial x^2} &= \frac{1}{\sqrt{1-v^2}}\frac{\partial}{\partial x}\left(\frac{\partial f'}{\partial x'} - v\frac{\partial f'}{\partial t'}\right) = \\
&= \frac{1}{\sqrt{1-v^2}}\left(\frac{\partial^2 f'}{\partial x'^2}\frac{\partial x'}{\partial x} + \frac{\partial^2 f'}{\partial t'\partial x'}\frac{\partial t'}{\partial x} - v\frac{\partial^2 f'}{\partial x'\partial t'}\frac{\partial x'}{\partial x} - v\frac{\partial^2 f'}{\partial t'^2}\frac{\partial t'}{\partial x}\right) = \\
&= \frac{1}{1-v^2}\left(\frac{\partial^2 f'}{\partial x'^2} - v\frac{\partial^2 f'}{\partial x'\partial t'} - v\frac{\partial^2 f'}{\partial t'\partial x'} + v^2\frac{\partial^2 f'}{\partial t'^2}\right) = \\
&= \frac{1}{1-v^2}\left(\frac{\partial^2 f'}{\partial x'^2} - 2v\frac{\partial^2 f'}{\partial t'\partial x'} + v^2\frac{\partial^2 f'}{\partial t'^2}\right).
\end{aligned}$$

After subtracting the second derivatives from both sides, we get

$$\frac{\partial^2 f'}{\partial t^2} - \frac{\partial^2 f'}{\partial x^2} = \frac{\partial^2 f'}{\partial t'^2} - \frac{\partial^2 f'}{\partial x'^2},$$

which means that if we convert the coordinates according to the formulas of the one-dimensional Lorentz transformation, **the form of the wave equation does not change** which was to be proved.

□

1.1.2 Three-dimensional space

At the beginning, the simplest case was selected for calculations, where the velocity vector is parallel to the X axis. In this case the proof of invariance of the wave equation was obtained, so we decided to check it for any velocity direction. For this purpose, a vector transformation was selected which, according to the literature [17] [19], has the form

$$t' = \frac{t - \mathbf{vx}}{\sqrt{1-v^2}} \qquad , \qquad \mathbf{x}' = \mathbf{x} - \mathbf{v}\left[\frac{\mathbf{xv}}{v^2}\left(1 - \frac{1}{\sqrt{1-v^2}}\right) + \frac{t}{\sqrt{1-v^2}}\right], \tag{1.9}$$

where $v^2 = v_x^2 + v_y^2 + v_z^2$. In this case, the proof also came out OK, although it was very laborious.

For the general case, Ugarow [19] derives invariance of d'Alembertian from 4-divergence invariance, which in turn he proves using tensor calculus. Since for the velocity $\mathbf{v} = (v, 0, 0)$ the proof of 4-divergence invariance is not laborious, it was checked in the same way as above and we got the result:

$$\partial A = \partial' A', \quad \text{when } A \text{ transforms like a 4-vector,}$$

which means that the 4-divergence of the 4-vector field is invariant.

Everything looks fine. An electromagnetic wave in any frame is an electromagnetic wave, so it must be described by a similar equation. In the tensor notation it looks even more simple and aesthetic.

$$\text{For} \qquad x'_\alpha = a_\alpha^\beta x_\beta, \qquad \partial^\alpha \partial_\alpha A_\beta = 0 \qquad \Rightarrow \qquad \partial'^\alpha \partial'_\alpha A'_\beta = 0 \tag{1.10}$$

However, unlike vector notation, tensor formulas are completely unintuitive, as they do not show the difference between the quantities behind the indexes. And yet, time and space have completely different physical properties because time is constantly moving forward, while in space one can stand still or, for example, return home after work.

1.2 Complex relativistic transformation

By analysing a system of wave equations

$$\frac{\partial^2\varphi(t,\mathbf{x})}{\partial t^2}-\nabla^2\varphi(t,\mathbf{x})=0 \qquad (1.11)$$
$$\frac{\partial^2\mathbf{A}(t,\mathbf{x})}{\partial t^2}-\nabla^2\mathbf{A}(t,\mathbf{x})=0$$

one can check that it is invariant under the transformation:

$$t'=\frac{t+\mathbf{v}\mathbf{x}}{\sqrt{1-v^2}}, \qquad \mathbf{x}'=\frac{\mathbf{x}+\mathbf{v}t\pm i\mathbf{v}\times\mathbf{x}}{\sqrt{1-v^2}}, \qquad (1.12)$$

where $i=\sqrt{-1}$, and $v^2=\mathbf{v}\mathbf{v}=v_x^2+v_y^2+v_z^2$.

Note that in a special 1-dimensional case (in the spatial sense), this transformation is equivalent to the 1-dimensional Lorentz transformation.

We will prove the invariance of the wave equation system in the same way as before.

Proof. In the equation (1.11), we convert non-primed arguments to primed ones according to (1.12)

$$\frac{\partial^2\varphi'}{\partial t^2}-\frac{\partial^2\varphi'}{\partial x^2}-\frac{\partial^2\varphi'}{\partial y^2}-\frac{\partial^2\varphi'}{\partial z^2}=0,$$

where, as before, φ' and ρ' mean the same scalar functions on primed arguments ($\varphi'=\varphi(t',\mathbf{x}')$ and $\rho'=\rho(t',\mathbf{x}')$), and φ is a double-differentiable function.

We obtain:

$$\frac{\partial^2\varphi'}{\partial t^2}=\frac{1}{\sqrt{1-v^2}}\frac{\partial}{\partial t}\left(\frac{\partial\varphi'}{\partial t'}+\mathbf{v}\nabla'\varphi'\right)=\frac{1}{1-v^2}\left(\frac{\partial^2\varphi'}{\partial t'^2}+v_x^2\frac{\partial^2\varphi'}{\partial x'^2}+v_y^2\frac{\partial^2\varphi'}{\partial y'^2}+v_z^2\frac{\partial^2\varphi'}{\partial z'^2}\right)+$$
$$+\frac{2}{1-v^2}\left(v_xv_y\frac{\partial^2\varphi'}{\partial x'\partial y'}+v_xv_z\frac{\partial^2\varphi'}{\partial x'\partial z'}+v_yv_z\frac{\partial^2\varphi'}{\partial y'\partial z'}+v_x\frac{\partial^2\varphi'}{\partial t'\partial x'}+v_y\frac{\partial^2\varphi'}{\partial t'\partial y'}+v_z\frac{\partial^2\varphi'}{\partial t'\partial z'}\right)$$

$$\frac{\partial^2\varphi'}{\partial x^2}=\frac{1}{\sqrt{1-v^2}}\frac{\partial}{\partial x}\left(v_x\frac{\partial\varphi'}{\partial t'}+\frac{\partial\varphi'}{\partial x'}+iv_z\frac{\partial\varphi'}{\partial y'}-iv_y\frac{\partial\varphi'}{\partial z'}\right)=$$
$$=\frac{1}{1-v^2}\left(v_x^2\frac{\partial^2\varphi'}{\partial t'^2}+\frac{\partial^2\varphi'}{\partial x'^2}-v_z^2\frac{\partial^2\varphi'}{\partial y'^2}-v_y^2\frac{\partial^2\varphi'}{\partial z'^2}\right)+\frac{2}{1-v^2}\left(v_x\frac{\partial^2\varphi'}{\partial t'\partial x'}+v_yv_z\frac{\partial^2\varphi'}{\partial y'\partial z'}\right)+$$
$$+\frac{2i}{1-v^2}\left(v_xv_z\frac{\partial^2\varphi'}{\partial t'\partial y'}-v_xv_y\frac{\partial^2\varphi'}{\partial t'\partial z'}+v_z\frac{\partial^2\varphi'}{\partial x'\partial y'}-v_y\frac{\partial^2\varphi'}{\partial x'\partial z'}\right)$$

$$\frac{\partial^2\varphi'}{\partial y^2}=\frac{1}{1-v^2}\left(v_y^2\frac{\partial^2\varphi'}{\partial t'^2}+\frac{\partial^2\varphi'}{\partial y'^2}-v_z^2\frac{\partial^2\varphi'}{\partial x'^2}-v_x^2\frac{\partial^2\varphi'}{\partial z'^2}\right)+\frac{2}{1-v^2}\left(v_y\frac{\partial^2\varphi'}{\partial t'yx'}+v_xv_z\frac{\partial^2\varphi'}{\partial x'\partial z'}\right)+$$
$$+\frac{2i}{1-v^2}\left(v_xv_y\frac{\partial^2\varphi'}{\partial t'\partial z'}-v_zv_y\frac{\partial^2\varphi'}{\partial t'\partial x'}+v_x\frac{\partial^2\varphi'}{\partial z'\partial y'}-v_z\frac{\partial^2\varphi'}{\partial x'\partial y'}\right)$$

$$\frac{\partial^2\varphi'}{\partial z^2}=\frac{1}{1-v^2}\left(v_z^2\frac{\partial^2\varphi'}{\partial t'^2}+\frac{\partial^2\varphi'}{\partial z'^2}-v_y^2\frac{\partial^2\varphi'}{\partial x'^2}-v_x^2\frac{\partial^2\varphi'}{\partial y'^2}\right)+\frac{2}{1-v^2}\left(v_z\frac{\partial^2\varphi'}{\partial t'\partial z'}+v_xv_y\frac{\partial^2\varphi'}{\partial x'\partial y'}\right)+$$
$$+\frac{2i}{1-v^2}\left(v_zv_y\frac{\partial^2\varphi'}{\partial t'\partial x'}-v_xv_z\frac{\partial^2\varphi'}{\partial t'\partial y'}+v_y\frac{\partial^2\varphi'}{\partial z'\partial x'}-v_x\frac{\partial^2\varphi'}{\partial z'\partial y'}\right)$$

By subtracting the above equations from both sides, we get the result:

$$\frac{\partial^2\varphi'}{\partial t^2}-\frac{\partial^2\varphi'}{\partial x^2}-\frac{\partial^2\varphi'}{\partial y^2}-\frac{\partial^2\varphi'}{\partial z^2}=\frac{\partial^2\varphi'}{\partial t'^2}-\frac{\partial^2\varphi'}{\partial x'^2}-\frac{\partial^2\varphi'}{\partial y'^2}-\frac{\partial^2\varphi'}{\partial z'^2}$$

We act similarly, for each of the components of the vector function $\mathbf{A}(t,\mathbf{x})$. □

The reader may be surprised at first by the fact that although the transformation is described by a complex equation, we differentiate using the rules of the calculus of real functions. In our case, however, we not only can, but we must do this, because with constant transformation parameters (the assumption about inertial frames) the transformation of $\{X\}\overset{on}{\longrightarrow}\{X\}'$ should be mutually unique. The sets $\{X\} \subset R^4$ and $\{X'\}$ are contained in some subset of the set C^4, about which we know nothing yet. Nevertheless, we will conduct further research to determine the set that is the domain we are looking for, which is space-time. If a contradiction is encountered, the result will also have cognitive value, as it will confirm that there is no other model than the classical SR.

The equation (1.12) can be written using a slightly modified notation used in [1]- [3] as follows:

$$\begin{pmatrix} t' \\ \mathbf{x}' \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} t+\mathbf{v}\mathbf{x} \\ \mathbf{x}+\mathbf{v}t+i\mathbf{v}\times\mathbf{x} \end{pmatrix} \tag{1.13}$$

The expressions in parentheses are matrices that are paravectors, but this is discussed in the next chapter, which is the key to understanding our idea.

Chapter 2

Algebra of paravectors

Presented here is the basic information about paravectors, their algebraic structure and geometric properties. The structure of the ring makes paravectors similar to numbers, but after introducing the concept of an integrated product, paravectors obtain geometrical properties and become similar to vectors. The concepts of parallelism, perpendicularity and angles between paravectors look the same as their vector equivalents known from the Euclidean geometry. The concept of a paravector determinant, which is a complex number, is introduced. Due to the properties of the determinant, paravectors are divided into three groups. The matrix representation of the paravectors has been presented, thanks to which many properties become obvious and do not require any proofs. Particular attention is paid to showing the similarity of the geometrical properties of the paravectors to vectors in the Euclidean geometry.

The reader who has had contact with Clifford Algebras, W. Baylis' Physical Space Algebra, D. Hestenes' Geometric Algebra or Grassmann Algebra, should read Appendix 2, which explains the relationship between our formalism and some formalisms that are used by other authors.

2.1 Basic definitions

Definition 2.1.1. The term **paravector** means a pair consisting of a complex number (α) and a vector ($\boldsymbol{\beta}$) belonging to a three-dimensional complex space.

$$\Gamma =: \begin{bmatrix} \alpha \\ \boldsymbol{\beta} \end{bmatrix} = \begin{bmatrix} a + id \\ \mathbf{b} + i\mathbf{c} \end{bmatrix} \tag{2.1}$$

The number is called a **scalar**. Paravectors will be denoted with capital letters, e.g.: A, X, Γ. Greek letters will mean a complex size, and Roman letters - a real one. We will use column notation to separate the scalar part from the vector part of the paravector. Sometimes the scalar component of paravector Γ will be denoted with index 'S', and the vector component with 'V', i.e.: $\Gamma_S = \alpha$ and $\Gamma_V = \boldsymbol{\beta}$.

Definition 2.1.2. A **reversed** element of paravector (2.1) is the paravector

$$\Gamma^{-} =: \begin{bmatrix} a + id \\ -\mathbf{b} - i\mathbf{c} \end{bmatrix}$$

Definition 2.1.3. A **conjugated** element of paravector (2.1) is the paravector

$$\Gamma^{*} =: \begin{bmatrix} a - id \\ \mathbf{b} - i\mathbf{c} \end{bmatrix}$$

Definition 2.1.4. of **summation**:

$$\begin{bmatrix}\alpha_1\\ \boldsymbol{\beta}_1\end{bmatrix}+\begin{bmatrix}\alpha_2\\ \boldsymbol{\beta}_2\end{bmatrix}=:\begin{bmatrix}\alpha_1+\alpha_2\\ \boldsymbol{\beta}_1+\boldsymbol{\beta}_2\end{bmatrix}$$

Conclusion 2.1.1. A **neutral** element under addition (**null element**) is the paravector

$$\begin{bmatrix}0\\ \mathbf{0}\end{bmatrix}$$

Definition 2.1.5. An **opposite** element of paravector (2.1) with respect to the addition is

$$-\Gamma=:\begin{bmatrix}-\alpha\\ -\boldsymbol{\beta}\end{bmatrix}$$

Definition 2.1.6. of the **multiplication**:

$$\begin{bmatrix}\alpha_1\\ \boldsymbol{\beta}_1\end{bmatrix}\begin{bmatrix}\alpha_2\\ \boldsymbol{\beta}_2\end{bmatrix}=:\begin{bmatrix}\alpha_1\alpha_2+\boldsymbol{\beta}_1\boldsymbol{\beta}_2\\ \alpha_2\boldsymbol{\beta}_1+\alpha_1\boldsymbol{\beta}_2+i\boldsymbol{\beta}_1\times\boldsymbol{\beta}_2\end{bmatrix}$$

where $\boldsymbol{\beta}_1\boldsymbol{\beta}_2$ is a dot product, and $\boldsymbol{\beta}_1\times\boldsymbol{\beta}_2$ is a cross product of vectors.

Conclusion 2.1.2. A **neutral** element under multiplication is the paravector

$$\begin{bmatrix}1\\ \mathbf{0}\end{bmatrix}$$

Note: There is no difference whether we write number α or paravector $\begin{bmatrix}\alpha\\ \mathbf{0}\end{bmatrix}$.

Conclusion 2.1.3. The operation of multiplication is associative but not commutative, because the vector product is non-commutative.

Note: Paravectors have the same structure as complex quaternions (biquaternions), they only differ in their multiplication. In the vector part of the quaternions product, there is no imaginary one i at the vector product, so the biquaternion multiplication is not associative unlike the paravector multiplication.

Definition 2.1.7. We call an outcome of multiplication of any paravector Γ by the element conjugate Γ^* the **vigor of paravector**:

$$\mathrm{vig}\Gamma:=\Gamma\Gamma^*$$

Conclusion 2.1.4. To each paravector we can assign a vigor which is a real paravector and its scalar component is a positive number.

Proof.

$$\Gamma\Gamma^*=\begin{bmatrix}\alpha\\ \boldsymbol{\beta}\end{bmatrix}\begin{bmatrix}\alpha^*\\ \boldsymbol{\beta^*}\end{bmatrix}=\begin{bmatrix}\alpha\alpha^*+\boldsymbol{\beta}\boldsymbol{\beta}^*\\ \alpha\boldsymbol{\beta}^*+\alpha^*\boldsymbol{\beta}+i\boldsymbol{\beta}\times\boldsymbol{\beta}^*\end{bmatrix}=$$

$$=\begin{bmatrix}(a+id)(a-id)+(\mathbf{b}+i\mathbf{c})(\mathbf{b}-i\mathbf{c})\\ (a+id)(\mathbf{b}-i\mathbf{c})+(a-id)(\mathbf{b}+i\mathbf{c})+i(\mathbf{b}+i\mathbf{c})\times(\mathbf{b}-i\mathbf{c})\end{bmatrix}=\begin{bmatrix}a^2+b^2+c^2+d^2\\ +2(a\mathbf{b}+d\mathbf{c}+\mathbf{b}\times\mathbf{c})\end{bmatrix}$$

□

Definition 2.1.8. We call an outcome of multiplication of any paravector Γ by the reverse element Γ^- the **determinant of a paravector**

$$\det\Gamma:=\Gamma\Gamma^-=\Gamma^-\Gamma$$

Conclusion 2.1.5. Each paravector has a determinant which is a complex number.

Proof.

$$\Gamma\Gamma^{-} = \begin{bmatrix}\alpha\\ \boldsymbol{\beta}\end{bmatrix}\begin{bmatrix}\alpha\\ -\boldsymbol{\beta}\end{bmatrix} = \begin{bmatrix}\alpha^2-\beta^2\\ \alpha\boldsymbol{\beta}-\alpha\boldsymbol{\beta}-i\boldsymbol{\beta}\times\boldsymbol{\beta}\end{bmatrix} = \begin{bmatrix}(a+id)^2-(\mathbf{b}+i\mathbf{c})^2\\ 0\end{bmatrix} = \begin{bmatrix}a^2-b^2+c^2-d^2+2i(ad-\mathbf{bc})\\ 0\end{bmatrix}$$

□

Conclusion 2.1.6. The reversion and conjugation have the following properties:

	Reversion	Conjugation
1	$(\Gamma^{-})^{-}=\Gamma$	$(\Gamma^{*})^{*}=\Gamma$
2	$(\Gamma+\Psi)^{-}=\Gamma^{-}+\Psi^{-}$	$(\Gamma+\Psi)^{*}=\Gamma^{*}+\Psi^{*}$
3	$(\Gamma\Psi)^{-}=\Psi^{-}\Gamma^{-}$	$(\Gamma\Psi)^{*}=\Psi^{*}\Gamma^{*}$
4	$\Gamma\Gamma^{-}\in C$	$\Gamma\Gamma^{*}\in R_{+}\times R^3$
5	$(\Gamma^{-})^{*}=(\Gamma^{*})^{-}$	

Conclusion 2.1.7. A set of pravectors together with an operation of summation forms an Abelian group, and with multiplication forms a semigroup. Therefore, we can conclude that a set of paravectors together with operations of summation and multiplication gives a ring with multiplicative identity.

Definition 2.1.9. We call the paravector Γ **proper** if $\det\Gamma\in R_{+}\setminus\{0\}$ (the determinant is a positive real number).

Definition 2.1.10. We call the paravector Γ **singular** if $\det\Gamma=0$.

By definition of the determinant it follows:

Conclusion 2.1.8. Each proper or singular paravector (2.1) must fulfil the following condition:

$$ad=\mathbf{bc}$$

Note: It is advisable to remember the above equation, because it will be used in many proofs.

Conclusion 2.1.9. Let Γ_1,Γ_2 be paravectors, then the following statements are true:

- $\det(\Gamma_1\Gamma_2)=\det\Gamma_1\det\Gamma_2$
- $\det\Gamma^{-}=\det\Gamma$
- $\det\Gamma^{*}=(\det\Gamma)^{*}$

Definition 2.1.11. For each non-singular paravector Γ, there exists an **inverse** element under multiplication:

$$\Gamma^{-1}:=\frac{\Gamma^{-}}{\det\Gamma}$$

Conclusion 2.1.10. A set of non-singular paravectors together with multiplication is a non-commutative group.

Conclusion 2.1.11. A set of proper paravectors together with multiplication is a non-commutative group.

Definition 2.1.12. For each proper and/or singular paravector we define a **module** of paravector:

$$|\Gamma|:=\sqrt{\det\Gamma}$$

Conclusion 2.1.12. The module of paravector (proper or singular) satisfies the following conditions:

1. $|s\Gamma|=|s|\,|\Gamma|$ where $s\in R$

2. $|\Gamma_1||\Gamma_2| = |\Gamma_1\Gamma_2|$

Definition 2.1.13. We call the paravector Λ **orthogonal** if $\det\Lambda = 1$, or equivalently:

$$\Lambda := \frac{\Gamma}{\sqrt{\det\Gamma}}, \qquad \text{where } \Gamma \text{ is a proper paravector}$$

Directly from the above definition it follows:

Conclusion 2.1.13. If Λ is an orthogonal paravector, then

$$\Lambda^{-1} = \Lambda^{-}$$

Definition 2.1.14. We call the paravector Γ **special** if $\Gamma^{-} = \Gamma^{*}$, or equivalently:

$$\Gamma = \begin{bmatrix} a \\ i\mathbf{c} \end{bmatrix}, \qquad \text{where } a \in R, \text{ and } \mathbf{c} \in R^3$$

Definition 2.1.15. We call the paravector Γ **unitary** if $\Gamma\Gamma^{*} = 1$

To summarize the current knowledge about paravectors, we can say that very little is missing so that a set of paravectors with summation and multiplication operations becomes a field: multiplication of paravectors is not commutative, and the role of the null element under multiplication is played by singular paravectors. Multiplying any paravector by a singular one, we get a singular paravector. Note that although there are many null elements under multiplication, only one element is neutral with respect to summation.

Conclusion 2.1.14. The set of special paravectors together with operations summation and multiplication is a division ring.

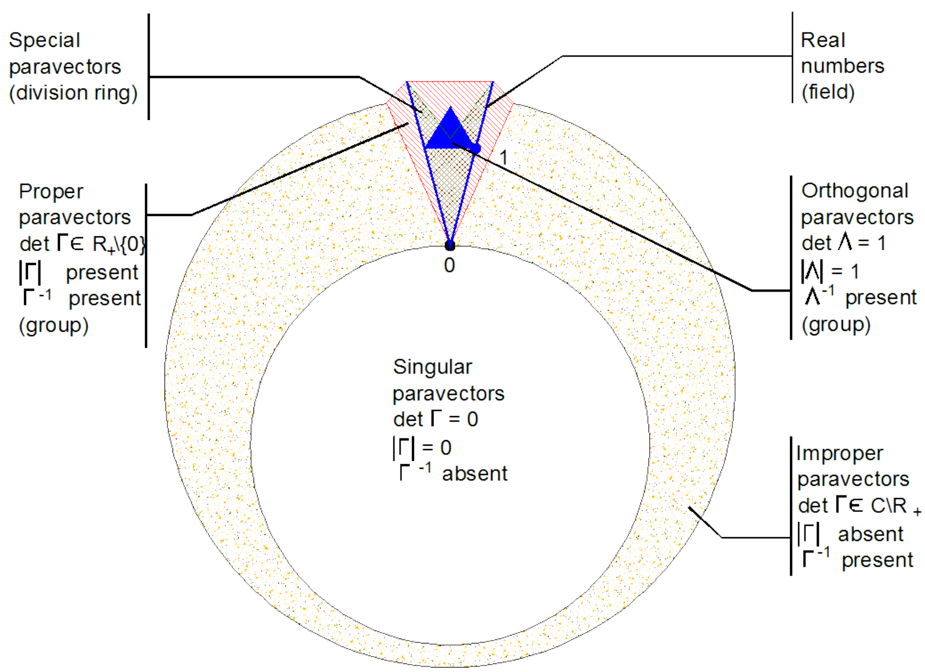

Figure 2.1: The ring of paravectors with some substructures

The black point means zero, and the blue point is one.

A set of proper paravectors together with multiplication is a noncommutative group.

A set of orthogonal paravectors together with multiplication is a subgroup of a proper paravectors group.

Note: To avoid misunderstandings, we specify the following names which we will use:

Γ^- - The paravector reverse of Γ

Γ^{-1} - The paravector inverse of Γ

2.2 Integrated product of two paravectors

Synge [18] defines the scalar product of complex quaternions as $\left(\Gamma_1\Gamma_2^- + \Gamma_2\Gamma_1^-\right)/2$, Hestenes [11] as a scalar part of the product of multivectors $\Gamma_1\Gamma_2$. Baylis [6] defines the scalar product of paravectors as $\left\langle p\bar{q}\right\rangle_S = (p\bar{q} + q\bar{p})/2$. Analysing the expression $\Gamma_1\Gamma_2^-$, we can see that it plays a similar but more universal role in the paravectors algebra as the dot product in vector space. Properties of its scalar component are the generalised properties of the scalar product of vectors, and its vector part corresponds to the cross product of vectors. The trouble is that there can be two different products which have the same properties (the second one is $\Gamma_1^-\Gamma_2$), so we define two integrated products:

Definition 2.2.1. A right integrated product of paravectors we call an expression

$$(\Gamma_1,\Gamma_2\rangle := \Gamma_1\Gamma_2^-$$

$$\text{or} \qquad (\Gamma_1,\Gamma_2\rangle = \Gamma_1\Gamma_2^- = \begin{bmatrix}\alpha_1\\ \boldsymbol{\beta}_1\end{bmatrix}\begin{bmatrix}\alpha_2\\ -\boldsymbol{\beta}_2\end{bmatrix} = \begin{bmatrix}\alpha_1\alpha_2 - \boldsymbol{\beta}_1\boldsymbol{\beta}_2\\ -\alpha_1\boldsymbol{\beta}_2 + \alpha_2\boldsymbol{\beta}_1 - i\boldsymbol{\beta}_1\times\boldsymbol{\beta}_2\end{bmatrix} \tag{2.2}$$

Definition 2.2.2. A left integrated product of paravectors we call an expression

$$\langle\Gamma_1,\Gamma_2) := \Gamma_1^-\Gamma_2$$

$$\text{or} \qquad \langle\Gamma_1,\Gamma_2) = \Gamma_1^-\Gamma_2 = \begin{bmatrix}\alpha_1\\ -\boldsymbol{\beta}_1\end{bmatrix}\begin{bmatrix}\alpha_2\\ \boldsymbol{\beta}_2\end{bmatrix} = \begin{bmatrix}\alpha_1\alpha_2 - \boldsymbol{\beta}_1\boldsymbol{\beta}_2\\ \alpha_1\boldsymbol{\beta}_2 - \alpha_2\boldsymbol{\beta}_1 - i\boldsymbol{\beta}_1\times\boldsymbol{\beta}_2\end{bmatrix} \tag{2.3}$$

In both cases, the scalar part is the same, therefore

Definition 2.2.3. The scalar component of an integrated product of paravectors Γ_1 and Γ_2 is called a **scalar product** of paravectors. The scalar product will be denoted $\langle\Gamma_1,\Gamma_2\rangle$, which is equivalent to $\langle\Gamma_1,\Gamma_2)_S$ or $(\Gamma_1,\Gamma_2\rangle_S$.

Definition 2.2.4. The vector component of an integrated product we call a **vector product** of paravectors.

It is necessary to distinguish the orientation of a vector product, like in the case of an integrated product. Therefore, we denote

Right vector product	$(\Gamma_1,\Gamma_2\} = -\alpha_1\boldsymbol{\beta}_2 + \alpha_2\boldsymbol{\beta}_1 - i\boldsymbol{\beta}_1\times\boldsymbol{\beta}_2$
Left vector product	$\{\Gamma_1,\Gamma_2) = \alpha_1\boldsymbol{\beta}_2 - \alpha_2\boldsymbol{\beta}_1 - i\boldsymbol{\beta}_1\times\boldsymbol{\beta}_2$

For further considerations it is of little importance if a product is right or left, so talking about an integrated product we will mean the right product. This could be the left product as well, but it is important to use only one product consistently. Hence the integrated product can be denoted as follows:

$$(\Gamma_1,\Gamma_2\rangle = \begin{bmatrix}\langle\Gamma_1,\Gamma_2\rangle\\ (\Gamma_1,\Gamma_2\}\end{bmatrix} \tag{2.4}$$

and

$$\det(\Gamma_1,\Gamma_2\rangle = \langle\Gamma_1,\Gamma_2\rangle^2 - (\Gamma_1,\Gamma_2\}^2 = \det\Gamma_1\det\Gamma_2 \tag{2.5}$$

Theorem 2.2.1. An integrated product of paravectors has the following properties:

	Left integrated product	Right integrated product
1	Integrated product is a paravector	
2	$\langle\Gamma_1+\Gamma_2,\Gamma_3)=\langle\Gamma_1,\Gamma_3)+\langle\Gamma_2,\Gamma_3)$	$(\Gamma_1+\Gamma_2,\Gamma_3\rangle=(\Gamma_1,\Gamma_3\rangle+(\Gamma_2,\Gamma_3\rangle$
3	$\langle\alpha\Gamma_1,\Gamma_2)=\alpha\langle\Gamma_1,\Gamma_2)=\langle\Gamma_1,\alpha\Gamma_2)$	$(\alpha\Gamma_1,\Gamma_2\rangle=\alpha(\Gamma_1,\Gamma_2\rangle=(\Gamma_1,\alpha\Gamma_2\rangle$
4	$\langle\Gamma_1,\Gamma_2)^-=\langle\Gamma_2,\Gamma_1)$	$(\Gamma_1,\Gamma_2\rangle^-=(\Gamma_2,\Gamma_1\rangle$
5	$\langle\Gamma,\Gamma)=(\Gamma,\Gamma\rangle=\det\Gamma\in C$	
6	$\det\langle\Gamma_1,\Gamma_2)=\det(\Gamma_1,\Gamma_2\rangle=\det\Gamma_1\det\Gamma_2$	

Conclusion 2.2.1. Let Γ_1,Γ_2 and Γ_3 be any paravectors, then the following table shows the properties of the scalar product of paravectors compared to the properties of the scalar product of vectors in Euclidean space:

	Scalar product of paravectors	Scalar product of vectors
1	$\langle\Gamma_1,\Gamma_2\rangle\in C$	$\langle\mathbf{x}_1,\mathbf{x}_2\rangle\in R$
2	$\langle\Gamma_1+\Gamma_2,\Gamma_3\rangle=\langle\Gamma_1,\Gamma_3\rangle+\langle\Gamma_2,\Gamma_3\rangle$	same
3	$\langle\alpha\Gamma_1,\Gamma_2\rangle=\alpha\langle\Gamma_1,\Gamma_2\rangle=\langle\Gamma_1,\alpha\Gamma_2\rangle$	same
4	$\langle\Gamma_1,\Gamma_2\rangle=\langle\Gamma_2,\Gamma_1\rangle$	same
5	$\langle\Gamma,\Gamma\rangle\in C$	$\langle\mathbf{x},\mathbf{x}\rangle\in R_+$
6	If $\langle\Gamma,\Gamma\rangle=0$, then Γ is singular	If $\langle\mathbf{x},\mathbf{x}\rangle=0$, then $\mathbf{x}=0$

Rows 5 and 6 of Table 2.1 show a fundamental difference between the definition of the scalar product of paravectors and known algebraic definitions. These properties made me consider giving another name to a scalar product. However, the traditional name remained, because:

1. the value of the product is a scalar,
2. the role which the scalar product plays in the Paravector Algebra corresponds completely to the role of the dot product of vectors in Euclidean Geometry,
3. for spatial proper/singular paravectors ($\Gamma_S=0$ and $\det\Gamma\geq 0$), the scalar product of paravectors becomes the scalar product of vectors.

Theorem 2.2.2. For any two paravectors it is true that $(\Gamma_1,\Gamma_2\rangle^*=\left\langle\Gamma_1^*,\Gamma_2^*\right)^-$

Proof. $(\Gamma_1,\Gamma_2\rangle^*=(\Gamma_1,\Gamma_2^-)^*=\Gamma_2^{-*}\Gamma_1^*=\left(\Gamma_1^{-*},\Gamma_2^*\right)^-=\left\langle\Gamma_1^*,\Gamma_2^*\right)^-$ □

2.3 Geometrical properties of paravectors

When studying an integrated product of paravectors, anyone can see a close similarity between paravectors and vectors in Euclidean space. Using an integrated product we introduce geometric concepts into the algebra of paravectors, which contributes to its intuitive understanding.

2.3.1 Parallelism and perpendicularity relations

Definition 2.3.1. Non-singular paravectors Γ_1 and Γ_2 are **parallel** ($\Gamma_1\parallel\Gamma_2$) if the vector product $(\Gamma_1,\Gamma_2\}=0$.

Theorem 2.3.1. Two non-singular paravectors Γ_1 and Γ_2 are parallel if and only if there exists a number $\lambda\neq 0$ that $\Gamma_1=\lambda\Gamma_2$.

Proof.

Parallelism means that $\Gamma_1\Gamma_2^-=\alpha$, where α is a complex number. We multiply this equation on the right by paravector Γ_2

$$\Gamma_1(\Gamma_2^-\Gamma_2)=\alpha\Gamma_2$$

Since Γ_2 is non-singular then $\Gamma_2^-\Gamma_2 = \beta$ is a non-zero complex number. Hence we get $\Gamma_1 = \lambda\Gamma_2$, where $\lambda = \alpha/\beta$.

□

The above theorem shows that

Conclusion 2.3.1. The parallelism satisfies the conditions of equivalence relation.

Definition 2.3.2. Two non-singular paravectors Γ_1 and Γ_2 are **perpendicular** ($\Gamma_1 \perp \Gamma_2$) if the scalar product $\langle\Gamma_1,\Gamma_2\rangle = 0$.

The perpendicularity of paravectors has the same properties as the perpendicularity of vectors in Euclidean space.

Theorem 2.3.2. For each non-singular paravector:

1. $\backsim(\Gamma \perp \Gamma)$
2. If $\Gamma_1 \perp \Gamma_2$, then $\Gamma_2 \perp \Gamma_1$
3. If $\Gamma_1 \perp \Gamma_2$ and $\Gamma_2 \parallel \Gamma_3$, then $\Gamma_1 \perp \Gamma_3$

Proof.

1. It follows by definition of perpendicularity.
2. It follows by the fact that the scalar product is symmetrical ($\langle\Gamma_1,\Gamma_2\rangle = \langle\Gamma_2,\Gamma_1\rangle$ and $\langle\Gamma_1,\Gamma_2\rangle = 0$).
3. $$\Gamma_1 \perp \Gamma_2 \iff \langle\Gamma_1,\Gamma_2\rangle = \begin{bmatrix} 0 \\ \boldsymbol{\omega} \end{bmatrix}$$
   $$\Gamma_2 \parallel \Gamma_3 \iff \langle\Gamma_2,\Gamma_3\rangle = \lambda,$$
   $$\text{hence } \langle\Gamma_1,\Gamma_3\rangle = \Gamma_1\Gamma_3^- = \Gamma_1\Gamma_2^{-1}\Gamma_2\Gamma_3^- = \tfrac{\lambda}{\det\Gamma_2}\Gamma_1\Gamma_2^- = \begin{bmatrix} 0 \\ \frac{\lambda\boldsymbol{\omega}}{\det\Gamma_2} \end{bmatrix}$$

□

Conclusion 2.3.2. For each paravector

1. the scalar component is perpendicular to the spatial one,
2. a paravector is perpendicular to itself if and only if it is singular,
3. paravectors mutually reversed (inversed) are not parallel,
4. orthogonal paravectors are parallel if and only if they are equal or opposite.

Proof.

1. Let $\Gamma = \begin{bmatrix} \alpha \\ \boldsymbol{\beta} \end{bmatrix}$ and $\Gamma_S = \begin{bmatrix} \alpha \\ \mathbf{0} \end{bmatrix}$ and $\Gamma_V = \begin{bmatrix} 0 \\ \boldsymbol{\beta} \end{bmatrix}$, then $\langle\Gamma_S,\Gamma_V\rangle = \alpha\cdot 0 - \mathbf{0}\cdot\boldsymbol{\beta} = 0$
2. It follows by assumption and definition of the singular paravector.
3. $\langle\Gamma,\Gamma^-\rangle = \Gamma\Gamma = \begin{bmatrix} \alpha \\ \boldsymbol{\beta} \end{bmatrix}\begin{bmatrix} \alpha \\ \boldsymbol{\beta} \end{bmatrix} = \begin{bmatrix} \alpha^2+\beta^2 \\ 2\alpha\boldsymbol{\beta} \end{bmatrix}$
4. Parallelism means that $\Lambda_1\Lambda_2^- = \lambda$, hence $\det\Lambda_1 \det\Lambda_2 = \lambda^2$. Paravectors are orthogonal hence $\lambda = \pm 1$. From the first equality it follows that $\Lambda_1 = \Lambda_2$ or $\Lambda_1 = -\Lambda_2$.

□

Conclusion 2.3.3. Let Γ_1 and Γ_2 be proper paravectors and $\Gamma_1 \parallel \Gamma_2$, then $\Gamma_1/|\Gamma_1| = \Gamma_2/|\Gamma_2|$.

Proof. From the definition of parallelism of proper paravectors it follows that their integrated product is a number $\lambda \neq 0$.

$$\frac{\Gamma_1}{|\Gamma_1|} = \frac{\Gamma_1\Gamma_2^-\Gamma_2}{\sqrt{\Gamma_1\Gamma_1^-}\det\Gamma_2} = \frac{\lambda}{\sqrt{\Gamma_1\Gamma_1^-\Gamma_2\Gamma_2^-}}\frac{\Gamma_2}{|\Gamma_2|} = \frac{\lambda}{\sqrt{\Gamma_1\lambda\Gamma_2^-}}\frac{\Gamma_2}{|\Gamma_2|} = \frac{\lambda}{\sqrt{\lambda^2}}\frac{\Gamma_2}{|\Gamma_2|} = \frac{\Gamma_2}{|\Gamma_2|}$$

□

Theorem 2.3.3. For any non-singular paravectors Γ_1 and Γ_2 it occurs that:

1. If $\Gamma_1 \perp \Gamma_2$, then $\Gamma_1^* \perp \Gamma_2^*$
2. If $\Gamma_1 \parallel \Gamma_2$, then $\Gamma_1^* \parallel \Gamma_2^*$
3. If $\Gamma_1 \parallel \Gamma_2$, then $\mathrm{vig}\Gamma_1 \parallel \mathrm{vig}\Gamma_2$

Proof.

1. $\langle\Gamma_1,\Gamma_2\rangle = 0 \implies \langle\Gamma_2,\Gamma_1\rangle = 0$, so $\left\langle\Gamma_1^*,\Gamma_2^*\right\rangle = \langle\Gamma_2,\Gamma_1\rangle^* = 0$
2. $(\begin{bmatrix}\alpha_1\\ \boldsymbol{\beta}_1\end{bmatrix}\begin{bmatrix}\alpha_2\\ -\boldsymbol{\beta}_2\end{bmatrix})_V = -\alpha_1\boldsymbol{\beta}_2 + \alpha_2\boldsymbol{\beta}_1 - i\boldsymbol{\beta}_1\times\boldsymbol{\beta}_2 = \mathbf{0}$

   Since complex vectors are governed by the same laws as real ones, so it must be:

   $\alpha_2\boldsymbol{\beta}_1 - \alpha_1\boldsymbol{\beta}_2 = \mathbf{0}$ and $\boldsymbol{\beta}_1\times\boldsymbol{\beta}_2 = \mathbf{0}$

   On the other hand, we have $\left\langle\Gamma_1^*,\Gamma_2^*\right\rangle = \langle\Gamma_2,\Gamma_1\rangle^* = (\begin{bmatrix}\alpha_2\\ -\boldsymbol{\beta}_2\end{bmatrix}\begin{bmatrix}\alpha_1\\ \boldsymbol{\beta}_1\end{bmatrix})^*$

   hence, under the assumption

   $(\begin{bmatrix}\alpha_2\\ -\boldsymbol{\beta}_2\end{bmatrix}\begin{bmatrix}\alpha_1\\ \boldsymbol{\beta}_1\end{bmatrix})_V = \alpha_2\boldsymbol{\beta}_1 - \alpha_1\boldsymbol{\beta}_2 - i\boldsymbol{\beta}_2\times\boldsymbol{\beta}_1 = \mathbf{0}$
3. $\left\langle\Gamma_1\Gamma_1^*,\Gamma_2\Gamma_2^*\right\rangle = \Gamma_1\Gamma_1^*(\Gamma_2\Gamma_2^*)^- = \Gamma_1\Gamma_1^*\Gamma_2^{*-}\Gamma_2^- = \Gamma_1(\Gamma_2^-\Gamma_1)^*\Gamma_2^-$

   Under the assumption, the product in parentheses is a number, so we can move it in front of the product

   $\lambda^*\langle\Gamma_1,\Gamma_2\rangle = \lambda^*\lambda$

□

Theorem 2.3.4. For any paravectors Γ_1 and Γ_2 polarization identity occurs:

$$\det(\Gamma_1+\Gamma_2) = \det\Gamma_1 + 2\langle\Gamma_1,\Gamma_2\rangle + \det\Gamma_2$$

Proof.

$$\det(\Gamma_1+\Gamma_2) = (\Gamma_1+\Gamma_2)(\Gamma_1+\Gamma_2)^- = \Gamma_1\Gamma_1^- + \Gamma_1\Gamma_2^- + \Gamma_2\Gamma_1^- + \Gamma_2\Gamma_2^- =$$
$$= \det\Gamma_1 + \langle\Gamma_1,\Gamma_2\rangle + \langle\Gamma_2,\Gamma_1\rangle + \det\Gamma_2 = \det\Gamma_1 + 2\langle\Gamma_1,\Gamma_2\rangle + \det\Gamma_2$$

□

Conclusion 2.3.4. Let paravectors Γ_1 and Γ_2 be perpendicular, then the determinant of these paravectors sum equals the sum of their determinants:

$$\det(\Gamma_1+\Gamma_2) = \det\Gamma_1 + \det\Gamma_2$$

This conclusion complies with the Pythagorean theorem in Euclidean geometry.

Conclusion 2.3.5. For any paravectors Γ_1 and Γ_2 the parallelogram law occurs:

$$\det(\Gamma_1+\Gamma_2)+\det(\Gamma_1-\Gamma_2)=2\det\Gamma_1+2\det\Gamma_2$$

The parallelogram identity gives the structure we are building the character of a metric space.

Definition 2.3.3. Two paravectors $\begin{bmatrix}\alpha_1\\ \boldsymbol{\beta}_1\end{bmatrix}$ and $\begin{bmatrix}\alpha_2\\ \boldsymbol{\beta}_2\end{bmatrix}$ are **spatially parallel** if $\boldsymbol{\beta}_1\times\boldsymbol{\beta}_2=\mathbf{0}$

Theorem 2.3.5. If two non-singular paravectors are parallel, they are also spatially parallel.

Proof.

$\Gamma_1 \parallel \Gamma_2 \iff \alpha_1\boldsymbol{\beta}_2-\alpha_2\boldsymbol{\beta}_1-i\boldsymbol{\beta}_1\times\boldsymbol{\beta}_2=0$

First we multiply the above equation by $\alpha_1\boldsymbol{\beta}_2$, and then by $\alpha_2\boldsymbol{\beta}_1$. Hence we get two equations:

$\left(\alpha_1\boldsymbol{\beta}_2\right)^2-\alpha_1\alpha_2\boldsymbol{\beta}_1\boldsymbol{\beta}_2=0$

$\alpha_1\alpha_2\boldsymbol{\beta}_1\boldsymbol{\beta}_2-\left(\alpha_2\boldsymbol{\beta}_1\right)^2=0$

The difference of the above equations yields $\left(\alpha_1\boldsymbol{\beta}_2-\alpha_2\boldsymbol{\beta}_1\right)^2=0$, which is true when $\alpha_1\boldsymbol{\beta}_2=\alpha_2\boldsymbol{\beta}_1$.

Hence it follows that $\boldsymbol{\beta}_1\times\boldsymbol{\beta}_2=0$

□

As a consequence, we can say that

Conclusion 2.3.6. Spatial parallelism is an equivalence relation.

Conclusion 2.3.7. The spatial parallelism of paravectors is weaker than parallelism, i.e.: If two paravectors are parallel, they must be spatially parallel, too, but in the opposite direction, the implication does not have to occur.

Definition 2.3.4. We call two singular non-zero paravectors Γ_1 and Γ_2 **singularly parallel** if

$$(\Gamma_1,\Gamma_2\rangle=0$$

The following conclusions are easy to prove:

Conclusion 2.3.8. .

1. If two paravectors are non-zero and singularly parallel, then they are singular.
2. Singular parallelism is an equivalence relation.
3. The set of singularly parallel paravectors is an ideal of the ring of paravectors.

2.3.2 Angles

Definition 2.3.5. The term **right angle** between two proper paravectors, denoted by $\angle(\Gamma_1,\Gamma_2\rangle$, is used for the paravector:

$$\Phi=\angle(\Gamma_1,\Gamma_2\rangle:=\frac{(\Gamma_1,\Gamma_2\rangle}{|\Gamma_1||\Gamma_2|}:=\begin{bmatrix}\mathrm{cosi}\Phi\\ \mathbf{dex}\Phi\end{bmatrix},$$

where we call the scalar component **cosinis**, and the vector component – **dextis** of angle Φ.

Definition 2.3.6. The term **left angle** between two proper paravectors is used for the paravector:

$$\Phi = \angle\langle\Gamma_1,\Gamma_2) := \frac{\langle\Gamma_1,\Gamma_2)}{|\Gamma_1||\Gamma_2|} := \begin{bmatrix} \text{cosi}\Phi \\ \mathbf{sini}\Phi \end{bmatrix},$$

where we call the scalar component **cosinis**, and the vector component – **sinis** of angle Φ.

These names are derived from Latin. *Sinistram* means left, and *dextram* means right.

Note - Please note that the angle components are not trigonometric (hyperbolic) functions - these are just names, given because angle components have the same properties as the well-known trigonometric functions which makes them easier to imagine and to remember.

In paravector space (which has not been defined yet!) as well as in Euclidean space, we should identify a positive orientation. We don't do this in this chapter, not to impose any restrictions. It seems that it will sooner or later be necessary, but not now.

Definition 2.3.7. We call an angle $\Phi = \Phi_1\Phi_2$ the **composition of** (left) **angles** Φ_1 and Φ_2.

$$\begin{aligned}\Phi_1\Phi_2 &= \begin{bmatrix} \text{cosi}\Phi_1 \\ \mathbf{sini}\Phi_1 \end{bmatrix}\begin{bmatrix} \text{cosi}\Phi_2 \\ \mathbf{sini}\Phi_2 \end{bmatrix} = \\ &= \begin{bmatrix} \text{cosi}\Phi_1\text{cosi}\Phi_2 + \mathbf{sini}\Phi_1\mathbf{sini}\Phi_2 \\ \text{cosi}\Phi_1\mathbf{sini}\Phi_2 + \text{cosi}\Phi_2\mathbf{sini}\Phi_1 + i\,\mathbf{sini}\Phi_1 \times \mathbf{sini}\Phi_2 \end{bmatrix} = \\ &= \begin{bmatrix} \text{cosi}(\Phi_1\Phi_2) \\ \mathbf{sini}(\Phi_1\Phi_2) \end{bmatrix} = \begin{bmatrix} \text{cosi}\Phi \\ \mathbf{sini}\Phi \end{bmatrix} = \Phi\end{aligned}$$

We can write an analogous composition for the right angles.

As a consequence, we can see further analogy with Euclidean trigonometry:

Conclusion 2.3.9. The explement of an angle is:

$$\begin{aligned} &\angle\langle\Gamma_1,\Gamma_2)^- = \angle\langle\Gamma_2,\Gamma_1) && \text{for the left angle} \\ \text{or} \quad &\angle(\Gamma_1,\Gamma_2\rangle^- = \angle(\Gamma_2,\Gamma_1\rangle && \text{for the right angle,} \end{aligned} \tag{2.6}$$

which gives

- $\text{cosi}\Phi^- = \text{cosi}\Phi$
- $\mathbf{sini}\Phi^- = -\mathbf{sini}\Phi$
- $\mathbf{dex}\Phi^- = -\mathbf{dex}\Phi$

The left and right angles have an opposite orientation in space and they are not explementary.

$$\angle\langle\Gamma_1,\Gamma_2) = \frac{\Gamma_1^-\Gamma_2}{|\Gamma_1||\Gamma_2|}, \qquad \angle(\Gamma_1,\Gamma_2\rangle = \frac{\Gamma_1\Gamma_2^-}{|\Gamma_1||\Gamma_2|},$$

hence the composition of these angles is:

$$\angle\langle\Gamma_1,\Gamma_2)\angle(\Gamma_1,\Gamma_2\rangle = \frac{\Gamma_1^-\Gamma_2\Gamma_1\Gamma_2^-}{\det(\Gamma_1\Gamma_2)} \neq 1$$

The explement of the left angle $\angle\langle\Gamma_1,\Gamma_2)$ is the left angle $\angle\langle\Gamma_2,\Gamma_1)$, and the same occurs for the right angle.

In Table 2.3.1 there are shown the properties of left angle components in order to simply and intuitively justify the names chosen for them. The right angle has analogous formulas.

Tab. 2.3.1 General recurrence formulas for cosi and **sini** components of the left angle.

Φ - Orthogonal paravector	$\mathrm{cosi}\Phi \in C$ and $\mathbf{sini}\Phi \in C^3$
$\det\Phi$	$\mathrm{cosi}^2\Phi$ - $\mathrm{sini}^2\Phi = 1$
Doubling the angle	$\mathrm{cosi}(\Phi^2) = \mathrm{cosi}^2\Phi + \mathrm{sini}^2\Phi$ $\mathbf{sini}(\Phi^2) = 2\mathrm{cosi}\Phi\mathbf{sini}\Phi$
Angles composition	$\mathrm{cosi}(\Phi_1\Phi_2) = \mathrm{cosi}\Phi_1\mathrm{cosi}\Phi_2 + \mathbf{sini}\Phi_1\mathbf{sini}\Phi_2$ $\mathbf{sini}(\Phi_1\Phi_2) = \mathrm{cosi}\Phi_1\mathbf{sini}\Phi_2 + \mathrm{cosi}\Phi_2\mathbf{sini}\Phi_1 + i\mathbf{sini}\Phi_1 \times \mathbf{sini}\Phi_2$
Angle explement	$\mathrm{cosi}\Phi^- = \mathrm{cosi}\Phi$ $\mathbf{sini}\Phi^- = -\mathbf{sini}\Phi$

Conclusion 2.3.10. If $Im(\mathrm{cosi}\Phi) = 0$ and $Im(\mathbf{sini}\Phi) = \mathbf{0}$, then the nature of angle Φ is hyperbolic. If $Im(\mathrm{cosi}\Phi) = 0$ and $Re(\mathbf{sini}\Phi) = \mathbf{0}$, then the nature of the angle is trigonometric, which is shown by the determinant of this angle.

2.3.3 Similarity and rotation

Definition 2.3.8. Two paravectors Γ_1 and Γ_2 are **similar** if there exists a non-singular paravector Φ, such that $\Phi\Gamma_1 = \Gamma_2\Phi$. We call the paravector Φ an **axis of similarity**.

The similarity can be shown in another way as $\Gamma' = \Phi^{-1}\Gamma\Phi$, which is how many authors define the rotation. We would like to unequivocally associate the rotation with the angle between the rotated paravector and its image after the turning. If the paravector Φ is improper, then it's impossible to determine this angle. For this reason, it was necessary to clarify the definition of rotation.

Definition 2.3.9. A similarity whose axis is a proper paravector is called the **rotation**.

We also need to orient the rotation in accordance with the angles, therefore we will distinguish left and right rotations. The left rotation will take the form of $\Gamma' = \Phi^{-1}\Gamma\Phi$, and the right: $\Gamma' = \Phi\Gamma\Phi^{-1}$. Since the paravector Φ is proper, then the module $|\Phi|$ exists. So, we can exhibit the rotation in the following form:

$$\Gamma' = \Lambda^-\Gamma\Lambda, \tag{2.7}$$

where paravector $\Lambda = \Phi/|\Phi|$ is the axis of rotation and also determines the value of rotation. The axis of rotation in space is determined by the spatial component of paravector Λ.

The properties of rotations in the space built by us are so general that we cannot restrict them to the rotation in Euclidean sense only. In cases where $\mathrm{cosi}\Lambda$ is a real number, and $\mathbf{sini}\Lambda$ is an imaginary vector (or angle Λ is a special paravector), we deal with an Euclidean rotation. For the paravector $\Lambda = \begin{bmatrix} \cos\varphi \\ i\mathbf{n}\sin\varphi \end{bmatrix}$ we have spatial rotation by 2φ angle about the axis defined by vector $\mathbf{n}$. Here φ is a traditional angle, and sine/cosine are trigonometric functions. We can see that the paravector angle (despite the similarities) is something other than the Euclidean angle by examining the right angle between Γ and its rotated image $\Gamma' = \Lambda^-\Gamma\Lambda$. The right angle between paravectors Γ and Γ' can be denoted as

$$\angle(\Gamma,\Gamma'\rangle = \frac{\Gamma(\Lambda^-\Gamma\Lambda)^-}{\det\Gamma} = \frac{(\Gamma\Lambda^-)(\Gamma^-\Lambda)}{\det\Gamma} = \angle(\Gamma,\Lambda\rangle\angle\langle\Gamma,\Lambda) \tag{2.8}$$

Conclusion 2.3.11. In the complex space the angle between any proper paravector and its image after turning is a combination of the right and left angles between this paravector and the axis of rotation paravector.

In the case when the axis of rotation is a real paravector, W. Baylis says that such a rotation is a Lorentz transformation of the electric field.

Below we show the obvious properties of similarity:

Theorem 2.3.6. For each similar paravector:

1. Similar paravectors must have the same scalar components. In other words, similarity is a spatial relationship.

2. If any paravectors are spatially parallel to the axis of similarity and they are similar, then they are identical.

3. Parallel axes represent the same similarity.

4. Similarity is an equivalence relation.

Proof.

1. $\Phi^{-1}\Gamma\Phi = \frac{1}{\alpha^2-\beta^2}\begin{bmatrix}\alpha\\ \boldsymbol{\beta}\end{bmatrix}\begin{bmatrix}\tau\\ \boldsymbol{\varpi}\end{bmatrix}\begin{bmatrix}\alpha\\ -\boldsymbol{\beta}\end{bmatrix} = \frac{1}{\alpha^2-\beta^2}\begin{bmatrix}(\alpha^2-\beta^2)\tau\\ (\alpha^2-\beta^2)\boldsymbol{\varpi}-2\left[i\boldsymbol{\beta}\times\boldsymbol{\varpi}+\boldsymbol{\beta}\times(\boldsymbol{\beta}\times\boldsymbol{\varpi})\right]\end{bmatrix} =$

$$= \begin{bmatrix}\tau\\ \boldsymbol{\varpi}-\frac{2}{(\alpha^2-\beta^2)}\left[i\boldsymbol{\beta}\times\boldsymbol{\varpi}+\boldsymbol{\beta}\times(\boldsymbol{\beta}\times\boldsymbol{\varpi})\right]\end{bmatrix}$$

2. It follows from the previous property because $\boldsymbol{\beta}\times\boldsymbol{\varpi}=\mathbf{0}$

3. Let $\Phi_1 = \lambda\Phi_2$
   $\frac{1}{\det\Phi_1}\Phi_1^{-}\Gamma\Phi_1 = \frac{1}{\lambda^2\det\Phi_2}\lambda^2\Phi_2^{-}\Gamma\Phi_2 = \frac{1}{\det\Phi_2}\Phi_2^{-}\Gamma\Phi_2$

4. The proof is simple, so we leave it to the reader.

□

By the theorem 2.3.6 we draw the following conclusions for rotations:

Conclusion 2.3.12. For each rotated paravector:

1. Rotation does not change the scalar component. In other words, rotation is a spatial relationship.

2. Rotation does not change the paravector which is spatially parallel to the axis of this rotation.

3. Parallel axes represent the same rotation.

Using paravectors, we can easily introduce Euler angles, or the composition of angles on the planes with normal $\mathbf{n}_1$ and $\mathbf{n}_2$

$$\begin{bmatrix}\cos\alpha\\ i\mathbf{n}\sin\alpha\end{bmatrix} = \begin{bmatrix}\cos\alpha_1\\ i\mathbf{n}_1\sin\alpha_1\end{bmatrix}\begin{bmatrix}\cos\alpha_2\\ i\mathbf{n}_2\sin\alpha_2\end{bmatrix} \qquad (|\mathbf{n}_i|=1) \tag{2.9}$$

Note: Please note that the above angles φ and functions have a trigonometric sense in real Euclidean space!

Any vector in the space can be denoted as a paravector $\begin{bmatrix}0\\ i\mathbf{w}\end{bmatrix}$ (imaginary vector, because if it were real, then the paravector would not have a module). The angle between two vectors $\mathbf{w}_1$ and $\mathbf{w}_2$, is

$$\frac{1}{|\mathbf{w}_1|}\begin{bmatrix}0\\ -i\mathbf{w}_1\end{bmatrix}\frac{1}{|\mathbf{w}_2|}\begin{bmatrix}0\\ i\mathbf{w}_2\end{bmatrix} = \begin{bmatrix}\frac{\mathbf{w}_1\mathbf{w}_2}{|\mathbf{w}_1||\mathbf{w}_2|}\\ i\frac{\mathbf{w}_1\times\mathbf{w}_2}{|\mathbf{w}_1||\mathbf{w}_2|}\end{bmatrix} = \begin{bmatrix}\cos x\\ i\mathbf{n}\sin x\end{bmatrix} \tag{2.10}$$

In this case we have

$$\mathrm{cosi}\angle\langle\begin{bmatrix}0\\ i\mathbf{w}_1\end{bmatrix},\begin{bmatrix}0\\ i\mathbf{w}_2\end{bmatrix}\rangle = \cos x \qquad \text{and} \qquad \mathbf{sini}\angle\langle\begin{bmatrix}0\\ i\mathbf{w}_1\end{bmatrix},\begin{bmatrix}0\\ i\mathbf{w}_2\end{bmatrix}\rangle = i\mathbf{n}\sin x,$$

where x is an Euclidean angle between vectors $\mathbf{w}_1$ and $\mathbf{w}_2$.

Definition 2.3.10. Mirror symmetry with respect to the plane of normal **n** is called conversion such that:

$$\begin{bmatrix} 0 \\ i\mathbf{n} \end{bmatrix}\begin{bmatrix} a \\ \mathbf{w} \end{bmatrix}\begin{bmatrix} 0 \\ i\mathbf{n} \end{bmatrix} \tag{2.11}$$

Let **n** be a vector normal to plane N. Take any non-zero vector **w**. This vector can be decomposed into a vector parallel to **n** and perpendicular (orthogonal projection of vector **w** on plane N), respectively:

$$\mathbf{n}(\mathbf{wn}) \qquad \text{and} \qquad \mathbf{w}-\mathbf{n}(\mathbf{wn}) = \mathbf{n}^2\mathbf{w}-\mathbf{n}(\mathbf{wn}) = (\mathbf{n}\times\mathbf{w})\times\mathbf{n},$$

hence vector $\quad \mathbf{w} = \mathbf{n}(\mathbf{wn}) + (\mathbf{n}\times\mathbf{w})\times\mathbf{n}$

Mirror symmetry changes the sign of the component perpendicular to the plane of symmetry (parallel to **n**), but when we take a non-zero scalar, we get:

$$\begin{bmatrix} 0 \\ i\mathbf{n} \end{bmatrix}\begin{bmatrix} a \\ \mathbf{w} \end{bmatrix}\begin{bmatrix} 0 \\ i\mathbf{n} \end{bmatrix} = \begin{bmatrix} -a \\ -\mathbf{n}(\mathbf{wn}) + (\mathbf{n}\times\mathbf{w})\times\mathbf{n} \end{bmatrix}$$

Hence we see that mirror symmetry is not similarity in the meaning of definition 2.3.8, because it changes the sign of a scalar.

Mirror symmetry can be generalized to complex paravectors:

$$\frac{1}{-\omega^2}\begin{bmatrix} 0 \\ \boldsymbol{\omega} \end{bmatrix}\begin{bmatrix} \alpha \\ \boldsymbol{\beta} \end{bmatrix}\begin{bmatrix} 0 \\ \boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} -\alpha \\ \omega^{-2}[-\boldsymbol{\omega}(\boldsymbol{\beta}\boldsymbol{\omega}) + (\boldsymbol{\omega}\times\boldsymbol{\beta})\times\boldsymbol{\omega}] \end{bmatrix} \tag{2.12}$$

As was to be expected, rotation can be presented in the form of a composition of two mirror symmetries. A paravector parallel to both planes of symmetry sets the axis of rotation.

$$\frac{1}{\omega_1^2\omega_2^2}\begin{bmatrix} 0 \\ \boldsymbol{\omega}_2 \end{bmatrix}\begin{bmatrix} 0 \\ \boldsymbol{\omega}_1 \end{bmatrix}\begin{bmatrix} \alpha \\ \boldsymbol{\beta} \end{bmatrix}\begin{bmatrix} 0 \\ \boldsymbol{\omega}_1 \end{bmatrix}\begin{bmatrix} 0 \\ \boldsymbol{\omega}_2 \end{bmatrix} = \frac{1}{(\boldsymbol{\omega}_1\boldsymbol{\omega}_2)^2 + (\boldsymbol{\omega}_1\times\boldsymbol{\omega}_2)^2}\begin{bmatrix} \boldsymbol{\omega}_1\boldsymbol{\omega}_2 \\ -i\boldsymbol{\omega}_1\times\boldsymbol{\omega}_2 \end{bmatrix}\begin{bmatrix} \alpha \\ \boldsymbol{\beta} \end{bmatrix}\begin{bmatrix} \boldsymbol{\omega}_1\boldsymbol{\omega}_2 \\ i\boldsymbol{\omega}_1\times\boldsymbol{\omega}_2 \end{bmatrix}$$

Axial symmetry is nothing else but a straight angle rotation around the $\boldsymbol{\omega}$ vector

$$\frac{1}{\omega^2}\begin{bmatrix} 0 \\ -i\boldsymbol{\omega} \end{bmatrix}\begin{bmatrix} \alpha \\ \boldsymbol{\beta} \end{bmatrix}\begin{bmatrix} 0 \\ i\boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} \alpha \\ \omega^{-2}[\boldsymbol{\omega}(\boldsymbol{\beta}\boldsymbol{\omega}) - (\boldsymbol{\omega}\times\boldsymbol{\beta})\times\boldsymbol{\omega}] \end{bmatrix} \tag{2.13}$$

From the above discussion we can see that paravectors, despite their complex construction and lack of vector metrics, have geometrical features of vectors, so that they become imaginable.

2.4 Matrix representation of paravectors

Based on the definition of paravectors multiplication (2.1.6), the equation

$$X_2 = \Gamma X_1 = \begin{bmatrix} \alpha_2 \\ \boldsymbol{\beta}_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \boldsymbol{\beta} \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \boldsymbol{\beta}_1 \end{bmatrix} \tag{2.14}$$

can be described as

$$\begin{bmatrix} \alpha_2 \\ \boldsymbol{\beta}_2 \end{bmatrix} = \begin{bmatrix} \alpha\alpha_1 + \boldsymbol{\beta}\boldsymbol{\beta}_1 \\ \alpha\boldsymbol{\beta}_1 + \alpha_1\boldsymbol{\beta} + i\boldsymbol{\beta}\times\boldsymbol{\beta}_1 \end{bmatrix} \tag{2.15}$$

The above equation is a system of linear equations, which can be exhibited in a matrix form

$$\begin{bmatrix} \alpha_2 \\ \beta_{2x} \\ \beta_{2y} \\ \beta_{2z} \end{bmatrix} = \begin{bmatrix} \alpha & \beta_x & \beta_y & \beta_z \\ \beta_x & \alpha & -i\beta_z & i\beta_y \\ \beta_y & i\beta_z & \alpha & -i\beta_x \\ \beta_z & -i\beta_y & i\beta_x & \alpha \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \beta_{1x} \\ \beta_{1y} \\ \beta_{1z} \end{bmatrix} \tag{2.16}$$

Anyone can see that the above equation is equivalent to

$$\begin{bmatrix} \alpha_2 & \beta_{2x} & \beta_{2y} & \beta_{2z} \\ \beta_{2x} & \alpha_2 & -i\beta_{2z} & i\beta_{2y} \\ \beta_{2y} & i\beta_{2z} & \alpha_2 & -i\beta_{2x} \\ \beta_{2z} & -i\beta_{2y} & i\beta_{2x} & \alpha_2 \end{bmatrix} =$$

$$= \begin{bmatrix} \alpha & \beta_x & \beta_y & \beta_z \\ \beta_x & \alpha & -i\beta_z & i\beta_y \\ \beta_y & i\beta_z & \alpha & -i\beta_x \\ \beta_z & -i\beta_y & i\beta_x & \alpha \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_{1x} & \beta_{1y} & \beta_{1z} \\ \beta_{1x} & \alpha_1 & -i\beta_{1z} & i\beta_{1y} \\ \beta_{1y} & i\beta_{1z} & \alpha_1 & -i\beta_{1x} \\ \beta_{1z} & -i\beta_{1y} & i\beta_{1x} & \alpha_1 \end{bmatrix} \tag{2.17}$$

Therefore, each paravector $\Gamma = \begin{bmatrix} \alpha \\ \boldsymbol{\beta} \end{bmatrix}$ is equivalent to a matrix

$$\begin{bmatrix} \alpha & \beta_x & \beta_y & \beta_z \\ \beta_x & \alpha & -i\beta_z & i\beta_y \\ \beta_y & i\beta_z & \alpha & -i\beta_x \\ \beta_z & -i\beta_y & i\beta_x & \alpha \end{bmatrix} \tag{2.18}$$

The determinant of the above matrix is $\left(\alpha^2 - \beta^2\right)^2 = (\Gamma\Gamma^-)^2 = (\det\Gamma)^2$, hence paravector $\Gamma^{-1} = \Gamma^-/\det\Gamma$ corresponds to the matrix inverse to the above one. Since the inverse paravector should correspond to the transposed matrix, we were considering naming it a transposed paravector. But this transposition is not complete because the first row and the first column are not subject to transposition:

$$\Gamma = \begin{bmatrix} \alpha & \beta_x & \beta_y & \beta_z \\ \beta_x & \alpha & -i\beta_z & i\beta_y \\ \beta_y & i\beta_z & \alpha & -i\beta_x \\ \beta_z & -i\beta_y & i\beta_x & \alpha \end{bmatrix} \qquad \Gamma^- = \begin{bmatrix} \alpha & -\beta_x & -\beta_y & -\beta_z \\ -\beta_x & \alpha & i\beta_z & -i\beta_y \\ -\beta_y & -i\beta_z & \alpha & i\beta_x \\ -\beta_z & i\beta_y & -i\beta_x & \alpha \end{bmatrix} \tag{2.19}$$

The geometric meaning of this paravector corresponds to the reverse direction in space, so it was decided to leave the name: reverse paravector.

Conclusion 2.4.1. Some of the matrix counterparts:

1. A singular paravector corresponds to a singular matrix.
2. A conjugate paravector corresponds to a Hermitian conjugate matrix.

Proof. of the 2nd point.

$$\begin{bmatrix} \alpha^* & \beta_x^* & \beta_y^* & \beta_z^* \\ \beta_x^* & \alpha^* & -i\beta_z^* & i\beta_y^* \\ \beta_y^* & i\beta_z^* & \alpha^* & -i\beta_x^* \\ \beta_z^* & -i\beta_y^* & i\beta_x^* & \alpha^* \end{bmatrix} = \begin{bmatrix} a-id & b_x-ic_x & b_y-ic_y & b_z-ic_z \\ b_x-ic_x & a-id & -ib_z-c_z & ib_y+c_y \\ b_y-ic_y & ib_z+c_z & a-id & -ib_x-c_x \\ b_z-ic_z & -ib_y-c_y & ib_x+c_x & a-id \end{bmatrix} =$$

$$= \begin{bmatrix} a+id & b_x+ic_x & b_y+ic_y & b_z+ic_z \\ b_x+ic_x & a+id & ib_z-c_z & -ib_y+c_y \\ b_y+ic_y & -ib_z+c_z & a+id & ib_x-c_x \\ b_z+ic_z & ib_y-c_y & -ib_x+c_x & a+id \end{bmatrix}^* =$$

$$= \begin{bmatrix} a+id & b_x+ic_x & b_y+ic_y & b_z+ic_z \\ b_x+ic_x & a+id & i(b_z+ic_z) & -i(b_y+ic_y) \\ b_y+ic_y & -i(b_z+ic_z) & a+id & i(b_x+ic_x) \\ b_z+ic_z & i(b_y+ic_y) & -i(b_x+ic_x) & a+id \end{bmatrix}^* =$$

$$= \begin{bmatrix} \alpha & \beta_x & \beta_y & \beta_z \\ \beta_x & \alpha & -i\beta_z & i\beta_y \\ \beta_y & i\beta_z & \alpha & -i\beta_x \\ \beta_z & -i\beta_y & i\beta_x & \alpha \end{bmatrix}^{*T}$$

□

2.5 Orthogonal transformations

Definition 2.5.1. A linear transformation represented by a non-singular paravector will be called a **paravector transformation**.

Definition 2.5.2. A paravector transformation is called **orthogonal** if it preserves the scalar product of each paravector pair.

Conclusion 2.5.1. A paravector transformation is orthogonal if its determinant is equal to 1.

From the condition of orthogonal transformation it can be seen that its paravector must be proper (def.2.1.9), and thus it has a module (def.2.1.12). Hence,

Conclusion 2.5.2. An orthogonal transformation is represented by paravector:

$$\Lambda = \frac{1}{\sqrt{\alpha^2-\beta^2}}\begin{bmatrix}\alpha\\ \boldsymbol{\beta}\end{bmatrix} = \frac{1}{\sqrt{a^2-b^2+c^2-d^2}}\begin{bmatrix}a+id\\ \mathbf{b}+i\mathbf{c}\end{bmatrix} \tag{2.20}$$

such that $ad = \mathbf{bc}$

Definition 2.5.3. A transformation which preserves determinants is called an **isometric transformation**.

Conclusion 2.5.3. An orthogonal transformation is isometric.

Example 2.5.1. A paravector transformation does not change the shape of a sphere.

An equation of the sphere of r radius can be written in the following way:

$$r^2 - x^2 = \begin{bmatrix}r\\ \mathbf{x}\end{bmatrix}\begin{bmatrix}r\\ -\mathbf{x}\end{bmatrix} = \begin{bmatrix}r\\ \mathbf{x}\end{bmatrix}\begin{bmatrix}r\\ \mathbf{x}\end{bmatrix}^{-} = 0 \tag{2.21}$$

Since the sphere equation is the determinant of a singular paravector $\begin{bmatrix}r\\ \mathbf{x}\end{bmatrix}$, in the complex space the spherical shape must be invariant with respect to the discussed transformation.

Theorem 2.5.1. Let Λ be a orthogonal paravector, then transformation $\Gamma' = \Lambda\Gamma$ preserves a scalar product of the vigors.

Proof.

By definition $\mathrm{vig}\Gamma = \Gamma\Gamma^*$

Let $\Gamma' = \Lambda\Gamma$ and $\det\Lambda = 1$

$\left\langle \mathrm{vig}\Gamma_1', \mathrm{vig}\Gamma_2' \right\rangle = \left[\Gamma_1'\Gamma_1'^*(\Gamma_2'\Gamma_2'^*)^-\right]_S = [\Lambda\Gamma_1(\Lambda\Gamma_1)^*[\Lambda\Gamma_2(\Lambda\Gamma_2)^*]^-]_S =$

$= \left[\Lambda\Gamma_1\Gamma_1^*\Lambda^*\Lambda^{*-}\Gamma_2^{*-}\Gamma_2^-\Lambda^-\right]_S = [\Lambda(\Gamma_1\Gamma_1^*)(\Gamma_2\Gamma_2^*)^-\Lambda^-\big]_S = \left\langle \mathrm{vig}\Gamma_1, \mathrm{vig}\Gamma_2 \right\rangle$

which completes the proof, since based on the conclusion 2.3.12.1, a rotation does not change the scalar component of the rotated paravector.

□

Conclusion 2.5.4. A paravector transformation preserves the parallelism of vigors

2.6 Discussion

During school courses physical quantities are always divided into scalars and vectors. Their natural generalization are paravectors which have both the characteristics of integers (a ring), and the geometric properties of vectors. The properties of the analyzed structure of paravectors indicate that it is similar to a unitary space, but it is not such a space, because there are such paravectors for which the norm cannot be defined. Since paravectors together with the operation of summation form an Abelian group and there the scalar product of paravectors is defined, we can say that paravectors form a unitary space over the complex numbers field. However, there is a significant difference between the definition of the scalar product of paravectors and the definitions commonly known, because the product of any paravector with itself cannot be a real number. The conclusion is that space of paravectors is not normed, but we can define the function whose some properties are the same as the square of the norm. This function is a determinant which fulfills the parallelogram law and the polarization equation. Unfortunately, the determinant is not enough to introduce the ordering relation, because its values are complex numbers, and complex numbers are not ordered. The concept of a norm can only be entered for proper and singular paravectors (a module of paravector). The trouble is that set of proper paravectors together with the addition operation is not a group. As can be seen, the issue is so wide that its solution will be presented at the end of this monograph.

By exploring various properties of paravectors we have found that different groups of them have different properties, in spite of having the same construction. Some of them act as vectors, and other as matrices, therefore:

- Additive paravectors (i.e. coordinates or field functions), denoted in parentheses, are traditionally called **four-vectors**.

$$\mathbb{X} := \begin{pmatrix} \Delta t + is \\ \Delta \mathbf{x} + i\mathbf{y} \end{pmatrix} \in C^4 \tag{2.22}$$

- Paravectors which are not additive (transformation parameters i.e. speed or rotation) are denoted in brackets:

$$\Gamma := \begin{bmatrix} \alpha \\ \boldsymbol{\beta} \end{bmatrix} = \begin{bmatrix} a + id \\ \mathbf{b} + i\mathbf{c} \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} a \\ \mathbf{b} \end{bmatrix} \tag{2.23}$$

Besides, we also distinguish:

- Coordinates of points in space-time, which we will denote with capital letters X or Y

$$X = \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \in R^4 \tag{2.24}$$

  where $\mathbf{x}$ is not a vector, but represents three coordinates of the point in real space.

- Differentiation operators

$$\partial = \begin{bmatrix} \frac{\partial}{\partial t} \\ \nabla \end{bmatrix} \qquad \partial^- = \begin{bmatrix} \frac{\partial}{\partial t} \\ -\nabla \end{bmatrix} \tag{2.25}$$

A note is needed here, which is too early to explain. The problem will be discussed in the chapter on the structure of complex space-time. In general, the four-vector $\mathbb{X}$ can be complex in contrast to the coordinates of points X, so not always $\mathbb{X} = \Delta X$. The equation $\Delta X = X_2 - X_1$ only makes sense if $\Delta X \in R^4$. The imaginary coordinates of points in space-time make no sense, but the complex interval does make sense. The definition of (2.22) says Δt and $\Delta \mathbf{x}$ to make it clear that it is about the space-time interval.

To understand the following chapters it is necessary to have a good understanding of the paravector algebra, so we recommend that the reader read this chapter several times. To facilitate an intuitive understanding of complex space-time, we tried to show the similarity of paravectors to vectors in Euclidean space.

Chapter 3

Paravectors in physics

In this chapter, the orthogonal transformation (POT) in the real space-time and the geometric interpretation of the velocity paravector are presented. The electric field equations, the basic electrodynamics equations in the paravector notation, and the way POT impacts these equations are shown. POTs were divided into three groups of transformations and a preliminary study of the domain in which the theory will be built has been carried out.

To make the considerations understandable and relevant to textbook physics, the domain will be real space-time, and the boost will be represented by a real orthogonal paravector whose scalar component is equal to one. Although the results will be very similar to those we know from the current STR, it must be remembered that the idea of complex space is different. The Lorentz transformation can be represented by a composition of orthogonal paravector transformations, which is shown in Chapter 12.

3.1 Relativistic boost in a real space-time

In a rest frame, the described object only ages ($\Delta t \in R_+$). The same object seen from a frame moving at $\mathbf{v}$ speed, moves according to the formula:

$$\begin{pmatrix} \Delta t \\ 0 \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t' \\ \Delta \mathbf{x}' \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} \Delta t' - \mathbf{v}\Delta \mathbf{x}' \\ \Delta \mathbf{x}' - \mathbf{v}\Delta t' - i\mathbf{v}\times\Delta\mathbf{x}' \end{pmatrix} \quad . \tag{3.1}$$

From the vector part of the above formula, we get the equations

$$\Delta \mathbf{x}' - \mathbf{v}\Delta t' = 0 \qquad \text{and} \qquad \mathbf{v}\times\Delta\mathbf{x}' = 0 \quad ,$$

hence we get $\mathbf{x}'_1 = \mathbf{x}'_0 + \mathbf{v}\Delta t'$ and $\left(\mathbf{x}'_1 - \mathbf{x}'_0\right)\times\mathbf{v} = 0,$ which is obvious for the inertial motion.

It should be noted that the obtained primed coordinates are still real, and the equation of motion in the primed frame has the form of a Galilean transformation despite the fact that we started from a complex orthogonal transformation. We also see that for the description of motion in a given frame it does not matter whether the speed is close to light speed or not. There is no Lorentz factor in the obtained formulas, which was reduced because only the spatial component was transformed, since for observer in their primed frame only their primed time and their primed space are important. For a non-relativistic approximation, which will be better visible when we go to the SI system (i.e. we explicitly enter the speed of light as c) the formula (3.1) has the following form

$$\begin{pmatrix} c\Delta t \\ 0 \end{pmatrix} = \frac{1}{\sqrt{1-(v/c)^2}} \begin{bmatrix} 1 \\ -\mathbf{v}/c \end{bmatrix} \begin{pmatrix} c\Delta t' \\ \Delta \mathbf{x}' \end{pmatrix} = \frac{1}{\sqrt{1-(v/c)^2}} \begin{pmatrix} c\Delta t' - \Delta \mathbf{x}'\mathbf{v}/c \\ \Delta \mathbf{x}' - \mathbf{v}\Delta t' - i\mathbf{v}\times\Delta\mathbf{x}'/c \end{pmatrix} \quad . \tag{3.2}$$

When $v \ll c$ we get approximate equations:

- scalar $c\Delta t = c\Delta t'$
- and vector $0 = \Delta\mathbf{x}' - \mathbf{v}\Delta t'$.

Thus, placing the origin of the experience at the definite point in space-time $\left(t'_0, \mathbf{x}'_0\right)$, we have a Galilean transformation

$$\begin{aligned} \Delta t &= \Delta t' \\ \mathbf{x}'_1 &= \mathbf{x}'_0 + \mathbf{v}\Delta t'. \end{aligned}$$

As can be seen, for low speeds the POT turns into Galilean transformation. Everything looks fine.

The $\mathbf{v}$ vector in the orthogonal paravector

$$V = \frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix}, \qquad \text{where} \qquad \mathbf{v} \in R^3 \qquad \text{and} \qquad 0 \leqslant v^2 < 1 \tag{3.3}$$

is a velocity of the object in the observer's real space-time.

Definition 3.1.1. The paravector (3.3) will be called a **velocity paravector** and the transformation (3.1) will be called **boost** in real space-time.

As a result of composition of such paravectors, we obtain a complex paravector

$$\frac{1}{\sqrt{1-v_1^2}}\begin{bmatrix}1\\ \mathbf{v}_1\end{bmatrix}\frac{1}{\sqrt{1-v_2^2}}\begin{bmatrix}1\\ \mathbf{v}_2\end{bmatrix} = \frac{1}{\sqrt{1-\left(\frac{\mathbf{v}_1+\mathbf{v}_2}{1+\mathbf{v}_1\mathbf{v}_2}\right)^2+\left(\frac{\mathbf{v}_1\times\mathbf{v}_2}{1+\mathbf{v}_1\mathbf{v}_2}\right)^2}}\begin{bmatrix}1\\ \frac{\mathbf{v}_1+\mathbf{v}_2+i\mathbf{v}_1\times\mathbf{v}_2}{1+\mathbf{v}_1\mathbf{v}_2}\end{bmatrix} \tag{3.4}$$

From the above formula it can be seen that in the one-dimensional case ($\mathbf{v}_1 \parallel \mathbf{v}_2$) the compound velocity is $\mathbf{v} = (\mathbf{v}_1+\mathbf{v}_2)/(1+\mathbf{v}_1\mathbf{v}_2)$, which is in compliance with the classic SR.

Now let's take a closer look at speed. In the rest frame, the described point rests for the Δt time, which in our notation means $\mathbb{X} = \begin{pmatrix}\Delta t\\ \mathbf{0}\end{pmatrix}$. In a moving frame this point moves and its coordinates are described by a 4-vector

$$\mathbb{X}' = \begin{pmatrix}\Delta t'\\ \Delta\mathbf{x}'\end{pmatrix} = \frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix}\begin{pmatrix}\Delta t\\ \mathbf{0}\end{pmatrix} = V\mathbb{X}, \tag{3.5}$$

hence $\quad \Delta t'\begin{bmatrix}1\\ \frac{\Delta\mathbf{x}'}{\Delta t'}\end{bmatrix} = \frac{\Delta t}{\sqrt{1-v^2}}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix}$, that is, again we get the results consistent with the classic theory

$$\frac{\Delta t'}{\Delta t} = \frac{1}{\sqrt{1-v^2}} \qquad \text{and} \qquad \mathbf{v} = \frac{\Delta\mathbf{x}'}{\Delta t'} \quad .$$

Let us once again go to the formula (3.5), from which it follows that $V = \mathbb{X}'\mathbb{X}^-/|\mathbb{X}|^2$, and since the velocity paravector is orthogonal, then four-vectors $\mathbb{X}$ and $\mathbb{X}'$ have the same module, i.e.

$$V = \frac{\mathbb{X}'\mathbb{X}^-}{|\mathbb{X}'||\mathbb{X}|} = \angle\left(\mathbb{X}', \mathbb{X}\right\rangle \tag{3.6}$$

Therefore, the velocity paravector can be interpreted as the paravector angle between the four-vector in the rest frame and its image in the moving frame. The above interpretation is consistent with the assumption that the speed of material objects is lower than the speed of light. From the mathematical analysis point of view, derivatives in space-time must be directional because 4-vectors of the position changes of physical objects cannot take any values, but only such values where these 4-vectors are proper. Otherwise, they would not have a module. It is allowed to differentiate only on the proper coordinates. In the simplest case (3.5), we interpret the velocity as the space-time deviation of the observer's time axis (t') from the time axis of the observed object's proper time (t).

$$\Delta^2 t' - \Delta^2\mathbf{x}' = \Delta^2 t \qquad , \qquad \mathbf{v} = \frac{\Delta\mathbf{x}'}{\Delta t'} = \cos\alpha = \sin\beta \tag{3.7}$$

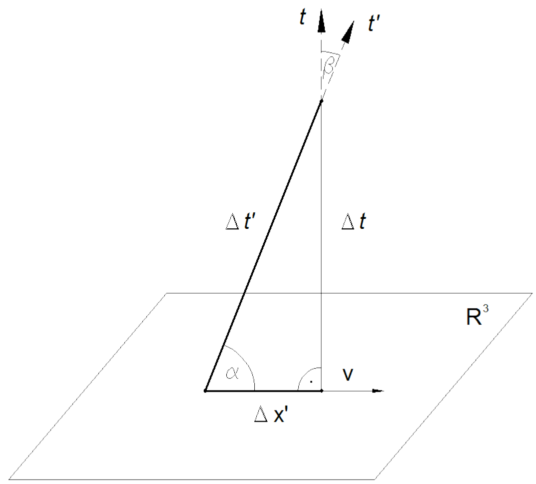

Figure 3.1: Interpretation of speed paravector in a real space-time

3.2 Four-vectors

While studying the algebra of paravectors, we introduced many concepts known from Euclidean geometry. We will first look at parallelism and check to what extent the parallelism of the four vectors differs from the parallelism of the Euclidean vectors.

We assume that $\mathbb{X}_1$ and $\mathbb{X}_2$ 4-vectors are both real and proper, which means that $\det\mathbb{X}_1, \det\mathbb{X}_2 \in R_+ \setminus \{0\}$ and are parallel to each other, hence

$$-\Delta t_1 \Delta \mathbf{x}_2 + \Delta t_2 \Delta \mathbf{x}_1 = 0$$

If we divide the above equation by $\Delta t_1 \Delta t_2$, we get that $\mathbf{v}_1 = \mathbf{v}_2$, where $\mathbf{v}_1 = \Delta \mathbf{x}_1/\Delta t_1$ and $\mathbf{v}_2 = \Delta \mathbf{x}_2/\Delta t_2$. The parallelism of the four-vectors describing the position of two objects in space-time means that these objects are moving at the same speed and in the same direction. It is easy to check that 4-vectors of the position of objects moving in the same direction but with different velocities are not parallel. The same goes for velocity paravectors before and after elastic collision perpendicular to an obstacle. It is obvious, because parallelism in space-time should be an invariant feature, and when switching to a moving frame, in both cases the paths of particle motion cease to be parallel. When dealing with purely geometric vectors (scalar component equal to 0), paravector parallelism is equivalent to Euclidean parallelism.

The perpendicularity of the four-vectors from the previous example means the dot product $\langle \mathbb{X}_1, \mathbb{X}_2 \rangle = 0$,

$$\text{or} \qquad \Delta t_1 \Delta t_2 - \Delta \mathbf{x}_1 \Delta \mathbf{x}_2 = 0$$

If we divide the above equation by $\Delta t_1 \Delta t_2$ as before, it turns out that the dot product of velocity vectors $\mathbf{v}_1 \mathbf{v}_2 = 1$. Since the velocity of objects with mass is less than 1, the position 4-vectors of the objects are never perpendicular to each other. However, this does not mean that the 4-vectors determining the mutual position of various physical objects in space-time cannot be perpendicular, but by conducting such geometric considerations we are leaving the realm of physics.

3.3 Electric field equations in vacuum and the equations of motion of an electrically charged particle in a paravector notation

Chapter 1 shows that the system of wave equations

$$\frac{\partial^2\varphi(t,\mathbf{x})}{\partial t^2}-\nabla^2\varphi(t,\mathbf{x})=\rho(t,\mathbf{x}) \tag{3.8}$$
$$\frac{\partial^2\mathbf{A}(t,\mathbf{x})}{\partial t^2}-\nabla^2\mathbf{A}(t,\mathbf{x})=\mathbf{j}(t,\mathbf{x})$$

does not change under the complex linear transformation

$$t'=\frac{t+\mathbf{vx}}{\sqrt{1-v^2}} \quad , \quad \mathbf{x}'=\frac{\mathbf{x}+\mathbf{v}t\pm i\mathbf{v}\times\mathbf{x}}{\sqrt{1-v^2}} \quad , \quad \text{where} \quad v=|\mathbf{v}| \tag{3.9}$$

which is correct in terms of calculations but is embedded in an unknown domain, which we will try to define at the end of the paper. In the special 1-dimensional case (in a spatial sense) this transformation is equivalent to the 1-dimensional Lorentz transformation. Let us remind you that we agreed to call the transformation (3.9) **paravector orthogonal transformation** (POT) and write it down in the form of paravectors:

$$\begin{pmatrix} t' \\ \mathbf{x}' \end{pmatrix}=\frac{1}{\sqrt{1-v^2}}\begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix}\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix}=\frac{1}{\sqrt{1-v^2}}\begin{pmatrix} t+\mathbf{vx} \\ \mathbf{x}+\mathbf{v}t+i\mathbf{v}\times\mathbf{x} \end{pmatrix} \tag{3.10}$$

In the formula (3.10) we have a plus sign for a vector product, but according to (3.9) there can be a minus sign, which in paravector notation means:

$$\begin{pmatrix} t' \\ \mathbf{x}' \end{pmatrix}=\frac{1}{\sqrt{1-v^2}}\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix}\begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix}=\frac{1}{\sqrt{1-v^2}}\begin{pmatrix} t+\mathbf{vx} \\ \mathbf{x}+\mathbf{v}t-i\mathbf{v}\times\mathbf{x} \end{pmatrix} \tag{3.11}$$

In the same notation, the equations of the electric and magnetic fields can be represented:

$$\begin{bmatrix} \frac{\partial}{\partial t} \\ \nabla \end{bmatrix}\begin{pmatrix} 0 \\ \mathbf{E}+i\mathbf{B} \end{pmatrix}=\begin{pmatrix} \rho \\ -\mathbf{j} \end{pmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{\partial}{\partial t} \\ -\nabla \end{bmatrix}\begin{pmatrix} \varphi \\ -\mathbf{A} \end{pmatrix}=\begin{pmatrix} 0 \\ \mathbf{E}+i\mathbf{B} \end{pmatrix}=\mathbb{E} \tag{3.12}$$

or

$$\begin{bmatrix} \frac{\partial}{\partial t} \\ -\nabla \end{bmatrix}\begin{pmatrix} 0 \\ -\mathbf{E}+i\mathbf{B} \end{pmatrix}=\begin{pmatrix} \rho \\ \mathbf{j} \end{pmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{\partial}{\partial t} \\ \nabla \end{bmatrix}\begin{pmatrix} \varphi \\ \mathbf{A} \end{pmatrix}=\begin{pmatrix} 0 \\ -\mathbf{E}+i\mathbf{B} \end{pmatrix}=\mathbb{E}_1=\mathbb{E}^{-*} \tag{3.13}$$

On the right side we have gauge conditions and on the left side Maxwell's equations in the paravector form.

$$\begin{pmatrix} \nabla\mathbf{E}+i\nabla\mathbf{B} \\ \frac{\partial\mathbf{E}}{\partial t}-\nabla\times\mathbf{B}+i(\frac{\partial\mathbf{B}}{\partial t}+\nabla\times\mathbf{E}) \end{pmatrix}=\begin{pmatrix} \rho \\ -\mathbf{j} \end{pmatrix} \tag{3.14}$$

or

$$\nabla\mathbf{E}=\rho \tag{3.15}$$
$$\nabla\mathbf{B}=0 \tag{3.16}$$
$$\nabla\times\mathbf{B}-\frac{\partial\mathbf{E}}{\partial t}=\mathbf{j} \tag{3.17}$$
$$\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0 \tag{3.18}$$

The system of equations (3.8) can also be described using the space-time differentiation operators in two ways:

- according to the equations (3.12)

$$\begin{bmatrix} \frac{\partial}{\partial t} \\ -\nabla \end{bmatrix}\begin{bmatrix} \frac{\partial}{\partial t} \\ \nabla \end{bmatrix}\begin{pmatrix} \varphi \\ -\mathbf{A} \end{pmatrix}=\begin{pmatrix} \rho \\ -\mathbf{j} \end{pmatrix} \tag{3.19}$$

- according to the equations (3.13)

$$\begin{bmatrix} \frac{\partial}{\partial t} \\ \nabla \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial t} \\ -\nabla \end{bmatrix} \begin{pmatrix} 0 \\ \mathbf{A} \end{pmatrix} = \begin{pmatrix} \rho \\ \mathbf{j} \end{pmatrix} \tag{3.20}$$

The field energy density is presented by the following formulas:

$$W = \begin{pmatrix} w \\ \mathbf{S} \end{pmatrix} = \frac{\mathbb{E}\mathbb{E}^*}{2} = \frac{1}{2} \begin{pmatrix} 0 \\ \mathbf{E} + i\mathbf{B} \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{E} - i\mathbf{B} \end{pmatrix} = \begin{pmatrix} \frac{\mathbf{E}^2 + \mathbf{B}^2}{2} \\ \mathbf{E} \times \mathbf{B} \end{pmatrix} \tag{3.21}$$

or

$$W_1 = \begin{pmatrix} w_1 \\ \mathbf{S}_1 \end{pmatrix} = \frac{\mathbb{E}_1^*\mathbb{E}_1}{2} = \frac{1}{2} \begin{pmatrix} 0 \\ -\mathbf{E} - i\mathbf{B} \end{pmatrix} \begin{pmatrix} 0 \\ -\mathbf{E} + i\mathbf{B} \end{pmatrix} = \frac{\mathbb{E}^-\mathbb{E}^{-*}}{2} = \begin{pmatrix} \frac{\mathbf{E}^2 + \mathbf{B}^2}{2} \\ \mathbf{E} \times \mathbf{B} \end{pmatrix} = W \tag{3.22}$$

In order to obtain the formulas for the field energy and Poyting's vector from equations (3.12) according to the classical theory, one has to multiply the paravectors $\mathbb{E}\mathbb{E}^*$, and in the case of the equations (3.13) the order of multiplication is reverse: $\mathbb{E}_1^*\mathbb{E}_1$. However, these dependencies are equivalent to each other.

Let's write down both systems of equations in a symbolic form:

$$\text{Equations (3.12)} \quad \begin{cases} \partial^-\mathbb{A}^- = \mathbb{E} \\ \partial\mathbb{E} = \mathbb{J}^- \end{cases} \quad \text{and wave equation} \quad \partial\partial^-\mathbb{A}^- = \mathbb{J}^- \tag{3.23}$$

$$\text{Equations (3.13)} \quad \begin{cases} \partial\mathbb{A} = \mathbb{E}^{-*} \\ \partial^-\mathbb{E}^{-*} = \mathbb{J} \end{cases} \quad \text{and wave equation} \quad \partial^-\partial\mathbb{A} = \mathbb{J} \tag{3.24}$$

Since $\partial\partial^- = \partial^-\partial$ is a scalar operator, the equations (3.24) can be obtained by reverting [1] equations (3.23). This proves that the equations (3.12) and (3.13) are equivalent. It is a matter of convention which equations we choose, so we choose the equations (3.12) (3.19) for dealing with the electric field later in this paper.

Non-relativistic equations of motion [12] for a particle with charge q in external electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields

$$\frac{dT}{dt} = q\mathbf{v}\mathbf{E} \tag{3.25}$$

$$\frac{d\mathbf{p}}{dt} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \tag{3.26}$$

are the real part of the paravector equation

$$\begin{bmatrix} \frac{d}{c\,dt} \\ 0 \end{bmatrix} \begin{pmatrix} T \\ c\mathbf{p} \end{pmatrix} = q \begin{pmatrix} 0 \\ \mathbf{E} + i\mathbf{B} \end{pmatrix} \begin{bmatrix} 1 \\ \mathbf{v}/c \end{bmatrix} \tag{3.27}$$

In the current STR, if the equations of motion are described by the Minkowski equation[2]

$$F_0 = m_0\gamma c \frac{d\gamma}{dt} \tag{3.28}$$

$$\mathbf{F} = m_0\gamma \frac{d(\gamma\mathbf{v})}{dt} \quad , \tag{3.29}$$

then kinetic energy and momentum of a particle with m_0 mass are components of one 4-vector

$$T = \frac{1}{2[1-(v/c)^2]}(m_0c^2 + m_0v^2) \tag{3.30}$$

$$\mathbf{p} = \frac{m_0\mathbf{v}}{1-(v/c)^2} \tag{3.31}$$

[1] In the sense of the definition 2.1.2

[2] The commonly used Planck's equation of motion is $\mathbf{F} = d(m_0\gamma\mathbf{v})/dt$ because it is consistent with Einstein's equation $E = mc^2$. However, it is not covariant with LT, a fact that is left unsaid. The Minkowski's equation of motion, covariant with LT, is in line with our theory. The subtle difference has serious consequences because in the first case the energy of a rest body is $E = m_0c^2$, and in the second case it is $E = m_0c^2/2$. The problem is described in detail in the article [14].

The above equations in the paravector notation take the form of:

$$\begin{pmatrix} T \\ c\mathbf{p} \end{pmatrix} = \frac{m_0 c^2}{2[1-(v/c)^2]} \begin{bmatrix} 1 \\ \mathbf{v}/c \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{v}/c \end{bmatrix} = m_0 c^2 \frac{VV}{2}, \quad \text{where } V = \frac{1}{\sqrt{1-(v/c)^2}} \begin{bmatrix} 1 \\ \mathbf{v}/c \end{bmatrix} \tag{3.32}$$

It can be seen from the above that the calculus of paravectors can be easily applied to write the valid equations of physics. Moreover, the formulas in the paravector notation are much clearer than the same formulas in the tensor form.

3.4 Discussion

The textbook physics formulas described in paravector notation look transparent, which encourages further research. However, there is a problem: the composition of velocity paravectors is not a velocity paravector, but a complex orthogonal paravector. The solution to this problem will be presented later.

When analysing transformations that preserve the wave equation invariance, we set ourselves new tasks to be solved. The most important of these is to specify the domain. Ideally, space-time should be real, but research into the complex domain is the most general and mathematically the easiest. The main course of our considerations will go in this direction. At this stage, we are not sure yet whether this is the right direction. Everything seems to indicate that physical objects cannot assume any position in space-time, but only such that four-vectors of their positions are proper or singular paravectors ($\det\mathbb{X} \in R_+$). Within this assumption, three domains can be selected.

1. $\mathbb{X}$ is complex paravector $\quad \mathbb{X} \to \mathbb{X}': \quad \mathbb{X}' = \Lambda\mathbb{X} \quad$ and $\quad \det\Lambda = 1$
2. $\mathbb{X}$ is real paravector $\quad \mathbb{X} \to \mathbb{X}': \quad \mathbb{X}' = \Lambda\mathbb{X}\Lambda^* \quad$ and $\quad \det\Lambda = 1$
3. $\mathbb{X}$ is special paravector $\quad \mathbb{X} \to \mathbb{X}': \quad \mathbb{X}' = \Lambda\mathbb{X} \quad$ and $\quad \Lambda$ is special and $\det\Lambda = 1$

There is still the $\quad \mathbb{X} \to \mathbb{X}': \quad \mathbb{X}' = \Lambda\mathbb{X}\Lambda^{-1}, \quad$ transformation but we will omit it because it is a rotation that does not change the scalar.

The second transformation has a huge advantage - it is inner in real space-time. This transformation has been studied by Professor William Baylis of the University of Windsor in Canada, and many accessibly written works on the subject can be found at his website. According to Professor Baylis's articles, this is a classical Lorentz transformation presented in the paravector formalism. Chapter 12 will devote to this issue. My research has moved towards the first transformation, which requires complex space-time. Although one can also distinguish special paravectors (3th case), which transform real space-time (strictly speaking, quasi-real space equinumerous to it) on itself, but in my opinion this is a dead end, because too many results are in contradiction with the textbook knowledge, which is shown in Appendix 3.

Chapter 4

Spatio-temporal differential operators

Four identities containing the spatio-temporal differentiation operators $\partial^{\pm} = [\partial/\partial t, \pm\nabla]$ are derived. Using the proven identities, the invariance of the wave equation under orthogonal transformations, such as boost and rotation, is shown.

The previous chapter showed that the electric field equations can be presented concisely and transparently with the spatio-temporal ∂ differential operators. It was also shown that the wave equation is invariant under the orthogonal paravector transformation, so it has become necessary to determine how these transformations affect differential operators. The identities we present below contain a four-dimensional differential operator called para-divergence or 4-divergence. Their proofs are not complicated but a bit tedious, so we will derive only the first equality in detail. For the next ones, we will present the results only. However, it is recommended that the reader prove the remaining identities on their own to gain confidence in the presented formulas, especially since they are of fundamental importance in the theory of the electric field. The paravector function should be understood as a function whose values belong to the subset of paravectors. Since we don't know the restrictions that should be imposed on the differentiation so that it does not contradict physics, we assume that the domain of the paravector function is complex space-time (set C^{1+3}). The fact that some additional assumptions are needed is indicated, for example, by the fact that time differentiation should have different rules then space differentiation, because time does not move back - time has a strictly defined direction. Determining when, why, and what restrictions should be imposed still requires careful examination, therefore we choose the most general case, where the domain is complex space-time, but the differentiation is directional due to the condition $(dt)^2 - (d\mathbf{x})^2 \in R_+$.

4.1 Transformational identities of the spatio-temporal differentiation operators

Theorem 4.1.1. Suppose that $A(X)$ is a paravector analytic function (field) defined on the set C^{1+3} and let the non-singular paravector Γ determines the automorphism in the set C^{1+3} such that $X' = \Gamma X$, then the following identities are true:

$$\partial A(X) = \partial' \Gamma A(\Gamma^{-1} X') \tag{4.1}$$

$$\partial^- A(X) = \Gamma^- \partial'^- A\left(\Gamma^{-1} X'\right) \tag{4.2}$$

Proof.

Let's expand the equation $X' = \Gamma X$:

$$\begin{aligned} t' &= \alpha t + x\beta_x + y\beta_y + z\beta_z \\ x' &= t\beta_x + \alpha x - iy\beta_z + iz\beta_y \\ y' &= t\beta_y + ix\beta_z + \alpha y - iz\beta_x \\ z' &= t\beta_z - ix\beta_y + iy\beta_x + \alpha z \end{aligned}$$

We transform the differential expression $\partial A(X)$

$$\partial A(X) = \partial A(\Gamma^{-1}\Gamma X) = \partial A(\Gamma^{-1}X') = \begin{bmatrix} \frac{\partial}{\partial t} \\ \nabla \end{bmatrix} \begin{bmatrix} \varphi(\Gamma^{-1}X') \\ \mathbf{\Phi}(\Gamma^{-1}X') \end{bmatrix} = \begin{bmatrix} \frac{\partial \varphi'}{\partial t} + \nabla \mathbf{\Phi}' \\ \frac{\partial \mathbf{\Phi}'}{\partial t} + \nabla \varphi' + i\nabla \times \mathbf{\Phi}' \end{bmatrix} \tag{4.3}$$

where prime at the symbol of function means that phase $\Gamma^{-1}X'$ is an argument. Using the formula for the derivative of a composite function we get:

$$\frac{\partial \varphi'}{\partial t} = \frac{\partial \varphi'}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial \varphi'}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial \varphi'}{\partial y'}\frac{\partial y'}{\partial t} + \frac{\partial \varphi'}{\partial z'}\frac{\partial z'}{\partial t} = \frac{\partial \varphi'}{\partial t'}\alpha + \frac{\partial \varphi'}{\partial x'}\beta_x + \frac{\partial \varphi'}{\partial y'}\beta_y + \frac{\partial \varphi'}{\partial z'}\beta_z =$$

$$= \frac{\partial \varphi'}{\partial t'}\alpha + \boldsymbol{\beta}\nabla'\varphi' \tag{4.4}$$

$$\begin{aligned} \nabla \mathbf{\Phi}' &= \frac{\partial \Phi'_x}{\partial t'}\frac{\partial t'}{\partial x} + \frac{\partial \Phi'_x}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial \Phi'_x}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial \Phi'_x}{\partial z'}\frac{\partial z'}{\partial x} + \\ &+ \frac{\partial \Phi'_y}{\partial t'}\frac{\partial t'}{\partial y} + \frac{\partial \Phi'_y}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial \Phi'_y}{\partial y'}\frac{\partial y'}{\partial y} + \frac{\partial \Phi'_y}{\partial z'}\frac{\partial z'}{\partial y} + \\ &+ \frac{\partial \Phi'_z}{\partial t'}\frac{\partial t'}{\partial z} + \frac{\partial \Phi'_z}{\partial x'}\frac{\partial x'}{\partial z} + \frac{\partial \Phi'_z}{\partial y'}\frac{\partial y'}{\partial z} + \frac{\partial \Phi'_z}{\partial z'}\frac{\partial z'}{\partial z} = \\ &= \frac{\partial \Phi'_x}{\partial t'}\beta_x + \frac{\partial \Phi'_x}{\partial x'}\alpha + i\beta_z\frac{\partial \Phi'_x}{\partial y'} - i\beta_y\frac{\partial \Phi'_x}{\partial z'} + \\ &+ \frac{\partial \Phi'_y}{\partial t'}\beta_y - i\beta_z\frac{\partial \Phi'_y}{\partial x'} + \frac{\partial \Phi'_y}{\partial y'}\alpha + i\beta_x\frac{\partial \Phi'_y}{\partial z'} + \\ &+ \frac{\partial \Phi'_z}{\partial t'}\beta_z + i\beta_y\frac{\partial \Phi'_z}{\partial x'} - i\beta_x\frac{\partial \Phi'_z}{\partial y'} + \frac{\partial \Phi'_z}{\partial z'}\alpha = \end{aligned}$$

$$= \boldsymbol{\beta}\frac{\partial \mathbf{\Phi}'}{\partial t'} + \alpha\nabla'\mathbf{\Phi}' - i\boldsymbol{\beta}\left(\nabla' \times \mathbf{\Phi}'\right) = \boldsymbol{\beta}\frac{\partial \mathbf{\Phi}'}{\partial t'} + \alpha\nabla'\mathbf{\Phi}' + \nabla' i\left(\boldsymbol{\beta} \times \mathbf{\Phi}'\right) \tag{4.5}$$

$$\nabla\varphi' = \begin{bmatrix} \frac{\partial \varphi'}{\partial t'}\frac{\partial t'}{\partial x} + \frac{\partial \varphi'}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial \varphi'}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial \varphi'}{\partial z'}\frac{\partial z'}{\partial x} \\ \frac{\partial \varphi'}{\partial t'}\frac{\partial t'}{\partial y} + \frac{\partial \varphi'}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial \varphi'}{\partial y'}\frac{\partial y'}{\partial y} + \frac{\partial \varphi'}{\partial z'}\frac{\partial z'}{\partial y} \\ \frac{\partial \varphi'}{\partial t'}\frac{\partial t'}{\partial z} + \frac{\partial \varphi'}{\partial x'}\frac{\partial x'}{\partial z} + \frac{\partial \varphi'}{\partial y'}\frac{\partial y'}{\partial z} + \frac{\partial \varphi'}{\partial z'}\frac{\partial z'}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial \varphi'}{\partial t'}\beta_x + \frac{\partial \varphi'}{\partial x'}\alpha + i\beta_z\frac{\partial \varphi'}{\partial y'} - i\beta_y\frac{\partial \varphi'}{\partial z'} \\ \frac{\partial \varphi'}{\partial t'}\beta_y - i\beta_z\frac{\partial \varphi'}{\partial x'} + \frac{\partial \varphi'}{\partial y'}\alpha + i\beta_x\frac{\partial \varphi'}{\partial z'} \\ \frac{\partial \varphi'}{\partial t'}\beta_z + i\beta_y\frac{\partial \varphi'}{\partial x'} - i\beta_x\frac{\partial \varphi'}{\partial y'} + \frac{\partial \varphi'}{\partial z'}\alpha \end{bmatrix} =$$

$$= \boldsymbol{\beta}\frac{\partial \varphi'}{\partial t'} + \alpha\nabla'\varphi' + i\left(\nabla'\varphi'\right) \times \boldsymbol{\beta} = \boldsymbol{\beta}\frac{\partial \varphi'}{\partial t'} + \alpha\nabla'\varphi' + i\nabla' \times \left(\boldsymbol{\beta}\varphi'\right) \tag{4.6}$$

$$\frac{\partial \mathbf{\Phi}'}{\partial t} + i\nabla \times \mathbf{\Phi}' = \frac{\partial \mathbf{\Phi}'}{\partial t} + i\begin{bmatrix} \frac{\partial \Phi'_z}{\partial y} - \frac{\partial \Phi'_y}{\partial z} \\ \frac{\partial \Phi'_x}{\partial z} - \frac{\partial \Phi'_z}{\partial x} \\ \frac{\partial \Phi'_y}{\partial x} - \frac{\partial \Phi'_x}{\partial y} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\partial \Phi'_x}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial \Phi'_x}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial \Phi'_x}{\partial y'}\frac{\partial y'}{\partial t} + \frac{\partial \Phi'_x}{\partial z'}\frac{\partial z'}{\partial t} \\ \frac{\partial \Phi'_y}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial \Phi'_y}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial \Phi'_y}{\partial y'}\frac{\partial y'}{\partial t} + \frac{\partial \Phi'_y}{\partial z'}\frac{\partial z'}{\partial t} \\ \frac{\partial \Phi'_z}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial \Phi'_z}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial \Phi'_z}{\partial y'}\frac{\partial y'}{\partial t} + \frac{\partial \Phi'_z}{\partial z'}\frac{\partial z'}{\partial t} \end{bmatrix} +$$

$$+i\begin{bmatrix} \frac{\partial\Phi'_z}{\partial t'}\frac{\partial t'}{\partial y}+\frac{\partial\Phi'_z}{\partial x'}\frac{\partial x'}{\partial y}+\frac{\partial\Phi'_z}{\partial y'}\frac{\partial y'}{\partial y}+\frac{\partial\Phi'_z}{\partial z'}\frac{\partial z'}{\partial y}-\frac{\partial\Phi'_y}{\partial t'}\frac{\partial t'}{\partial z}-\frac{\partial\Phi'_y}{\partial x'}\frac{\partial x'}{\partial z}-\frac{\partial\Phi'_y}{\partial y'}\frac{\partial y'}{\partial z}-\frac{\partial\Phi'_y}{\partial z'}\frac{\partial z'}{\partial z} \\ \frac{\partial\Phi'_x}{\partial t'}\frac{\partial t'}{\partial z}+\frac{\partial\Phi'_x}{\partial x'}\frac{\partial x'}{\partial z}+\frac{\partial\Phi'_x}{\partial y'}\frac{\partial y'}{\partial z}+\frac{\partial\Phi'_x}{\partial z'}\frac{\partial z'}{\partial z}-\frac{\partial\Phi'_z}{\partial t'}\frac{\partial t'}{\partial x}-\frac{\partial\Phi'_z}{\partial x'}\frac{\partial x'}{\partial x}-\frac{\partial\Phi'_z}{\partial y'}\frac{\partial y'}{\partial x}-\frac{\partial\Phi'_z}{\partial z'}\frac{\partial z'}{\partial x} \\ \frac{\partial\Phi'_y}{\partial t'}\frac{\partial t'}{\partial x}+\frac{\partial\Phi'_y}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial\Phi'_y}{\partial y'}\frac{\partial y'}{\partial x}+\frac{\partial\Phi'_y}{\partial z'}\frac{\partial z'}{\partial x}-\frac{\partial\Phi'_x}{\partial t'}\frac{\partial t'}{\partial y}-\frac{\partial\Phi'_x}{\partial x'}\frac{\partial x'}{\partial y}-\frac{\partial\Phi'_x}{\partial y'}\frac{\partial y'}{\partial y}-\frac{\partial\Phi'_x}{\partial z'}\frac{\partial z'}{\partial y} \end{bmatrix}=$$

$$=\begin{bmatrix} \frac{\partial\Phi'_x}{\partial t'}\alpha+\frac{\partial\Phi'_x}{\partial x'}\beta_x+\frac{\partial\Phi'_x}{\partial y'}\beta_y+\frac{\partial\Phi'_x}{\partial z'}\beta_z \\ \frac{\partial\Phi'_y}{\partial t'}\alpha+\frac{\partial\Phi'_y}{\partial x'}\beta_x+\frac{\partial\Phi'_y}{\partial y'}\beta_y+\frac{\partial\Phi'_y}{\partial z'}\beta_z \\ \frac{\partial\Phi'_z}{\partial t'}\alpha+\frac{\partial\Phi'_z}{\partial x'}\beta_x+\frac{\partial\Phi'_z}{\partial y'}\beta_y+\frac{\partial\Phi'_z}{\partial z'}\beta_z \end{bmatrix}+$$

$$+i\begin{bmatrix} \frac{\partial\Phi'_z}{\partial t'}\beta_y-i\beta_z\frac{\partial\Phi'_z}{\partial x'}+\frac{\partial\Phi'_z}{\partial y'}\alpha+i\beta_x\frac{\partial\Phi'_z}{\partial z'}-\frac{\partial\Phi'_y}{\partial t'}\beta_z-i\beta_y\frac{\partial\Phi'_y}{\partial x'}+i\beta_x\frac{\partial\Phi'_y}{\partial y'}-\frac{\partial\Phi'_y}{\partial z'}\alpha \\ \frac{\partial\Phi'_x}{\partial t'}\beta_z+i\beta_y\frac{\partial\Phi'_x}{\partial x'}-i\beta_x\frac{\partial\Phi'_x}{\partial y'}+\frac{\partial\Phi'_x}{\partial z'}\alpha-\frac{\partial\Phi'_z}{\partial t'}\beta_x-\frac{\partial\Phi'_z}{\partial x'}\alpha-i\beta_z\frac{\partial\Phi'_z}{\partial y'}+i\beta_y\frac{\partial\Phi'_z}{\partial z'} \\ \frac{\partial\Phi'_y}{\partial t'}\beta_x+\frac{\partial\Phi'_y}{\partial x'}\alpha+i\beta_z\frac{\partial\Phi'_y}{\partial y'}-i\beta_y\frac{\partial\Phi'_y}{\partial z'}-\frac{\partial\Phi'_x}{\partial t'}\beta_y+i\beta_z\frac{\partial\Phi'_x}{\partial x'}-\frac{\partial\Phi'_x}{\partial y'}\alpha-i\beta_x\frac{\partial\Phi'_x}{\partial z'} \end{bmatrix}=$$

$$=\alpha\frac{\partial\mathbf{\Phi}'}{\partial t'}+i\alpha\nabla'\times\mathbf{\Phi}'-i\frac{\partial\mathbf{\Phi}'}{\partial t'}\times\boldsymbol{\beta}+\nabla'(\boldsymbol{\beta}\mathbf{\Phi}')+\nabla'\times(\mathbf{\Phi}'\times\boldsymbol{\beta}) \tag{4.7}$$

Substituting partial results (4.4)-(4.7) into the equation (4.3) we receive

$$\begin{bmatrix} \frac{\partial(\alpha\varphi'+\boldsymbol{\beta}\mathbf{\Phi}')}{\partial t'}+\nabla'\left(\boldsymbol{\beta}\varphi'+\alpha\mathbf{\Phi}'+i\boldsymbol{\beta}\times\mathbf{\Phi}'\right) \\ \frac{\partial}{\partial t'}\left(\boldsymbol{\beta}\varphi'+\alpha\mathbf{\Phi}'+i\boldsymbol{\beta}\times\mathbf{\Phi}'\right)+\nabla'\left(\alpha\varphi'+\boldsymbol{\beta}\mathbf{\Phi}'\right)+i\nabla'\times\left(\boldsymbol{\beta}\varphi'+\alpha\mathbf{\Phi}'+i\boldsymbol{\beta}\times\mathbf{\Phi}'\right) \end{bmatrix}=$$

$$=\begin{bmatrix} \frac{\partial}{\partial t'} \\ \nabla' \end{bmatrix}(\begin{bmatrix} \alpha \\ \boldsymbol{\beta} \end{bmatrix}\begin{bmatrix} \varphi' \\ \mathbf{\Phi}' \end{bmatrix})$$

which completes the proof of the 1st identity.

To prove the 2nd identity (4.2)

$$\partial^- A(X)=\Gamma^-\partial'^- A\left(\Gamma^{-1}X'\right),$$

we have to use the formulas (4.4) - (4.6), and instead of (4.7) we prove that:

$$\frac{\partial\mathbf{\Phi}'}{\partial t}-i\nabla\times\mathbf{\Phi}'=\alpha\frac{\partial\mathbf{\Phi}'}{\partial t'}-i\alpha\nabla'\times\mathbf{\Phi}'-i\boldsymbol{\beta}\times\frac{\partial\mathbf{\Phi}'}{\partial t'}+\boldsymbol{\beta}(\nabla'\mathbf{\Phi}')+\boldsymbol{\beta}\times(\nabla'\times\mathbf{\Phi}')$$

□

For the transformation $X'=X\Gamma$ we have analogous identities.

Theorem 4.1.2. Suppose that $A(X)$ is a paravector analytic function defined on the set C^{1+3} and let the non-singular paravector Γ determine the automorphism in the set C^{1+3} so that $X'=X\Gamma$, then the following identities are true:

$$\partial A(X) = \Gamma\partial' A\left(X'\Gamma^{-1}\right) \tag{4.8}$$

$$\partial^- A(X) = \partial'^-\Gamma^- A\left(X'\Gamma^{-1}\right) \tag{4.9}$$

Proofs of the above identities are left to the readers.

Theorem 4.1.3. Let $A(X)$ be an analytic paravector function defined on the set C^{1+3}, then for each paravector Γ it is true that:

$$[\partial A(X)]\Gamma = \partial[A(X)\Gamma] \tag{4.10}$$

$$\left[\partial^- A(X)\right]\Gamma = \partial^-[A(X)\Gamma] \tag{4.11}$$

Proof. of the equation (4.10)

$$\partial[A(X)\Gamma]=\begin{bmatrix} \frac{\partial}{\partial t} \\ \nabla \end{bmatrix}(\begin{bmatrix} \varphi(X) \\ \mathbf{\Phi}(X) \end{bmatrix}\begin{bmatrix} \alpha \\ \boldsymbol{\beta} \end{bmatrix})=\begin{bmatrix} \frac{\partial}{\partial t} \\ \nabla \end{bmatrix}\begin{bmatrix} \alpha\varphi(X)+\mathbf{\Phi}(X)\boldsymbol{\beta} \\ \alpha\mathbf{\Phi}(X)+\boldsymbol{\beta}\varphi(X)+i\mathbf{\Phi}(X)\times\boldsymbol{\beta} \end{bmatrix}=$$

$$=\begin{bmatrix} \alpha\frac{\partial\varphi(X)}{\partial t}+\boldsymbol{\beta}\frac{\partial\boldsymbol{\Phi}(X)}{\partial t}+\alpha\nabla\boldsymbol{\Phi}(X)+\boldsymbol{\beta}\nabla\varphi(X)+i\nabla[\boldsymbol{\Phi}(X)\times\boldsymbol{\beta}] \\ \alpha\frac{\partial\boldsymbol{\Phi}(X)}{\partial t}+\boldsymbol{\beta}\frac{\partial\varphi(X)}{\partial t}+\alpha\nabla\varphi(X)+i[\frac{\partial\boldsymbol{\Phi}(X)}{\partial t}\times\boldsymbol{\beta}+\alpha\nabla\times\boldsymbol{\Phi}(X)+\nabla\varphi(X)\times\boldsymbol{\beta}]+\nabla[\boldsymbol{\Phi}(X)\boldsymbol{\beta}]-\nabla\times[\boldsymbol{\Phi}(X)\times\boldsymbol{\beta}] \end{bmatrix}=$$

hence, under property of the nabla operator we obtain

$$=\begin{bmatrix} [\frac{\partial\varphi(X)}{\partial t}+\nabla\boldsymbol{\Phi}(X)]\alpha+[\frac{\partial\boldsymbol{\Phi}(X)}{\partial t}+\nabla\varphi(X)+i\nabla\times\boldsymbol{\Phi}(X)]\boldsymbol{\beta} \\ [\frac{\partial\boldsymbol{\Phi}(X)}{\partial t}+\nabla\varphi(X)+i\nabla\times\boldsymbol{\Phi}(X)]\alpha+[\frac{\partial\varphi(X)}{\partial t}+\nabla\boldsymbol{\Phi}(X)]\boldsymbol{\beta}+i[\frac{\partial\boldsymbol{\Phi}(X)}{\partial t}+\nabla\varphi(X)+i\nabla\times\boldsymbol{\Phi}(X)]\times\boldsymbol{\beta} \end{bmatrix}=$$

$=[\partial A(X)]\Gamma$

The reader can prove the equation (4.11) in a similar way as above. □

Formulas of transformation of the field by the rotation of reference system $X'=\Gamma X\Gamma^{-1}$ follow from above results, where the rotation means a more general transformation then Euclidean rotation (definition 2.3.9)

Example 4.1.1. Rotation of the observer in the field

Let Λ be an orthogonal paravector (ie. $\det\Lambda=1$) and let the fields $A(X)$ and $B(X)$ satisfy the relationship $\partial A(X)=B(X)$, where $X\in C^{1+3}$. The observer rotates:

$$\partial A(\Lambda^{-}X'\Lambda)=B(\Lambda^{-}X'\Lambda) \qquad , \qquad \text{where} \qquad X'=\Lambda X\Lambda^{-}$$

In the turned frame the above equation has the following form (by Theorems 4.1.1 and 4.1.2)

$$\Lambda^{-}\partial'\Lambda A(\Lambda^{-}X'\Lambda)=B(\Lambda^{-}X'\Lambda)$$

Multiplying this equation on the left-side by Λ and on the right-side by Λ^{-}, on the basis of the Theorem 4.1.3, we obtain an equation of the field after rotation.

$$\partial'[\Lambda A(\Lambda^{-}X'\Lambda)\Lambda^{-}]=\Lambda[B(\Lambda^{-}X'\Lambda)]\Lambda^{-} \tag{4.12}$$

Similarly for the reversed operator (4-gradient).

$$\partial'^{-}\left[\Lambda A\left(\Lambda^{-}X'\Lambda\right)\Lambda^{-}\right]=\Lambda\left[B\left(\Lambda^{-}X'\Lambda\right)\right]\Lambda^{-} \tag{4.13}$$

The conclusion is obvious: **If the observer turns to one side, the field around them will turn by the same amount in the opposite direction**.

4.2 Invariance of wave equation under orthogonal transformation

Using theorems 4.1.1 - 4.1.3, we can easily demonstrate the invariance of the wave equation $\Box A(X)=B(X)$ under the transformation represented by the orthogonal paravector. We narrow down the formulas (4.1), (4.2) and (4.8), (4.9) by replacing the non-singular paravector Γ with the orthogonal paravector Λ. It is visible that with the orthogonal transformation we can transform the wave equation $\Box A(X)=B(X)$ in four ways:

1. $\Box A(X)=\partial^{-}\partial A(X)=\partial'^{-}\Lambda^{-}\Lambda\partial' A(X'\Lambda^{-})=\Box' A(X'\Lambda^{-})=B(X'\Lambda^{-})$, hence

$$\Box A(X)=B(X) \qquad \Longleftrightarrow \qquad \Box' A(X'\Lambda^{-})=B(X'\Lambda^{-}) \tag{4.14}$$

2. $\Box A(X)=\partial\partial^{-}A(X)=\Lambda\partial'\partial'^{-}[\Lambda^{-}A(X'\Lambda^{-})]=\Lambda\Box'[\Lambda^{-}A(X'\Lambda^{-})]=B(X'\Lambda^{-})$, hence

$$\Box A(X)=B(X) \qquad \Longleftrightarrow \qquad \Box'[\Lambda^{-}A(X'\Lambda^{-})]=\Lambda^{-}B(X'\Lambda^{-}) \tag{4.15}$$

3. $\Box A(X) = \partial^- \partial A(X) = \Lambda^- \partial'^- \partial' [\Lambda A(\Lambda^- X')] = \Lambda^- \Box' [\Lambda A(\Lambda^- X')] = B(\Lambda^- X')$, hence

$$\Box A(X) = B(X) \qquad \Longleftrightarrow \qquad \Box' [\Lambda A(\Lambda^- X')] = \Lambda B(\Lambda^- X') \tag{4.16}$$

4. $\Box A(X) = \partial \partial^- A(X) = \partial' \Lambda \Lambda^- \partial'^- A(\Lambda^- X') = \Box' A(\Lambda^- X') = B(\Lambda^- X')$, hence

$$\Box A(X) = B(X) \qquad \Longleftrightarrow \qquad \Box' A(\Lambda^- X') = B(\Lambda^- X') \tag{4.17}$$

From the above relationships it can be seen that further discussion can be carried out in different directions. In points 1 and 4 both the equation and the value of the function are invariant.

$$A' = A \qquad \text{and} \qquad B' = B$$

In points 2 and 3 the form of wave equation is invariant, while the values of the function of a field undergo changes:

- contravariant
$$A' = \Lambda^- A \qquad \text{and} \qquad B' = \Lambda^- B$$
- or covariant
$$A' = \Lambda A \qquad \text{and} \qquad B' = \Lambda B$$

We encounter an interesting problem that goes beyond the scope of this chapter, and enters the field of physics, so its analysis will be presented further.

As for the rotation of the observer in the field meeting the wave equation, we get the same result as in example 4.1.1.

$$\Box A(X) = B(X) \qquad \Longleftrightarrow \qquad \Box' \Lambda [A(\Lambda^- X' \Lambda)] \Lambda^- = \Lambda [B(\Lambda^- X' \Lambda)] \Lambda^-$$

4.3 Discussion

The wave equation is one of the most important relationships in physics. It underlies the theory of the electromagnetic field and relativistic quantum mechanics, and is applicable in all fields of physics. The considerations presented above and the simplicity of calculation show that the paravector calculus fits into this equation naturally, and thus it is also natural for relativistic physics. In the following chapters, we will continue to convince the reader of the parawector calculus and show that with its help one can look at seemingly well-known physical phenomena differently.

Chapter 5

Phase interval

In this chapter, the concept of phase interval, which is invariant and equal to proper time interval, is introduced. Solutions of the wave equation in the form of a plane and a spherical wave are analysed. The concept of complex velocity is explored.

In Chapter 3 we made the assumption that the relativistic transformation is a linear transformation in the space of physical objects whose coordinates are proper or singular paravectors. An object in space-time is defined by its state, i.e. the coordinates of the change of position in time and the parameters characterizing this motion, e.g. velocity. All these parameters for inertial motion are described by a pair of paravectors connected by a $V^{-}\mathbb{X}$ multiplication operation.

Definition 5.0.1. We call a pair $V^{-}\mathbb{X} = \Delta t$ a **phase interval**.

The theory of relativity describes objects in motion at speeds comparable to the speed of light. For dynamic phenomena it is important what happens in the time interval. This explains where the assumption we made in the introduction that time is a discrete quantity, had come.

The phase interval is a non-negative real number and it is equivalent to the proper time interval.

Definition 5.0.2. The following expression is called the **phase interval** in real space-time

$$\Delta\Theta = \frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\-\mathbf{v}\end{bmatrix}\begin{pmatrix}\Delta t'\\\Delta\mathbf{x}'\end{pmatrix} = \begin{pmatrix}\Delta t\\0\end{pmatrix} \tag{5.1}$$

The individual elements of the phase interval are called:

- $\begin{pmatrix}\Delta t'\\\Delta\mathbf{x}'\end{pmatrix}$ - space-time interval (which is a four-vector)
- $\mathbf{v}$ - phase velocity. The direction of this vector determines the phase direction.

In the case when the above phase interval relates to an object whose motion is described in relation to several independent objects, individual elements of the phase interval can be complex. Then it has the form:

$$\Delta\Theta = \Lambda^{-}\mathbb{X} = \frac{1}{\sqrt{a^2-b^2+c^2-d^2}}\begin{bmatrix}a+id\\-\mathbf{b}-i\mathbf{c}\end{bmatrix}\begin{pmatrix}\Delta t'+is\\\Delta\mathbf{x}'+i\mathbf{y}\end{pmatrix} = \begin{pmatrix}\Delta t\\0\end{pmatrix} \in R_+ \tag{5.2}$$

The imaginary components of the space-time interval are interpreted as dynamic properties of the real 4-vector $(\Delta t, \Delta\mathbf{x})$ imaginable in the real affine space-time of the observer. We are used to minimizing the number of coordinates to such a number that they are all independent of each other. In our case, complex variables t, x^1, x^2, x^3 are independent, but their imaginary and real parts depend on each other. So we consider issues in a special 4-dimensional complex space. Our aim is not to consider the most general considerations, but to achieve computational simplicity and physical interpretability. The results obtained so far confirm the sense of this idea.

5.1 Wave

As can easily be checked, the solution of a homogeneous wave equation is fulfilled by any double-differentiable function:

$$A(\begin{bmatrix}\alpha\\ \boldsymbol{\beta}\end{bmatrix}\begin{pmatrix}t\\ \mathbf{x}\end{pmatrix}) \qquad \text{or} \qquad A(\begin{pmatrix}t\\ \mathbf{x}\end{pmatrix}\begin{bmatrix}\alpha\\ \boldsymbol{\beta}\end{bmatrix}) \tag{5.3}$$

such that the value of A is a paravector and paravector $\begin{bmatrix}\alpha\\ \boldsymbol{\beta}\end{bmatrix}$ is singular.

Conclusion 5.1.1. In the case when $\alpha = 1$, $\boldsymbol{\beta} = -\mathbf{c} \in R^3$ and $|\mathbf{c}| = 1$, the $\mathbf{c}$ vector is interpreted as the speed of the wave, that is the speed of light in real space-time, and it is d'Alembert's canonical solution.

If we are talking about a wave front, then we have to impose a phase agreement condition:

$$\begin{bmatrix}1\\ -\mathbf{c}\end{bmatrix}\begin{pmatrix}t-t_0\\ \mathbf{x}-\mathbf{x}_0\end{pmatrix} = C^-\mathbb{X} = 0 \tag{5.4}$$

We talk of a **plane wave** when the $\mathbf{c}$ vector is given. When a point with the coordinates $\mathbf{x}_0$ is given and $\mathbf{c}$ has any direction then we talk of a **spherical wave**.

The argument of the wave function is a pair of paravectors connected with each other by the operation of multiplication which, due to its similarity to the appropriate combination in the classical theory of the electric field and to the definition used in the theory of control, we called the phase. It is invariant in a paravector orthogonal transformation (relativistic boost). We say the coordinates are implicit in a phase. When examining this class of problems, we can use properties of singular parallelism, presented in section 2.3, from which it follows:

Conclusion 5.1.2. For the periodic function $f(C^-\mathbb{X})$ 4-vector $\mathbb{T}$ is the period of f if and only if $\mathbb{T}$ is singularly parallel to paravector C.

Proof.

The function is periodic with the period $\mathbb{T}$ when $f(C^-(\mathbb{X}+\mathbb{T})) = f(C^-\mathbb{X})$, and this is when $\langle C,\mathbb{T}\rangle = 0$.

The order of the paravectors does not matter in this case, because if $(\mathbb{T},C\rangle = 0$ then also $\langle C,\mathbb{T}\rangle = 0$.

□

It follows from the above conclusion that if $\langle C,\mathbb{T}\rangle = \begin{bmatrix}1\\ -\mathbf{c}\end{bmatrix}\begin{bmatrix}T\\ \mathbf{p}\end{bmatrix} = 0$ then

$$\begin{cases} T = \mathbf{cp}\\ \mathbf{c} = \mathbf{p}/T\\ \mathbf{c}\times\mathbf{p} = \mathbf{0}\end{cases}$$

where T is the period and $\mathbf{p}$ is the wavelength. The last equations are self-explanatory. If the function (5.3) is periodic and we extract the period T from the phase, then we get a different phase

$$\Theta = \begin{bmatrix}1/T\\ -\mathbf{c}/T\end{bmatrix}\begin{pmatrix}t\\ \mathbf{x}\end{pmatrix} = \begin{bmatrix}\omega\\ -\mathbf{k}\end{bmatrix}\begin{pmatrix}t\\ \mathbf{x}\end{pmatrix}$$

We transform this phase with an orthogonal transformation:

$$\Theta = \begin{bmatrix}\omega\\ -\mathbf{k}\end{bmatrix}\begin{pmatrix}t\\ \mathbf{x}\end{pmatrix} = \frac{1}{\sqrt{1-v^2}}\begin{bmatrix}\omega\\ -\mathbf{k}\end{bmatrix}\begin{bmatrix}1\\ -\mathbf{v}\end{bmatrix}\begin{pmatrix}t'\\ \mathbf{x}'\end{pmatrix} = \begin{bmatrix}\omega'\\ -\mathbf{k}'\end{bmatrix}\begin{pmatrix}t'\\ \mathbf{x}'\end{pmatrix},$$

then

$$\begin{bmatrix}\omega'\\ -\mathbf{k}'\end{bmatrix} = \frac{1}{\sqrt{1-v^2}}\begin{bmatrix}\omega+\mathbf{vk}\\ -(\mathbf{k}+\omega\mathbf{v}+i\mathbf{v}\times\mathbf{k})\end{bmatrix} \tag{5.5}$$

We obtain **Doppler law** in the paravector notation.

5.2 Complex velocity of light

Below we will check what the image of the speed of light C paravector looks like in the frame moving with speed $\mathbb{X}' = V\mathbb{X}$ in relation to the light source. Passing to the frame moving at the speed of $-\mathbf{v}$, the phase function arguments (5.4) of the field function transform according to the formula:

$$\begin{bmatrix}1\\-\mathbf{c}\end{bmatrix}\begin{pmatrix}\Delta t\\\Delta\mathbf{x}\end{pmatrix}=0 \qquad \to \qquad \frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\-\mathbf{c}\end{bmatrix}\begin{bmatrix}1\\-\mathbf{v}\end{bmatrix}\frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\\mathbf{v}\end{bmatrix}\begin{pmatrix}\Delta t\\\Delta\mathbf{x}\end{pmatrix}= \tag{5.6}$$

$$=\frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1+\mathbf{vc}\\-\mathbf{v}-\mathbf{c}-i\mathbf{v}\times\mathbf{c}\end{bmatrix}\begin{pmatrix}\Delta t'\\\Delta\mathbf{x}'+i\mathbf{y}'\end{pmatrix}=0$$

After extracting time and scalar $1+\mathbf{vc}$, we get the condition that must be met by the complex speed of light in relation to the speed of light in the rest frame.

$$\begin{bmatrix}1\\-\frac{\mathbf{v}+\mathbf{c}+i\mathbf{v}\times\mathbf{c}}{1+\mathbf{vc}}\end{bmatrix}\begin{bmatrix}1\\\mathbf{v}'+i\mathbf{w}'\end{bmatrix}=0 \qquad \text{where} \qquad \mathbf{v}'+i\mathbf{w}'=\mathbf{c}' \tag{5.7}$$

We conclude from this that:

$$\mathbf{c}'=\frac{\mathbf{v}+\mathbf{c}+i\mathbf{v}\times\mathbf{c}}{1+\mathbf{vc}} \qquad \text{and} \qquad c'^2=1 \tag{5.8}$$

Note that although the real part of the vector of light velocity $\mathbf{c}'$ may be greater than 1, the complex vector always has a length of 1. There is no contradiction with the Michelson-Morley experiment. Remember that the Michelson-Morley experiment only tells us that there is no aether and that it is performed in a light source frame. The complex model we are creating apparently gives the possibility of exceeding the speed of light, while being in line with the results of the Michelson-Morley experiment.

The problem is how to interpret the imaginary part of a velocity vector? From the computational side, the imaginary component of a vector is an element needed to properly balance the vector's coordinates so that its length is equal to 1. From the physical side, one can look at the real velocity vector as if it be twisting, because the imaginary vector perpendicular to the plane defined by vectors **v** and **c** gives the real vector such a feature. However, this is not a twist in space ... but we will talk about it many more times, because this is the key to understanding complex space-time.

5.3 Complex velocity

Now we will try to interpret complex vectors. We will start with the simplest cases and see what simple phases look like after the transformation:

5.3.1 Time interval

$$\begin{pmatrix}\Delta t\\\mathbf{0}\end{pmatrix}=\frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\-\mathbf{v}\end{bmatrix}\begin{pmatrix}\Delta t'\\\Delta\mathbf{x}'\end{pmatrix} \tag{5.9}$$

hence

$$\Delta\mathbf{x}'-\mathbf{v}\Delta t'=0 \qquad \text{and} \qquad \mathbf{v}\times\Delta\mathbf{x}'=0,$$

that is, we get the classical Galilean result $\quad \mathbf{x}_1'=\mathbf{x}_0'+\mathbf{v}\Delta t'$.

As observers in the primed frame, we have our own time and during this time we observe a moving object. The relation between the times of both frames is represented by the following relationship: $\Delta t=(\Delta t'-\mathbf{v}\Delta\mathbf{x}')/\sqrt{1-v^2}$. And here we need to make note that contributes a lot to the understanding of the essence of the so-called time dilation. Calculating the time in the primed frame from the last formula, it would appear that the time is shorter than in the non-primed frame. This would mean that the life of a moving particle is shorter than that of a rest particle, while the opposite is actually true. Let us return to formula (5.9), which after moving the velocity paravector to the left side has the following form:

$$\frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix}\begin{pmatrix}\Delta t\\ \mathbf{0}\end{pmatrix}=\begin{pmatrix}\Delta t'\\ \Delta \mathbf{x}'\end{pmatrix},$$

where we get another relationship between $\Delta t' = (\Delta t)/\sqrt{1-v^2}$, which is the correct result. But there is no contradiction in this. A paravector is also a matrix, so we cannot transform a part of a paravector equation, we can only do so with the entire equation. This results in a very important and obvious conclusion confirming the compliance of our theory with the postulates of the classical STR: **no inertial frame is privileged**. If from the primed frame we see that time in the non-primed frame is slower, then the observer from the non-primed frame sees time in the primed frame in the same way. In other words: **the so-called time dilation is symmetrical**. Therefore, one should treat it as an illusion, and not draw the conclusion that in two frames moving in relation to each other, time flows at a different pace.

5.3.2 Space-time interval

Let's complicate the exercise and check what the space-time interval will look like when viewed from a moving frame, i.e. what the combination of two velocities looks like.

$$\begin{pmatrix}\Delta t\\ \mathbf{0}\end{pmatrix}=\frac{1}{\sqrt{1-v_1^2}\sqrt{1-v_2^2}}\begin{bmatrix}1\\ -\mathbf{v}_1\end{bmatrix}\begin{bmatrix}1\\ -\mathbf{v}_2\end{bmatrix}\begin{pmatrix}\Delta t''\\ \Delta \mathbf{x}''+i\mathbf{y}''\end{pmatrix}= \tag{5.10}$$

$$=\frac{1}{\sqrt{1-v_1^2}\sqrt{1-v_2^2}}\begin{bmatrix}1+\mathbf{v}_1\mathbf{v}_2\\ -\mathbf{v}_1-\mathbf{v}_2+i\mathbf{v}_1\times\mathbf{v}_2\end{bmatrix}\begin{pmatrix}\Delta t''\\ \Delta \mathbf{x}''+i\mathbf{y}''\end{pmatrix}$$

After transforming the above equation, we get

$$\frac{1}{\sqrt{1-v_1^2}\sqrt{1-v_2^2}}\begin{bmatrix}1+\mathbf{v}_1\mathbf{v}_2\\ \mathbf{v}_1+\mathbf{v}_2-i\mathbf{v}_1\times\mathbf{v}_2\end{bmatrix}\begin{pmatrix}\Delta t\\ \mathbf{0}\end{pmatrix}=\begin{pmatrix}\Delta t''\\ \Delta \mathbf{x}''+i\mathbf{y}''\end{pmatrix}, \tag{5.11}$$

and since Δt is a positive real number, time $\Delta t'' > 0$. We divide both sides of the last equation by the real $(1+\mathbf{v}_1\mathbf{v}_2)\Delta t$, and on the right we pull out $\Delta t''$ before the 4-vector.

$$\frac{1}{\sqrt{1-v_1^2}\sqrt{1-v_2^2}}\begin{bmatrix}1\\ \frac{\mathbf{v}_1+\mathbf{v}_2-i\mathbf{v}_1\times\mathbf{v}_2}{1+\mathbf{v}_1\mathbf{v}_2}\end{bmatrix}=\frac{\Delta t''}{(1+\mathbf{v}_1\mathbf{v}_2)\Delta t}\begin{bmatrix}1\\ \mathbf{v}''\end{bmatrix} \tag{5.12}$$

Hence we get the complex resultant velocity

$$\mathbf{v}''=\frac{\mathbf{v}_1+\mathbf{v}_2-i\mathbf{v}_1\times\mathbf{v}_2}{1+\mathbf{v}_1\mathbf{v}_2} \tag{5.13}$$

and the dilatation factor

$$\frac{\Delta t''}{\Delta t}=\frac{(1+\mathbf{v}_1\mathbf{v}_2)}{\sqrt{1-v_1^2}\sqrt{1-v_2^2}}=\frac{1}{\sqrt{1-v_{re}''^2+v_{im}''^2}} \tag{5.14}$$

Real coordinates of the velocity vector

$$Re\mathbf{v}''=\frac{\mathbf{v}_1+\mathbf{v}_2}{1+\mathbf{v}_1\mathbf{v}_2}$$

are obvious to us, as learned from Descartes. However, vectors have imaginary components

$$Im\mathbf{v}''=-\frac{\mathbf{v}_1\times\mathbf{v}_2}{1+\mathbf{v}_1\mathbf{v}_2}$$

How to interpret them? If you think about it, there is nothing strange in it! In a similar way, we 'see' a moving electric charge - through the imaginary component of its electric field - the magnetic field. If we are on the straight path of the charge, we do not 'see' the magnetic field, but the real electric field only. If we move away from the wire, we will 'see' that the charge has a real electric field and an imaginary magnetic field. The magnetic field is a dynamic property of the electric field. Likewise, an imaginary vector is a dynamic property of a real vector in space-time due to the fact that time does not stand still.

We will try to allow for situations in which it is possible for a physical object to exceed the speed of light in real space, but only such that in the complex space the complex speed is not greater than c. Then, the whole problem is to find meaningful interpretations of the imaginary components. For now, we treat the imaginary coordinates of vectors as dependent on real ones, related to relativistic motion, and needed to balance the calculations.

5.4 Summary of the considerations so far and setting directions for the future

Summing up the considerations on the phase interval so far, we can say that the orthogonal transformation is a paravector transformation of the four-vectors $\Lambda\mathbb{X} = \mathbb{X}'$ which maintains the phase interval, i.e. $\Gamma^{-}\mathbb{X} = \Gamma'^{-}\mathbb{X}'$ (or $\mathbb{X}\Gamma^{-} = \mathbb{X}'\Gamma'^{-}$), where $\mathbb{X}$ and $\mathbb{X}' \in C^{1+3}$ and Λ is an orthogonal paravector and Γ is a proper or a singular one. Ideally, all the paravectors should be real. So we will conduct the research as if the boost be represented by the velocity paravector and the position coordinates be real, because they are physically interpretable. On the other hand, we are looking for a transformation that makes a complex orthogonal paravector to a velocity paravector real and also has a physical justification. However, we should not be concerned with the coordinates of the points in spacetime itself. In the phenomena under consideration, motion is of fundamental importance, i.e. changes in coordinates ($\Delta\mathbf{x}$) over time (Δt) and the constraints imposed on these quantities are important. Note that in the calculations so far, there were no object coordinates, but their differences, i.e. vectors. Mathematically, this means that we are dealing with a vector space, not an affine space. The field disturbances move in space at the speed of light, or in other words: the arguments of the field function are phase intervals, which are also concepts of the vector space. For this reason, all the time we talk about frames, and not coordinate systems.

If we want the components of phase intervals to describe physical phenomena, it seems obvious that the paravectors that represent them must be proper, that is, such that have a module. If there was a superluminal speed, its paravector would have a negative determinant, so it would be improper and it would not be possible to present it unequivocally in the form of an orthogonal paravector. We should stick to this direction because it gives us a mathematical confirmation of an empirical fact:

In physical phenomena there is no speed greater than the speed of light.

Since it is difficult for us to accept complex vectors, for the sake of simplicity we assume that in the future we will be able to find such a physically interpretable transformation of $\mathbb{X} \to \mathbb{X}'$ that $\mathbb{X} \in C^{1+3}$ and $\mathbb{X}' \in R^{1+3}$. For now, we will be checking how the electric field and STR change as a result of adopting the relativistic transformation described by the real velocity paravector, as such considerations will be understandable. On the other hand, we will extend the knowledge about paravector transformations on complex paravectors, because they are complex by nature. Later, we will try to reconcile the complex model of mathematical structure with (mathematically) real physical phenomena.

We wrote above that $\mathbb{X} \in R^{1+3}$ and not R^4 for example. However, it seems necessary to separate the scalar part from the spatial one in the position four-vector even more, because the scalar part is time and time does not run backwards. The time structure along with the addition should be a monoid. On the other hand, the spatial part is a 3-dimensional vector space.

Chapter 6

First steps in complex space-time

In this chapter the transformation of the spatial vector, proper and improper four-vectors, and selected space-time phenomena are explained through simple examples. It has been shown that simultaneity in a rest frame does not have to correspond to simultaneity in a moving frame, and imaginary vectors are the spatial effect of this desynchronization. A hypothesis is made on how to understand the imaginary components of vectors and time.

So far, there has been talk of the movement of the electromagnetic wave front and electric charges, i.e. objects possessing energy. According to the STR postulates, these objects can move at a speed not exceeding light, which mathematically means that state paravectors describing such motion must have real non-negative determinants.

Imagine an eruption on the Sun taking place right now. We do not know anything about it yet, because the flash will reach Earth in a few minutes. It does not matter for us at this moment, but the wave front is already approaching and we want to note this distance. A zero-time paravector has a negative determinant, so it makes no physical sense, but it does have a geometric significance. In this chapter, we will extend our discussion of coordinates without restricting their determinants.

6.1 Spatial vector

The problem presented below is a completely theoretical one, because the vectors in question are outside of time, so it is a geometric problem, not a physical one.

In the OX frame, in the same moment ($\Delta t = 0$) we have two different points. These points determine a vector $\Delta\mathbf{x}$, which we denote as a 4-vector $\mathbb{X} = \begin{pmatrix} 0 \\ \Delta\mathbf{x} \end{pmatrix}$.

To make the presentation more vivid, although we are talking about one vector, we show a bunch of vectors with a common origin (the cross) and equal lengths (the dot ends). The ends of these vectors form a circle. After passing to a frame that moves at the $-\mathbf{v}$ speed, the above 4-vector transforms:

$$\mathbb{X}' = \begin{pmatrix} \Delta t' \\ \Delta\mathbf{x}' + i\mathbf{y}' \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} \begin{pmatrix} 0 \\ \Delta\mathbf{x} \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} \mathbf{v}\Delta\mathbf{x} \\ \Delta\mathbf{x} + i\mathbf{v}\times\Delta\mathbf{x} \end{pmatrix} \tag{6.1}$$

In this example we are not dealing with the imaginary vector yet, we are only concerned with the interpretation of the temporal component. We assume that the imaginary vector is an auxiliary quantity used to balance the calculations - the dependent variable.

The last equation shows that the time ($\Delta t'$) between the beginning and the end of each of the vectors $\Delta\mathbf{x}'$ is proportional to $\mathbf{v}\Delta\mathbf{x}$. The time interval is positive when the end time comes after the start time. If the moment

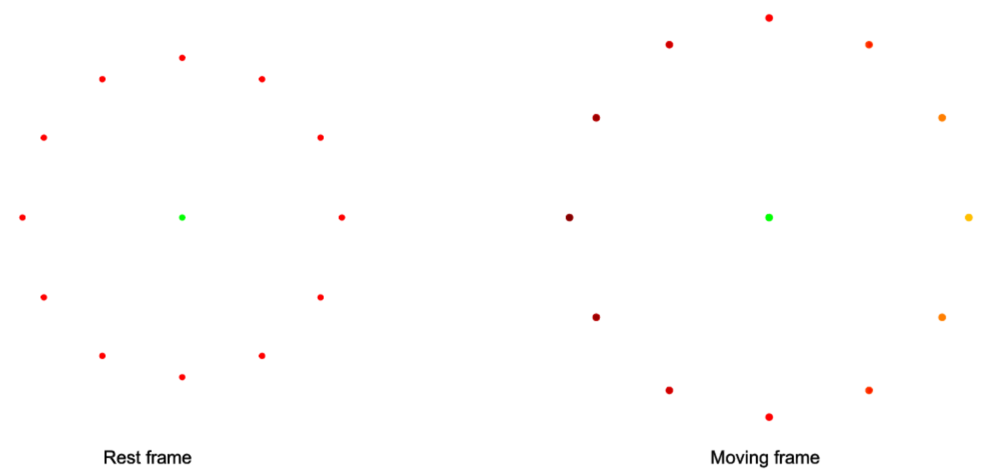

Figure 6.1: An image of a bunch of vectors

of vector's end is before the moment of its beginning, then the time interval between its end and its beginning is negative. If the vectors **v** and $\Delta\mathbf{x}$ are opposite, then their dot product is negative. This means that the end of the vector $\Delta\mathbf{x}'$ is before its beginning. In Fig 6.1 times are marked with different shades of red. The ends of the vectors that are in the front in the direction of the observer's movement precede their common beginning. Negative time obviously has no physical significance because energy cannot move in this way. If we assume that the point of reference is the center of the sphere (at time 0), then from the equation (6.1) we know which points occur earlier and which later. The delays we are talking about occur on a cosmic scale, because after switching to the SI (6.2) system it turns out that they are scaled inversely to the square of the speed of light.

Below (Fig 6.2) is shown an image of a 'flashing' disk as seen from a frame moving at relativistic speed. The real vector tells us about the shape. So we can say that the image of a circle (sphere) is a larger circle (sphere) whose points are shifted in time. Segment points are simultaneous[1] . We can see that the simultaneity in the OX frame does not correspond to the simultaneity in the OX' frame. The use of the 'we see' is a bit of an abuse, because seeing involves the eye receiving light that is moving at a certain speed. When we use the 'we see' expression, we mean mathematical formulas.

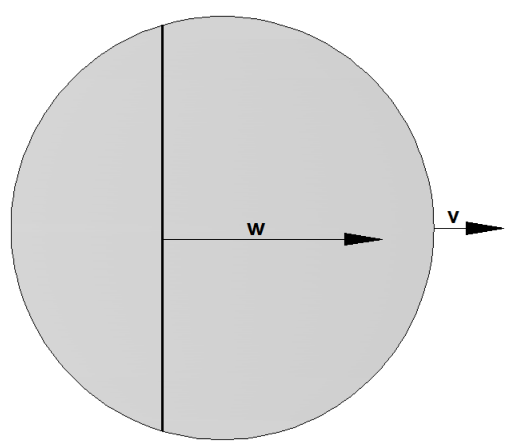

Figure 6.2: Mathematical image of a flashing disk

[1]Relation simultanity of two events X_1 and X_2 holds when a scalar coordinate of four-vector $\mathbb{X} = X_2 - X_1$ is equal to 0, that is $\mathbb{X}_S = (X_2 - X_1)_S = 0$

From the real vector part we conclude that if the observed object does not change its shape over time, that relativistic transformation scales the objects with the dilation factor without deforming it. At this point, we will turn to the SI system for a moment and see what the formula (6.1) looks like in this units system.

$$\mathbb{X}' = \begin{pmatrix} c\Delta t' \\ \Delta\mathbf{x}' + i\mathbf{y}' \end{pmatrix} = \frac{1}{\sqrt{1-(v/c)^2}} \begin{pmatrix} \mathbf{v}\Delta\mathbf{x}/c \\ \Delta\mathbf{x} + i\mathbf{v}\times\Delta\mathbf{x}/c \end{pmatrix} \tag{6.2}$$

It can be seen from the above that for non-relativistic velocities, the imaginary time and vector components disappear. The segment travel speed in Figure 6.2 is

$$\frac{\mathbf{w}}{c} = \frac{\Delta\mathbf{x}'}{c\Delta t'} = \frac{c\Delta\mathbf{x}}{\mathbf{v}\Delta\mathbf{x}}, \tag{6.3}$$

which in the natural system gives: $w = \Delta x'/\Delta t' = 1/v > 1$, and in the SI system: $\Delta x'/\Delta t' = c^2/v > c$. So not only is the speed of the moving segment higher than the speed of light, but it is also inversely proportional to the relative speed of the disk and the observer. Therefore, the phenomenon is not a physical one, but a geometric one. Considering the physical problems, we can assume that a purely spatial vector corresponds to a vector scaled by the dilation factor. We also assume that a flashing disk corresponds to a blurry flashing disk. The higher the speed of the disk, the more blurry the disk is.

In the above examples we pay attention to the scalar component $\mathbf{v}\Delta\mathbf{x}$, which indicates spatial desynchronization of phenomena observed from the moving frame. We'll deal with the imaginary vector later!

6.2 Growing vector

In the next example, we will complicate our considerations. We have two points A and B in the rest frame. At the initial t_0 moment both points overlap, then point B moves away at a constant relativistic speed **w** from point A, which is still stationary. We will describe this phenomenon in the OX' frame that moves at the speed of -**v** in relation to the rest frame (OX). Since simultaneity in both frames does not coincide, we will calculate the equations of point B motion in two ways:

1. simultaneously in a moving (primed) frame, i.e. as the observer sees,
2. so that the description in the moving frame corresponds to simultaneity in the rest one.

In the OX frame the coordinates of the four-vector (Fig. 6.3) determined by both points can be written down in the following way:

Point A: $\mathbb{X}_A = \begin{pmatrix} \Delta t \\ 0 \end{pmatrix}$

Point B: $\mathbb{X}_B = \begin{pmatrix} \Delta t \\ \Delta\mathbf{x}_B \end{pmatrix}$, whose coordinates are related by the dependence $\begin{pmatrix} \Delta t^0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{1-w^2}} \begin{bmatrix} 1 \\ -\mathbf{w} \end{bmatrix} \begin{pmatrix} \Delta t \\ \Delta\mathbf{x}_B \end{pmatrix}$

$$\mathbb{X} = \mathbb{X}_B - \mathbb{X}_A = \begin{pmatrix} 0 \\ \Delta\mathbf{x} \end{pmatrix} \tag{6.4}$$

Since the A point is resting, then AB vector $\Delta\mathbf{x} = \Delta\mathbf{x}_B$.

Point B moves according to the formula

$$\begin{bmatrix} 1 \\ -\mathbf{w} \end{bmatrix} \begin{pmatrix} \Delta t \\ \Delta\mathbf{x}_B \end{pmatrix} = \begin{pmatrix} p \\ 0 \end{pmatrix}, \tag{6.5}$$

where p is a positive real number - a parameter proportional to time. The dilation factor has been omitted because it only makes sense when reference is made to the coordinates of the second frame (in this case, the proper time of the object named 'B point').

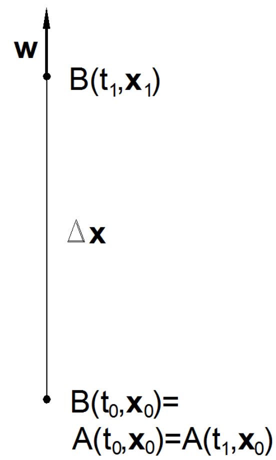

Figure 6.3: Vector $\Delta\mathbf{x}$ increasing at $\mathbf{w}$ velocity in the rest frame.

The first way. Vector AB in a moving OX' frame, where points A and B are simultaneous. After relocating to the frame moving at the $-\mathbf{v}$ speed the observer gets an image as in Fig. 6.4.

In the primed frame, the movement of point B will be described by the following relationship:

$$\mathbb{X}'_B = \begin{pmatrix} \Delta t'_B \\ \Delta\mathbf{x}'_B + i\mathbf{y}'_B \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t \\ \Delta\mathbf{x}_B \end{pmatrix} = \frac{\Delta t}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{w} \end{bmatrix} \tag{6.6}$$

The observer has moved to the primed frame, so he can observe the world only in this frame. Since the time t' applies in the OX' frame, he sees the point B primed coordinates changes at this time. Therefore, the above equation should be transformed in such a way as to obtain a proportional dependence of $\Delta\mathbf{x}'$ on t'. We transfer all redundant factors to the other side of the equality and replace them with the p_B parameter.

$$\frac{1}{1-w^2} \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ -\mathbf{w} \end{bmatrix} \begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t'_B \\ \Delta\mathbf{x}'_B + i\mathbf{y}'_B \end{pmatrix} = \begin{pmatrix} \Delta t \\ 0 \end{pmatrix} \tag{6.7}$$

$$\frac{1}{1-w^2} \frac{1+\mathbf{vw}}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ -\frac{\mathbf{w}+\mathbf{v}}{1+\mathbf{vw}} + i\,\frac{\mathbf{w}\times\mathbf{v}}{1+\mathbf{vw}} \end{bmatrix} \begin{pmatrix} \Delta t'_B \\ \Delta\mathbf{x}'_B + i\mathbf{y}'_B \end{pmatrix} = \begin{pmatrix} \Delta t \\ 0 \end{pmatrix} \tag{6.8}$$

$$\begin{bmatrix} 1 \\ -\mathbf{w}'_{re} + i\mathbf{w}'_{im} \end{bmatrix} \begin{pmatrix} \Delta t'_B \\ \Delta\mathbf{x}'_B + i\mathbf{y}'_B \end{pmatrix} = \begin{pmatrix} p_B \\ 0 \end{pmatrix} \tag{6.9}$$

The motion of point B in the primed frame is determined by the vector equation, hence

$$\begin{cases} \Delta\mathbf{x}'_B - \mathbf{w}'_{re}\Delta t'_B - \mathbf{w}'_{im} \times \mathbf{y}' = 0 \\ \mathbf{y}' + \mathbf{w}'_{im}\Delta t'_B = 0 \end{cases} \tag{6.10}$$

After substituting y' from the second equation to the first one, the third term disappears and we obtain

$$\Delta\mathbf{x}'_B = \frac{\mathbf{v}+\mathbf{w}}{1+\mathbf{vw}}\Delta t'_B \tag{6.11}$$

Point A in the OX frame is resting, which in the OX' frame is visible as a movement according to the formula

$$\mathbb{X}'_A = \begin{pmatrix} \Delta t'_A \\ \Delta\mathbf{x}'_A \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t \\ 0 \end{pmatrix} \tag{6.12}$$

In the primed frame, the same as before, we calculate the dependence $\Delta\mathbf{x}'_A$ on $\Delta t'_A$.

$$\begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t'_A \\ \Delta\mathbf{x}'_A \end{pmatrix} = \begin{pmatrix} p_A \\ 0 \end{pmatrix} \tag{6.13}$$

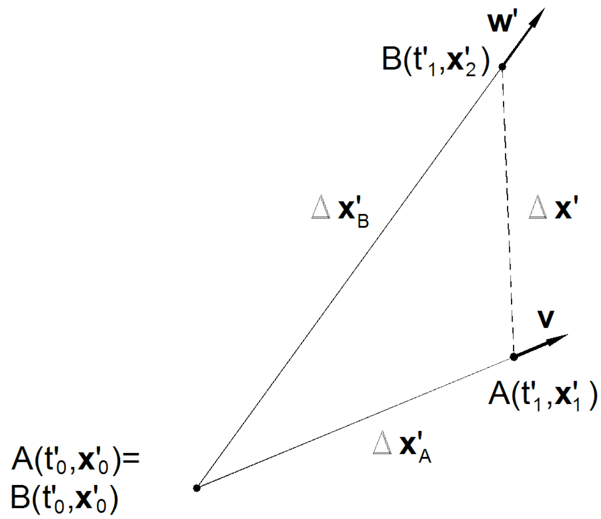

Figure 6.4: The real component of the AB vector in a frame moving at the speed -**v**. Simultaneous interpretation.

from above we get the dependence of interest to us

$$\Delta\mathbf{x}'_A = \mathbf{v}\Delta t'_A \tag{6.14}$$

In the primed frame, the observer sees points A and B simultaneously, i.e. $\Delta t'_A = \Delta t'_B = \Delta t'$. Simultaneity in both frames does not correspond, but in this case we do not care. The movement of points A and B in the primed frame is determined by the formulas

$$\Delta\mathbf{x}'_B = \frac{\mathbf{v}+\mathbf{w}}{1+\mathbf{vw}}\Delta t' \quad , \quad \Delta\mathbf{x}'_A = \mathbf{v}\Delta t' \tag{6.15}$$

The image of a real vector is a complex vector consisting of parts

$$\text{real} \quad \Delta\mathbf{x}' = \frac{\mathbf{w}-\mathbf{v}(\mathbf{vw})}{1+\mathbf{vw}}\Delta t' \qquad \text{and imaginary} \quad \mathbf{y}' = \frac{\mathbf{v}\times\mathbf{w}}{1+\mathbf{vw}}\Delta t'. \tag{6.16}$$

Simultaneous points in a moving frame are shifted in time in the rest frame.

$$\mathbb{X}' = \begin{pmatrix} 0 \\ \Delta\mathbf{x}' + i\mathbf{y}' \end{pmatrix} = \frac{1}{\sqrt{1-v^2}}\begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix}\begin{pmatrix} \Delta t \\ \Delta\mathbf{x} \end{pmatrix} = \frac{1}{\sqrt{1-v^2}}\begin{pmatrix} \Delta t + \mathbf{v}\Delta\mathbf{x} \\ \Delta\mathbf{x} + \mathbf{v}\Delta t + i\mathbf{v}\times\Delta\mathbf{x} \end{pmatrix} \tag{6.17}$$

The scalar equation shows that $\Delta t = -\mathbf{v}\Delta\mathbf{x}$. After substituting them into the vector equation, we get

$$\Delta\mathbf{x}' = \frac{1}{\sqrt{1-v^2}}[\Delta\mathbf{x} - \mathbf{v}(\mathbf{v}\Delta\mathbf{x})] \tag{6.18}$$

After obtaining a dot and cross product of the above formula from multiplication by the vector **v**, we get two conditions that the $\Delta\mathbf{x}'$ and $\Delta\mathbf{x}$ vectors must satisfy:

$$\begin{aligned} \Delta\mathbf{x}'\mathbf{v} &= (\Delta\mathbf{x}\mathbf{v})\sqrt{1-v^2} \\ \Delta\mathbf{x}'\times\mathbf{v} &= \frac{\Delta\mathbf{x}\times\mathbf{v}}{\sqrt{1-v^2}} \end{aligned}$$

Analysing the above system of equations, one might think that the image of an object in motion may be deformed. However, the relation of parallelism and perpendicularity to the direction of motion is preserved (If $\Delta\mathbf{x}\perp\mathbf{v}$, then $\Delta\mathbf{x}'\perp\mathbf{v}$ and if $\Delta\mathbf{x}\parallel\mathbf{v}$, then $\Delta\mathbf{x}'\parallel\mathbf{v}$). Still, we must remember that we always have an imaginary path and shifts in time, and we may perceive spherical points as a sphere. We do not know this at this point, but we will explain it further when we consider more compound systems.

The second way. The AB vector described in the OX' frame at the assumption of the simultaneity of the vector ends in the OX frame.

Let us describe the phase interval between these points in the OX system and the corresponding interval in the OX' system. For point B it is the following relationship:

$$\mathbb{X}'_B = \begin{pmatrix} \Delta t'_B \\ \Delta \mathbf{x}'_B + i\mathbf{y}_B \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t \\ \Delta \mathbf{x}_B \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} \Delta t + \mathbf{v}\Delta \mathbf{x}_B \\ \Delta \mathbf{x}_B + \mathbf{v}\Delta t + i\mathbf{v} \times \Delta \mathbf{x}_B \end{pmatrix} \tag{6.19}$$

After taking Δt out, we get the complex velocity of point B in the moving frame, which is the same as before.

$$\mathbb{X}'_B = \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t \\ \Delta \mathbf{x}_B \end{pmatrix} = \tag{6.20}$$

$$\frac{\Delta t}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{w} \end{bmatrix} = \frac{(1+\mathbf{vw})\Delta t}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \frac{\mathbf{v}+\mathbf{w}}{1+\mathbf{vw}} + i\,\frac{\mathbf{v}\times\mathbf{w}}{1+\mathbf{vw}} \end{bmatrix} = g \begin{bmatrix} 1 \\ \mathbf{w}'_{re} + i\mathbf{w}'_{im} \end{bmatrix} \begin{pmatrix} \Delta t \\ 0 \end{pmatrix},$$

where g is the new dilation factor. We did not write the factor γ adopted in the textbooks, due to the assumption made at the beginning of the monograph that we reserve Greek letters for complex quantities.

Point A in the OX frame is standing, which in the OX' frame is visible as movement according to the formula

$$\mathbb{X}'_A = \begin{pmatrix} \Delta t'_A \\ \Delta \mathbf{x}'_A \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t \\ 0 \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} \Delta t \\ \mathbf{v}\Delta t \end{pmatrix} \tag{6.21}$$

We calculate an image of the interval $\mathbb{X} = \mathbb{X}_B - \mathbb{X}_A$. In the original frame it is a spatial vector! Although it moves in time and grows simultaneously, the beginning and the end of it are at the same moment (they are simultaneous), so it is a purely spatial vector, and thus its determinant is negative.

$$\mathbb{X}_B - \mathbb{X}_A = \begin{pmatrix} \Delta t \\ \Delta \mathbf{x}_B \end{pmatrix} - \begin{pmatrix} \Delta t \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \Delta \mathbf{x} \end{pmatrix} \tag{6.22}$$

Starting from the above formulas, we get the vector image in the OX' frame:

$$\mathbb{X}' = \mathbb{X}'_A - \mathbb{X}'_B = V(\mathbb{X}_A - \mathbb{X}_B)$$

or

$$\mathbb{X}' = \begin{pmatrix} \Delta t'_B - \Delta t'_A \\ \Delta \mathbf{x}'_B - \Delta \mathbf{x}'_A + i\mathbf{y}' \end{pmatrix} = \begin{pmatrix} \Delta t'_{AB} \\ \Delta \mathbf{x}' + i\mathbf{y}' \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} \begin{pmatrix} 0 \\ \Delta \mathbf{x} \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} \mathbf{v}\Delta \mathbf{x} \\ \Delta \mathbf{x} + i\mathbf{v}\times\Delta \mathbf{x} \end{pmatrix} \tag{6.23}$$

Here we see (Fig. 6.5) that the image of the space vector, as before, is the space-time interval (the beginning and the end may be at a different moment). It has a purely geometric sense. As in the previous example, we emphasize that it has no physical significance because its determinant is negative.

If $\mathbf{v} \perp \Delta \mathbf{x}$ then

$$\mathbb{X}' = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} \mathbf{0} \\ \Delta \mathbf{x} + i\mathbf{v}\times\Delta \mathbf{x} \end{pmatrix}, \tag{6.24}$$

and if $\mathbf{v} \parallel \Delta \mathbf{x}$ then

$$\mathbb{X}' = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} \mathbf{v}\Delta \mathbf{x} \\ \Delta \mathbf{x} \end{pmatrix} \tag{6.25}$$

The real vector of the image is traditionally interpreted as the difference between the coordinates of two points and it is proportional (with the dilation factor) to the original vector. The scalar is time. We also see that the ends of the vectors $\Delta \mathbf{x}'$ (A points) are not at the same moment. At the same moment as B point there are points lying on the plane containing B and perpendicular to the direction of motion of B point. The ends that are before the common onset are time lagged, while those that are behind it, precede it in time. We also get an imaginary vector, which is a property of a real vector related to its dynamics.

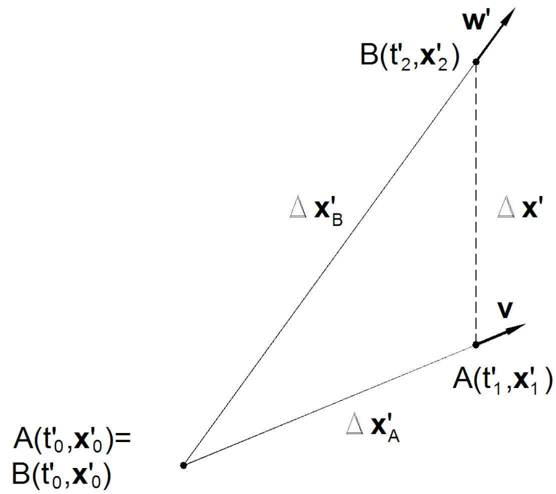

Figure 6.5: The real component of the AB vector in a frame moving at the speed -**v**

6.3 Swelling spheres

Let us now pass to the third case (Fig. 6.6). In the rest frame, from the point O($\mathbf{x}_0$) at the t_0 moment a soap bubble (sphere) pops out and its radius grows steadily with the relativistic speed of $w = 0{,}4$. The observer describes the movement of the points up to a moment t.

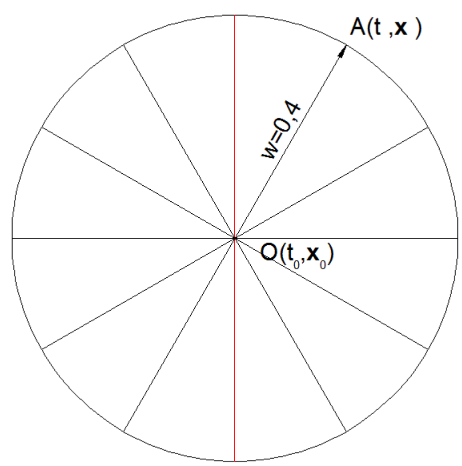

Figure 6.6: Swelling real sphere

In the OX rest frame, coordinates of four-vectors [point $A(t, \mathbf{x})$ - origin of sphere $O(t_0, \mathbf{x}_0)$] are

$$\mathbb{X} = \begin{pmatrix} \Delta t \\ \Delta \mathbf{x} \end{pmatrix} = \begin{pmatrix} t - t_0 \\ \mathbf{x} - \mathbf{x}_0 \end{pmatrix} \quad , \qquad \text{where} \quad \begin{bmatrix} 1 \\ -\mathbf{w} \end{bmatrix} \begin{pmatrix} \Delta t \\ \Delta \mathbf{x} \end{pmatrix} = \begin{pmatrix} p \\ 0 \end{pmatrix}. \tag{6.26}$$

In the OX' frame moving at relativistic speed $\mathbf{v}$ ($v = 0.8$) we denote the right equation as:

$$\begin{bmatrix} 1 \\ -\mathbf{w} \end{bmatrix} \begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t' \\ \Delta \mathbf{x}' + i\mathbf{y}' \end{pmatrix} = \begin{bmatrix} 1 + \mathbf{v}\mathbf{w} \\ -\mathbf{v} - \mathbf{w} + i\mathbf{w} \times \mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t' \\ \Delta \mathbf{x}' + i\mathbf{y}' \end{pmatrix} = \begin{pmatrix} p' \\ 0 \end{pmatrix} \quad , \qquad \text{where } p' = p\sqrt{1 - v^2} \tag{6.27}$$

In the OX' frame we describe the sphere at t' time, so we are not interested in the scalar equation as a reference to t time. We are interested in the real part of the vector equation. After replacing the compound velocity paravector to the right side, we have:

$$\begin{pmatrix} \Delta t' \\ \Delta \mathbf{x}' + i\mathbf{y}' \end{pmatrix} = \frac{p'}{(1+\mathbf{vw})^2 - (\mathbf{v}+\mathbf{w})^2 + (\mathbf{v}\times\mathbf{w})^2} \begin{bmatrix} 1+\mathbf{vw} \\ \mathbf{v}+\mathbf{w}+i\mathbf{v}\times\mathbf{w} \end{bmatrix} \tag{6.28}$$

From the scalar formula we calculate the dependence of p' on $\Delta t'$ and we insert it into the vector formula, from where we get

$$\Delta \mathbf{x}' + i\mathbf{y}' = \frac{\mathbf{v}+\mathbf{w}+i\mathbf{v}\times\mathbf{w}}{1+\mathbf{vw}}\Delta t' \tag{6.29}$$

For a bundle of vectors $\mathbf{w}$ and a constant vector $\mathbf{v}$, the ends of the real vectors $\Delta\mathbf{x}'$, proportional to $\mathbf{w}' = (\mathbf{v}+\mathbf{w})/(1+\mathbf{vw})$, form a flattened sphere (Fig. 6.7).

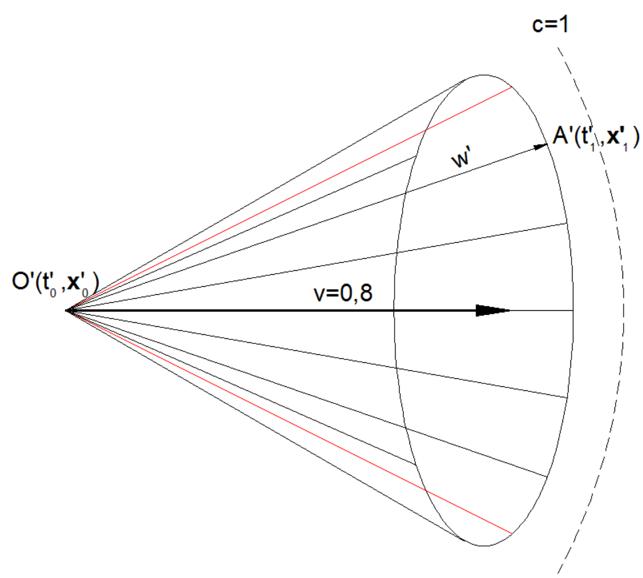

Figure 6.7: The real components of the resultant velocities on the X'O'Y' plane

However, it should be remembered that the deformation of the real component does not mean that the complex sphere is deformed because, as we remember from the parawector algebra, scalar product is invariant. Vectors that have not undergone relativistic deformation are marked in red. Additionally, Figure 6.7 is not a photograph because the points on the ellipsoid are shifted in time. So it is possible that if we took a photo, we would see... a ball in it. And it is indeed so, the difference being that the points we photographed in the moving frame were not simultaneous in the rest one. For this example the image of the distribution of imaginary components of compound velocities depending on the angle between the $\mathbf{w}$ and $\mathbf{v}$ vectors is shown in Figure 6.8. Note that all imaginary vectors lie in one direction perpendicular to the plane defined by the $\mathbf{v}$ and $\mathbf{w}$ vectors.

Let's go even further. In the rest frame, from point $O(t, \mathbf{x}_0)$ another bubble pops out after the first one and they both grow steadily at the relativistic speed of $\mathbf{w} = 0.4$ (Fig. 6.9). The first bubble (a) popped out at a t_0 moment and the second one (b) a little later, at t_1. The observer writes the equation at t_2 moment. In the OX rest frame, the first sphere (a) is described by the equation

$$\mathbb{X}_a = \begin{pmatrix} \Delta t_a \\ \Delta \mathbf{x}_a \end{pmatrix}, \qquad \text{where} \quad \begin{bmatrix} 1 \\ -\mathbf{w} \end{bmatrix} \begin{pmatrix} \Delta t_a \\ \Delta \mathbf{x}_a \end{pmatrix} = \begin{pmatrix} p_a \\ 0 \end{pmatrix} \quad \text{, and} \quad \Delta t_a = t_2 - t_0 \tag{6.30}$$

The second sphere (b) is similarly described by the equation:

$$\mathbb{X}_b = \begin{pmatrix} \Delta t_b \\ \Delta \mathbf{x}_b \end{pmatrix}, \qquad \text{where} \begin{bmatrix} 1 \\ -\mathbf{w} \end{bmatrix} \begin{pmatrix} \Delta t_b \\ \Delta \mathbf{x}_b \end{pmatrix} = \begin{pmatrix} p_b \\ 0 \end{pmatrix} \quad \text{, and} \quad \Delta t_b = t_2 - t_1 \text{ and } t_0 < t_1 < t_2 \tag{6.31}$$

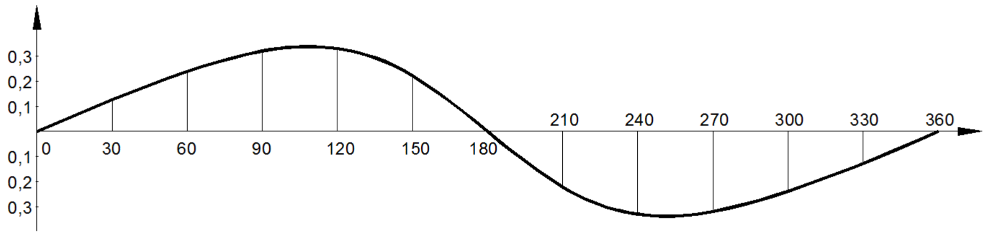

Figure 6.8: Imaginary components of resultant velocities in the O'Z' axis

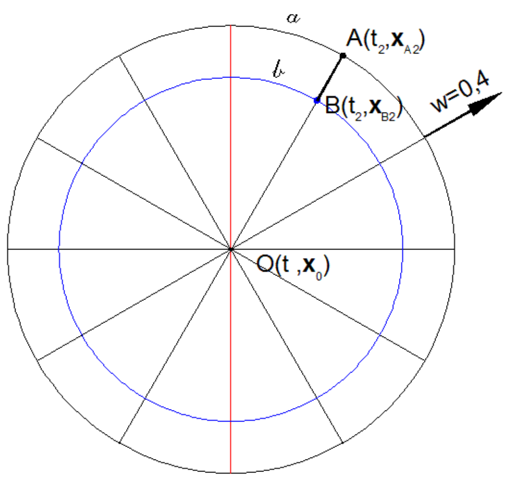

Figure 6.9: Concentrically swelling spheres in a rest frame

The segment between points A and B moving in the same direction and at the same moment is described by the following equation

$$\begin{pmatrix} 0 \\ \mathbf{x}_{A2}-\mathbf{x}_{B2} \end{pmatrix} = \begin{pmatrix} t_2-t_0 \\ \mathbf{x}_{A2}-\mathbf{x}_{O0} \end{pmatrix} - \begin{pmatrix} t_2-t_1 \\ \mathbf{x}_{B2}-\mathbf{x}_{O1} \end{pmatrix} - \begin{pmatrix} t_1-t_0 \\ 0 \end{pmatrix} = \begin{pmatrix} \Delta t_a \\ \Delta \mathbf{x}_a \end{pmatrix} - \begin{pmatrix} \Delta t_b \\ \Delta \mathbf{x}_b \end{pmatrix} - \begin{pmatrix} t_1-t_0 \\ 0 \end{pmatrix} \tag{6.32}$$

where:

$\begin{pmatrix} t_2-t_0 \\ \mathbf{x}_2-\mathbf{x}_0 \end{pmatrix}$ - are coordinates of the moving point A,

$\begin{pmatrix} t_2-t_1 \\ \mathbf{x}_1-\mathbf{x}_0 \end{pmatrix}$ - are coordinates of the moving point B,

$\begin{pmatrix} t_1-t_0 \\ 0 \end{pmatrix}$ - is the waiting time for b bubble to pop out,

$\mathbf{x}_{O0} = \mathbf{x}_{O1} = \mathbf{x}_{O2} = \mathbf{x}_0$, because the O point is standing still.

Going to the OX' frame moving at the speed of $-\mathbf{v}$, we transform the above equation:

$$\frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix}\begin{pmatrix}0\\ \mathbf{x}_{A2}-\mathbf{x}_{B2}\end{pmatrix}= \tag{6.33}$$
$$\frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix}\begin{pmatrix}t_2-t_0\\ \mathbf{x}_{A2}-\mathbf{x}_{O0}\end{pmatrix}-\frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix}\begin{pmatrix}t_2-t_1\\ \mathbf{x}_{B2}-\mathbf{x}_{O1}\end{pmatrix}-\frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix}\begin{pmatrix}t_1-t_0\\ 0\end{pmatrix},$$

hence

$$\begin{pmatrix}\Delta t'_d\\ \Delta\mathbf{x}'_d+i\mathbf{y}'_d\end{pmatrix}=\begin{pmatrix}\Delta t'_a\\ \Delta\mathbf{x}'_a+i\mathbf{y}'_a\end{pmatrix}-\begin{pmatrix}\Delta t'_b\\ \Delta\mathbf{x}'_b+i\mathbf{y}'_b\end{pmatrix}-\begin{pmatrix}\Delta t'_c\\ \Delta\mathbf{x}'_c\end{pmatrix} \tag{6.34}$$

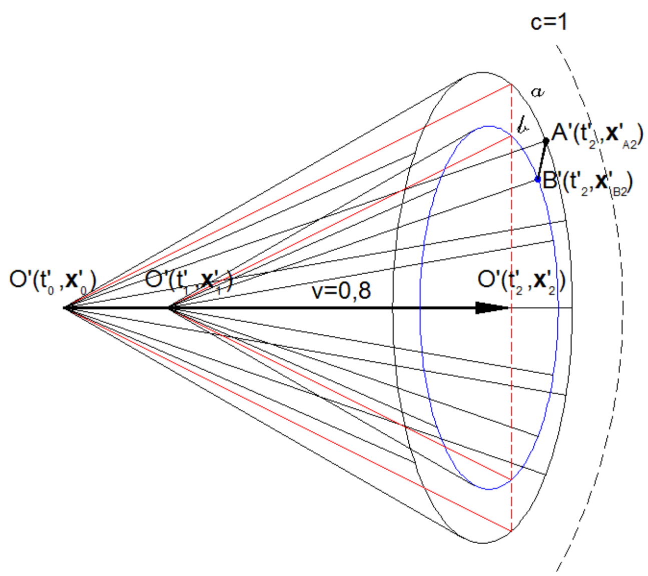

Figure 6.10: The image in the OX' moving frame of the real components of the spheres that were concentric in the rest frame

The formula (6.34) shows that in the OX' frame the $\Delta\mathbf{x}'_d$ vectors are the same [2] as in the OX frame (6.32), but the beginnings and ends of these vectors in the OX frame were simultaneous, while in the OX' frame they differ in time! The time difference between the end and the beginning of the $\Delta\mathbf{x}'_d$ vector is proportional to the product of $(\mathbf{x}_{A2}-\mathbf{x}_{B2})\mathbf{v}$. So, there was a deformation over time - getting out of sync. Thus Figure 6.10 does not show the image that would be seen by an experimenter taking a photograph in a moving frame. The observer in the OX frame 'photographs' what he 'sees' and he 'sees' the same points but at a different time than the observer from the OX frame. The 'photograph' of the *prime* observer seems to show a spatial deformation consisting in the mutual shift of the a and b spheres, because the camera is not able to 'see' at the same moment what is not simultaneous.

Thus, NOTE !: **What the O and O' observers 'photograph' is not the same**.

The desynchronization takes place along the direction of the observer's movement, but it can be seen from the drawing that the AB line was twisted in the perpendicular directions. The real space deformation took place and it is related to the spatial imaginary component $\mathbf{v}\times\Delta\mathbf{x}$. The direction of the $\overrightarrow{AB}$ vector follows the speed $\mathbf{w}$, and the direction of the $\overrightarrow{A'B'}$ vector follows the $\mathbf{w}'$ velocity. The twist of the $\overrightarrow{A'B'}$ vector in relation to the $\overrightarrow{AB}$ vector is proportional to the vector product $\mathbf{w}'\times\mathbf{w}=(\mathbf{w}+\mathbf{v})\times\mathbf{w}=\mathbf{v}\times\mathbf{w}$. The deformation is all the more interesting as it concerns not only space but also time. As we said before, the points that we 'see' simultaneously

[2] We ignore the dilation factor, because the (6.32) and (6.34) formulas are equivalent to each other, and by analysing the spatial image in the primed frame, we calculate the formula (6.34) where this factor is reduced

in a moving frame are not simultaneous in a rest frame. The deformation is temporal and spatial, but note that **no deformation occurred in the complex space-time** because the scalar product remains unchanged. So what is the interpretation of the imaginary vector? Since we think in terms of real space, we can assume that the imaginary vector is related to real quantities deformation. By observing a single point, we are not able to see it. We see it only by observing the vectors and doing it in time. We will explain this in detail with an example in the next section.

6.4 Movement of a point with elastic rebound

Let us imagine we have a sphere shaped laboratory. The wall of the laboratory is perforated (it is a sieve). In the very center of the laboratory we place an explosive charge. At the t_0 moment the charge explodes. The particles dissipate evenly in all directions at the $w = 0.4$ velocity. Some of them bounce elastically from the walls and return to the center of the sphere, and some of them fly freely into space through the holes in the wall. We end the observation when the rebounded particles meet in the center of the sphere. We will show how the real image of the experiment will look like when viewed from a frame moving at $v = 0.8$ speed. Figure 6.11 shows the experiment observed in the laboratory frame. Some of the particles bounce elastically and return towards the center (purple dashed lines), while others (which have passed through the sieve) escape into space (red dashed lines). The hitting of the walls is simultaneous and it leaves a green mark.

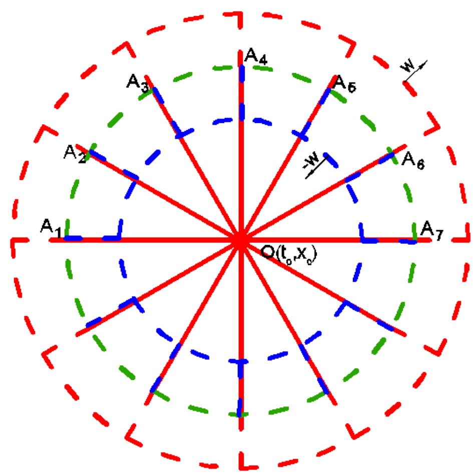

Figure 6.11: An explosion inside a perforated sphere in a rest frame

In Figure 6.11 the directions every 30 degrees have been distinguished in order to show how in the moving frame the time shift of the simultaneous phenomenon in the rest frame, such us an impact against the laboratory wall, 'is seen'. **Note:** In Figures 6.12 - 6.17 only real vectors are plotted.

The gray ellipse on the left shows the real part of the laboratory at the moment of the explosion (at t_0). The lab at the t_n moment is shown by black ellipses. Continuous red lines mark the paths of movement of the selected particles before the rebound. Tracks after the rebound are purple dashed lines. Paths of the particles that passed through the sieve are dashed red lines. The continuous part of the ellipse in red represents the wave front before reaching the laboratory wall. The dashed red line shows the particles that have passed through the sieve. The dashed purple line shows the blast front rebounded from the wall. The dashed green line marks the trace of particles bouncing off the laboratory wall. As time passes, the line draws a circle and points from A_1 to A_7 appear. We end the observation when the rebounded part of the blast front concentrates in the center of the laboratory, i.e. at the t_f moment. The lines were drawn with AutoCAD allowing for high precision, and the necessary calculations were made using formula (6.29). It should be noted that despite the deformation of the actual laboratory and the wave front, the trace of the collision of particles with the wall does not indicate any spatial deformation of our laboratory. We can therefore adopt a different interpretation of the obtained results:

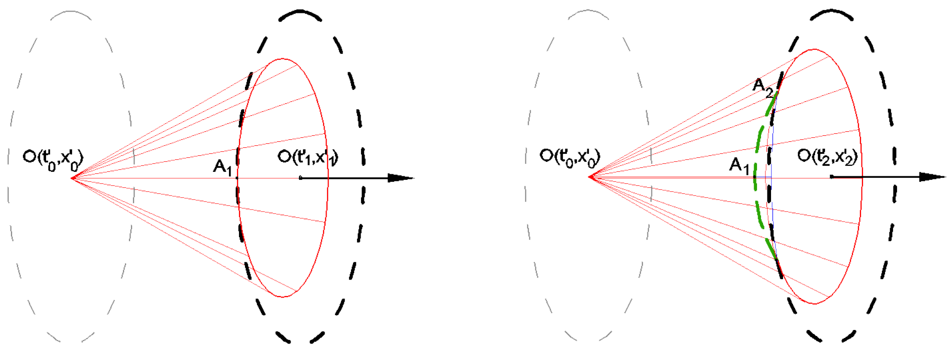

Figure 6.12: Moments t_1 and t_2. In the moving frame the first particles are reaching the laboratory wall.

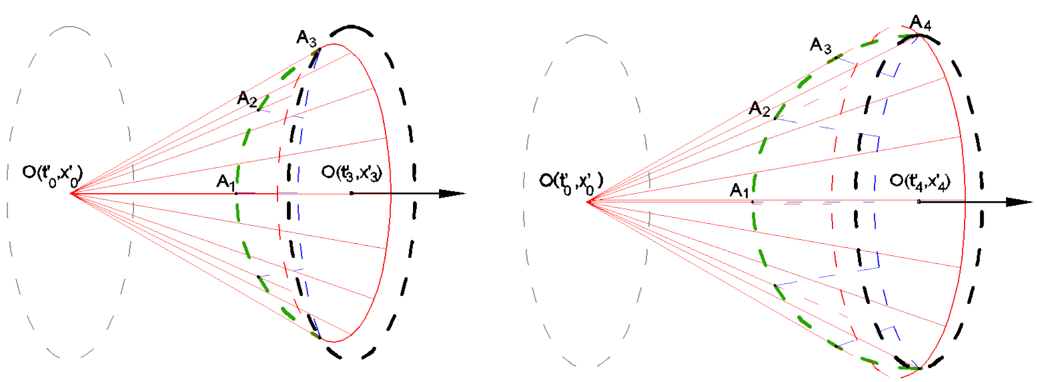

Figure 6.13: Moments t_3 and t_4

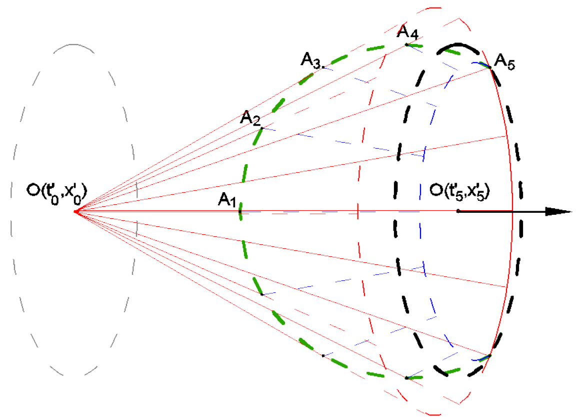

Figure 6.14: Moment t_5

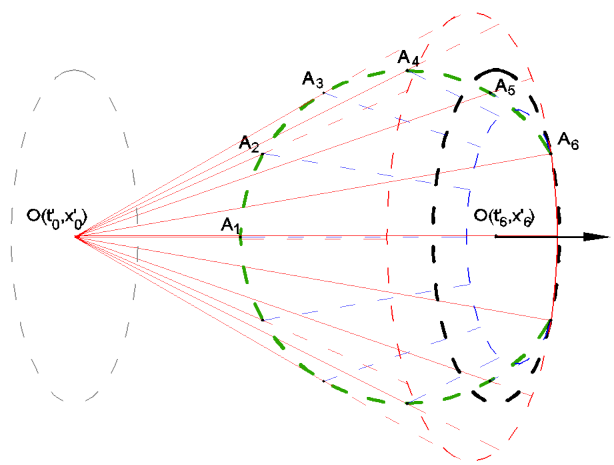

Figure 6.15: Moment t_6

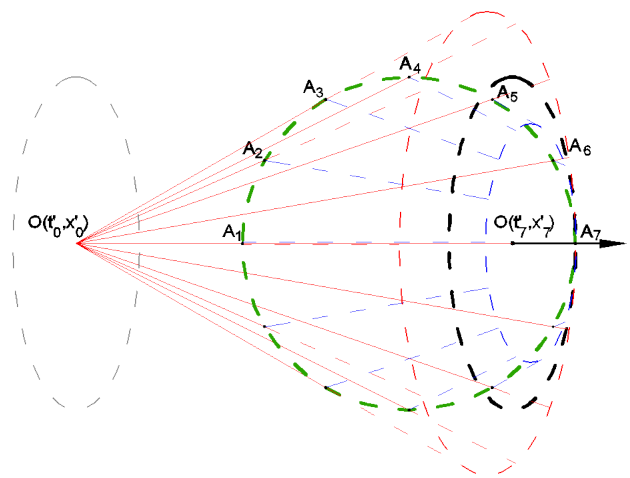

Figure 6.16: The last particles reach the wall. Moment t_7

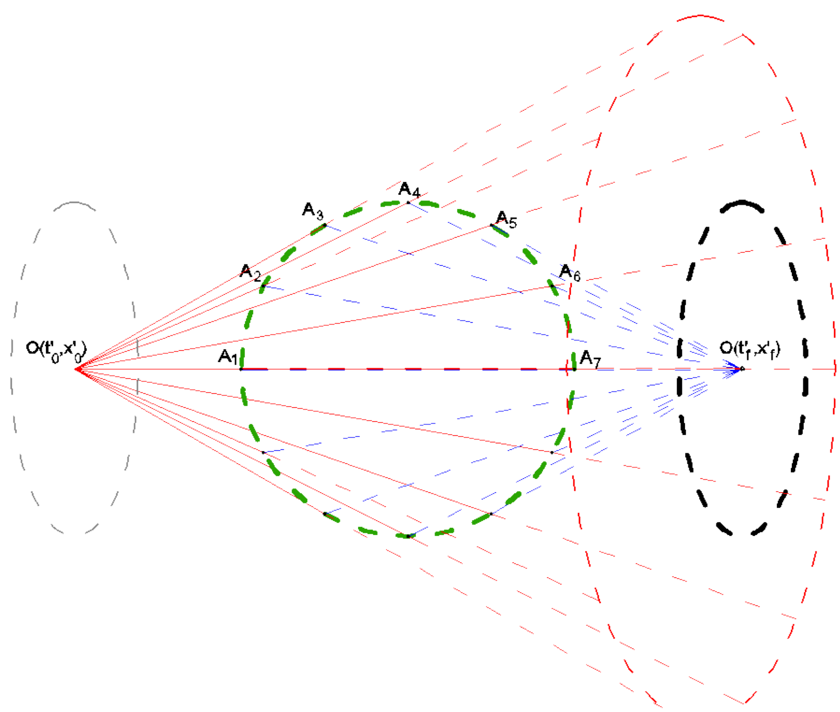

Figure 6.17: All the particles that bounced off the wall meet in the center of the laboratory. Moment t_f

The shape does not deform, but the individual fragments of the observed laboratory shift in time, that is, the deformation of the shape can be treated as an illusion resulting from our understanding of simultaneity. The moving observer should see the collision point moving in time and space, but is unable to do so because the collision point (as shown at the beginning of the chapter) travels faster than the speed of light. We emphasize here that we are not talking about a material point, but a geometric point. Each of the particles (material points) moves in a straight line at a speed lower than light. What the observer is able to register is the trace after the rebound, and it is an undeformed sphere. Once again, it has been confirmed that there is no deformation in complex space-time, but there is a definite necessity to abandon the use of the concept of simultaneity.

An attentive reader must have noticed that we use the words 'we observe', 'we see', 'we photograph'. These activities are related to the reception of light energy by an eye or a photographic film. We describe mathematical formulas with these words. We are aware of this enormous inaccuracy, but in some intuitive way we must name the theoretical results corresponding to the well known concept. The inaccuracy is even greater as the energy (which is the information carrier) not only moves at a limited speed, but is also always real, while we consider complex concepts. To indicate that we mean theoretical concepts, we mark them with apostrophes.

6.5 Apparent exceeding the speed of light

We will now consider a simple example that will show us that in complex space it is possible to apparently exceed the speed of light, but an observer performing measurements and located at the light source is unable to prove it.

Let's imagine a situation where we have two sources of directional light. One lamp O_R determines the origin of the rest frame. The second O lamp moves in the direction of the X axis and is in the center of the moving frame. In the rest frame, the A observer is on the Y axis. In the moving frame, the mirror M is on the Y axis. When the lamps pass each other, both lamps emit a light-beam directed toward the Y axis. Using the POT formulas, we will calculate how A observer will describe the motion of the light-beam sent from O lamp.

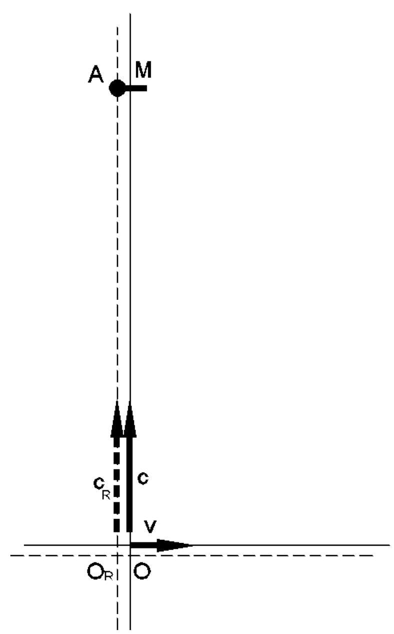

Figure 6.18: At the moment of meeting, lamps O and O1 emit light-beams in the direction perpendicular to the movement of O lamp

The paravector equation of motion of a light-beam in the A traveler's frame

$$\begin{bmatrix}1\\-\mathbf{c}\end{bmatrix}\begin{pmatrix}\Delta t\\\Delta\mathbf{x}\end{pmatrix}=0 \tag{6.35}$$

which is equivalent to a system of equations

$$\Delta t-\mathbf{c}\Delta\mathbf{x}=0 \tag{6.36}$$
$$\Delta\mathbf{x}-\mathbf{c}\Delta t=0 \tag{6.37}$$
$$\mathbf{c}\times\Delta\mathbf{x}=0 \tag{6.38}$$

This is a system of dependent equations because $\det\begin{bmatrix}1\\-\mathbf{c}\end{bmatrix}=0$. This system cannot be solved, but its elements can be interpreted.

The observer moves in the traveler's frame according to the equation

$$\begin{bmatrix}1\\\mathbf{v}\end{bmatrix}\begin{pmatrix}\Delta t\\\Delta\mathbf{x}\end{pmatrix}=\begin{pmatrix}\Delta t+\mathbf{v}\Delta\mathbf{x}\\\Delta\mathbf{x}+\mathbf{v}\Delta t\end{pmatrix}=p,\qquad \text{where } p\in R_+\setminus\{0\} \tag{6.39}$$

Since the vectors $\mathbf{v}$ and $\Delta\mathbf{x}$ are parallel, I don't write a cross product of vectors $\mathbf{c}$ and $\mathbf{v}$.

Below, I describe and comment on all the steps one by one.

I move the equation (6.35) to the observer frame.

$$\begin{bmatrix}1\\-\mathbf{c}\end{bmatrix}\frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\-\mathbf{v}\end{bmatrix}\frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\\mathbf{v}\end{bmatrix}\begin{pmatrix}\Delta t\\\Delta\mathbf{x}\end{pmatrix}=0 \tag{6.40}$$

Let's save

$$\frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\\mathbf{v}\end{bmatrix}\begin{pmatrix}\Delta t\\\Delta\mathbf{x}\end{pmatrix}=\begin{pmatrix}\Delta t'\\\Delta\mathbf{x}'+i\mathbf{y}'\end{pmatrix} \tag{6.41}$$

or

$$\Delta t' = \gamma(\Delta t+\mathbf{v}\Delta\mathbf{x}) \tag{6.42}$$
$$\Delta\mathbf{x}' = \gamma(\Delta\mathbf{x}+\mathbf{v}\Delta t) \tag{6.43}$$
$$\mathbf{y}' = \gamma\mathbf{v}\times\Delta\mathbf{x} \tag{6.44}$$

The equation (6.40) takes the form

$$\begin{bmatrix}1\\-\mathbf{c}\end{bmatrix}\frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\-\mathbf{v}\end{bmatrix}\begin{pmatrix}\Delta t'\\\Delta\mathbf{x}'+i\mathbf{y}'\end{pmatrix}=0 \tag{6.45}$$

I remove the Lorentz factor because it is not valid in the relationship between $\mathbf{x}$' and t', and the observer describes the experiment in such coordinates.

An important note here(!): Getting rid of the Lorentz factor is not possible in classical SR, because it does not act on all coordinates (contraction). In complex space-time this factor scales all of coordinates equally, so it has no significance for the relationships between these coordinates.

$$\begin{bmatrix}1\\-\mathbf{c}\end{bmatrix}\begin{bmatrix}1\\-\mathbf{v}\end{bmatrix}\begin{pmatrix}\Delta t'\\\Delta\mathbf{x}'+i\mathbf{y}'\end{pmatrix}=0, \tag{6.46}$$

hence

$$\begin{bmatrix}1\\-(\mathbf{c}+\mathbf{v})+i\mathbf{c}\times\mathbf{v}\end{bmatrix}\begin{pmatrix}\Delta t'\\\Delta\mathbf{x}'+i\mathbf{y}'\end{pmatrix}=0 \tag{6.47}$$

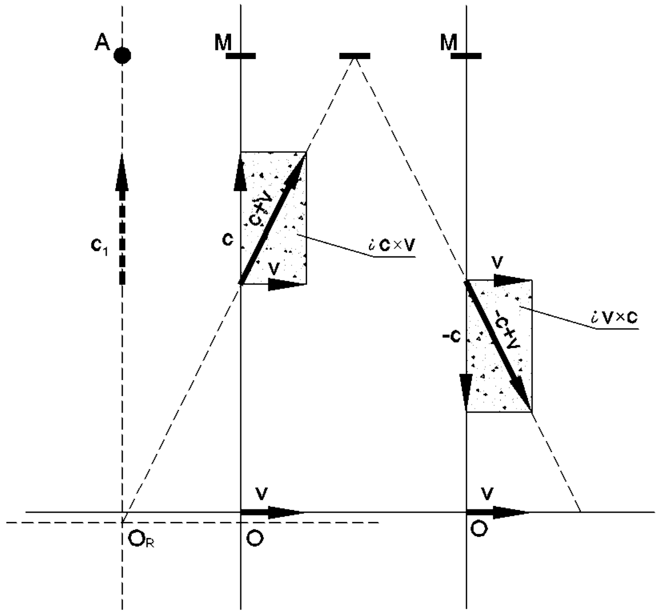

Figure 6.19: The position of the light-beams as the O lamp moves away from O_R the lamp

The paravector

$$\begin{bmatrix} 1 \\ -\mathbf{c}' \end{bmatrix} = \begin{bmatrix} 1 \\ -(\mathbf{c}+\mathbf{v}) + i\mathbf{c}\times\mathbf{v} \end{bmatrix} \tag{6.48}$$

is singular, so $|(\mathbf{c}+\mathbf{v}) + i\mathbf{c}\times\mathbf{v}| = 1$.

In this case, observer A will only see the flash of the lamp placed in his frame O_R and will not see the O lamp, and the light-beam from O lamp reflected from the M mirror will return to the source. Just like in Euclidean geometry, despite the fact that we are in agreement with the postulate of the constant speed of light. All thanks to imaginary dimensions, which are not independent of real dimensions, but complement them in such a way that the calculations balance out.

What is the imaginary component of the path? It is not the real path to be taken, so it is appropriately called 'imaginary path'. If we look around, we can find an analogy of our example with a well-known physical phenomenon. Let's imagine that the light beam is an electric charge, then an observer at point O with instruments for measuring electric and magnetic fields will measure the electric field of the receding charge but will not detect a magnetic field, whereas observer A will detect the presence of both electric and magnetic fields. The magnetic field is the electric field of this charge in the imaginary dimension.

6.6 Discussion

From the examples above, it can be seen that there are no mathematical contraindications to describe objects in complex space-time in a simply way. The problem is to accept that high velocity space-time is not real. By observing a single object, we are able to see neither complex space nor complex time. We should look for interpretations of imaginary spatial components by observing the relations between two mutually moving objects. We can find the imaginary time component only by examining the issues concerning at least three objects. But for now it is too early to explain this hypothesis. From the calculations done so far, it is obvious that the imaginary coordinate components are not independent of their real parts. Although the notation of transformation in complex space-time seems complicated, it can explain phenomena whose explanation in the classical STR led to paradoxes. The spatial deformation of the described objects is a consequence of the variability of time in the Lorentz transformation. In complex space-time, the relativistic transformation does not introduce any deformation. It was only in the twentieth century that people began to study phenomena taking place at speeds close to light, which is why they were beyond their imagination. Real space-time, which is rigid, is enough to properly describe slow phenomena. Complex space-time seems to be much more flexible and the description of the observed phenomenon can always be chosen so that it is orthogonal. Describing relativistic phenomena in real space-time has become very complicated, which rightly creates distrust in many people. The complex description, although completely foreign, is mathematically very simple, and most importantly: the relativistic transformations in complex space-time are orthogonal. However, we live and think in rigid real dimensions, so we should check if there is any way to reduce the complex description to the observer's real space-time - a way to project complex phenomena onto the observer's local real space-time. We'll cover this idea in more detail in Chapter 8.

Chapter 7

Maxwell's equations

In this chapter, several variants of the relativistic transformation of the wave equation system are analysed in terms of their application in the electric field theory, from which one variant is selected for further analysis. Inconsistencies with textbook physics are shown. Hypotheses are formulated on the Maxwell's equations and on the possibility of reducing complex orthogonal paravectors to real velocity paravectors.

Chapter 4 shows that the wave equation can be transformed in four basic ways (4.14)-(4.17), which makes it possible to take further discussion in different directions. However, we want to remain as close as possible to the formulas adopted in the electric field theory, so we will only consider equations (4.14) and (4.17) because only in these cases the wave equation is invariant due to POT and we obtain the magnetic field as a result of the transformation of the electric field. Looking at the obtained results, we can see that they are not entirely consistent with the classical theory because the potential and charge density are invariant quantities. For these two cases, a magnetic field can be interpreted, but something arises that does not exist according to the Lorenz gauge condition.

The transformation of equations (4.15) and (4.16) results in the invariance of the electric field intensity (the magnetic field has no mathematical justification!), but we obtain values such as a current density and vector potential. Also, the Lorenz gauge condition is justified. In order to obtain all the quantities occurring in TEM and satisfy the condition of invariance of the wave equation, we could multiply the equation (4.14) on the right hand by V^- which, however, would be an artificial move because we would obtain senseless Maxwell's equations. Besides, we could do this multiplication with any orthogonal paravector and we would get a mathematically correct expression, but with no physical sense. For example:

$$\partial^-(V^-\mathbb{E}) = \rho \qquad / \cdot V_1^- \qquad \text{hence} \qquad \partial^-(V^-\mathbb{E}V_1^-) = \rho V_1^-$$

Conclusion: **In our theory, it is impossible to reconcile the concepts of magnetic field with the current density in Maxwell's equations!**

Seeing such a drastic difference from the official theory, we could stop the search at this point and do something else, yet we will continue, curious about the next results. After all, these are just mathematical considerations. Scientists assume the existence of magnetic charges or superluminal velocities, which so far could not be determined. Why could we not try to modify the classical theories a bit, while sticking to the principles? In such a situation, we have no choice but to assume that there is no current density in Maxwell's equations. Consequently, we also have to give up the vector potential and the Lorenz gauge condition, which simplifies the theory very much. However, we are forced to introduce the scalar component of the intensity-induction 4-vector of the field. To make this task easier, let us note that such assumptions have historical and intuitive justification:

1. The vector potential was introduced in the search for general solutions to the wave equation when space seemed to be Euclidean, and only later it was given a physical meaning. In this work we assume that this structure is unknown. Only at the end will we try to define it.

2. We chose variants in which there is no vector potential in the field equations, but neither is there current density in Maxwell's equations. Looking at the transformation formulas known from STR, we come to the conclusion that it would not be bad if this was the case, because the charge density should be invariant just like the charge, since the shape would also be invariant.

3. The Lorenz gauge condition is a purely theoretical assumption used to define the abstract concept of electric field potential in such a way that the field is described by a wave equation. Since we are starting from the wave equation and the Lorenz gauge bothers us, we have a legitimate right to omit this assumption.

4. Experimentally, only the quantities related to the transmitted energy are available, because this is the information carrier, all other quantities are an abstraction used to build a transparent mathematical model.

Chapter 1 shows the system of electric field wave equations (1.3). It should be noted that it has a certain inaccuracy. Namely, the $\varphi(t,\mathbf{x})$ and $\rho(t,\mathbf{x})$ functions have their values at the same t and $\mathbf{x}$, but the source of this field is in a different time and place. The placement of the same $(t,\mathbf{x})$ arguments for the charge density and potential functions is inaccurate. In the field equations it should always be specified that the space-time distance $\mathbb{X} = X - X_0$ is between the source and the place of the given value of the field function, and it should be the argument, not the place in space-time itself.

The electrostatic field equations have the following form:

$$\partial\varphi(\mathrm{X}-X_0) = \begin{pmatrix} 0 \\ -\mathbf{E}(\mathrm{X}-X_0) \end{pmatrix} \qquad \text{and} \qquad \partial^-\begin{pmatrix} 0 \\ -\mathbf{E}(\mathrm{X}-X_0) \end{pmatrix} = \rho\,(X-\mathrm{X}_0) \tag{7.1}$$

or

$$\partial^-\varphi(\mathrm{X}-X_0) = \begin{pmatrix} 0 \\ \mathbf{E}(\mathrm{X}-X_0) \end{pmatrix} \qquad \text{and} \qquad \partial\begin{pmatrix} 0 \\ \mathbf{E}(\mathrm{X}-X_0) \end{pmatrix} = \rho\,(X-\mathrm{X}_0), \tag{7.2}$$

where X_0 is the place in space-time at which the source of the field is located, and X is the place where the field has a value specified by the function. These coordinates are related to each other by the following relationship:

$$\mathbb{X} = X - X_0 = \begin{pmatrix} t-t_0 \\ \mathbf{x}-\mathbf{x}_0 \end{pmatrix} = \Delta t \begin{bmatrix} 1 \\ \mathbf{c} \end{bmatrix}, \qquad \text{where } |\mathbf{c}| = 1 \tag{7.3}$$

In other words: $\mathbb{X}$ is a singular four-vector. In formulas (7.1) and (7.2) the straight letter denotes the coordinates of the point directly affects by the function value. The above formulas should be understood in the following way: $\mathbf{E}(\mathrm{X}-X_0)$ is the strength of the field at the point $\mathbf{x}$ and at t moment produced by charge $\rho\,(X-\mathrm{X}_0)$ located at the t_0 moment and at the point $\mathbf{x}_0$. The reader should always remember that the source of the field, as well as potentials, are spread over the entire space, but the domain is not the coordinates of the points, but their differences. From the mathematical point of view, only the space-time distance is important, that is the difference between coordinates as well as the condition that it should be a singular 4-vector.

At this point, further clarifications are needed to explain the notation we use. The 4-vector $\mathbb{X} = X - X_0$ should be understood as the difference between the coordinates of points in space-time only when these points are at rest relative to each other and the observer. In case the points are moving, we will write $\mathbb{X} = \Delta X + i\mathbf{y}$ and here $\mathbb{X}$ is a complex 4-vector. Such a distinction is necessary because everything seems to indicate that the coordinates of the points do not have imaginary components, while 4-vectors do. For this reason, it should be borne in mind that time is a quantum quantity and the mathematical structure of the space-time of motion is not an affine space. To simplify the considerations and facilitate the interpretation of the results, we assume for the time being that the velocity is described by the velocity paravector, i.e. it has a form of $V = \frac{1}{\sqrt{1-v^2}}\begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix}$.

The field described by the formulas (7.1) transforms according to the transformation $\mathbb{X}' = \mathbb{X}V$ (see the formula (4.14)):

Table 7.1	Rest frame	Moving frame
Charge density	$\rho(\mathbb{X})\in R$	invariant
Field strength	$\mathbb{E}(\mathbb{X})\in\{0\}\times R^3$	$\mathbb{E}'(\mathbb{X}'V^-)=V^-\mathbb{E}(\mathbb{X}'V^-)\in R\times C^3$
Potential	$\varphi(\mathbb{X})\in R$	invariant
Potential energy density	$\frac{\varphi(\mathbb{X})\rho(\mathbb{X})}{2}\in R$	invariant
Field energy density	$\frac{\mathbb{E}^*(\mathbb{X})\mathbb{E}(\mathbb{X})}{2}\in R_+$	$\frac{\mathbb{E}'^*(\mathbb{X}'V^-)\mathbb{E}'(\mathbb{X}'V^-)}{2}=\frac{\mathbb{E}^*(\mathbb{X}'V^-)V^-V^-\mathbb{E}(\mathbb{X}'V^-)}{2}\in R_+\times R^3$

However, field (7.2) transforms according to $\mathbb{X}'=V\mathbb{X}$ (see formula (4.17)):

Table 7.2	Rest frame	Moving frame
Charge density	$\rho(\mathbb{X})\in R$	invariant
Field strength	$\mathbb{E}(\mathbb{X})\in\{0\}\times R^3$	$\mathbb{E}'(V^-\mathbb{X}')=V\mathbb{E}(V^-\mathbb{X}')\in R\times C^3$
Potential	$\varphi(\mathbb{X})\in R$	invariant
Potential energy density	$\frac{\varphi(\mathbb{X})\rho(\mathbb{X})}{2}\in R$	invariant
Field energy density	$\frac{\mathbb{E}(\mathbb{X})\mathbb{E}^*(\mathbb{X})}{2}\in R_+$	$\frac{\mathbb{E}'(V^-\mathbb{X}')\mathbb{E}'^*(V^-\mathbb{X}')}{2}=\frac{V\mathbb{E}(V^-\mathbb{X}')\mathbb{E}^*(V^-\mathbb{X}')V}{2}\in R_+\times R^3$

It is worth noting that the expressions in the above tables were selected in such a way that by inserting them according to the formulas (7.1) or (7.2) one gets Maxwell equations (modified without the Ampere part!).

First, we select the values from Table 7.2 and substitute them into the formula (7.2)

$$\partial\mathbb{E}_0=\begin{bmatrix}\frac{\partial}{\partial t}\\ \nabla\end{bmatrix}\begin{pmatrix}0\\ \mathbf{E}_0(\mathbb{X})\end{pmatrix}=\begin{pmatrix}\rho(\mathbb{X})\\ 0\end{pmatrix} \tag{7.4}$$

Based on formula (4.17), we have Maxwell's equations in the new frame

$$\partial'\mathbb{E}'=\begin{bmatrix}\frac{\partial}{\partial t'}\\ \nabla'\end{bmatrix}\left(\frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix}\begin{pmatrix}0\\ \mathbf{E}_0(V^-\mathbb{X}')\end{pmatrix}\right)=\begin{pmatrix}\rho(V^-\mathbb{X}')\\ 0\end{pmatrix},$$

where

$$\frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix}\begin{pmatrix}0\\ \mathbf{E}_0'\end{pmatrix}=\frac{1}{\sqrt{1-v^2}}\begin{pmatrix}\mathbf{v}\mathbf{E}_0'\\ \mathbf{E}_0'+i\mathbf{v}\times\mathbf{E}_0'\end{pmatrix}=\begin{pmatrix}e'\\ \mathbf{E}'+i\mathbf{B}'\end{pmatrix},$$

and 'prime' means the new coordinates. The modified Maxwell equations follow from the above

$$\begin{bmatrix}\frac{\partial}{\partial t}\\ \nabla\end{bmatrix}\begin{pmatrix}e\\ \mathbf{E}+i\mathbf{B}\end{pmatrix}=\begin{pmatrix}\rho\\ 0\end{pmatrix} \tag{7.5}$$

which gives

$$\begin{aligned}\frac{\partial e}{\partial t}+\nabla\mathbf{E}&=\rho & \nabla\mathbf{B}&=0\\ \nabla\times\mathbf{B}-\frac{\partial\mathbf{E}}{\partial t}&=\nabla e & \frac{\partial\mathbf{B}}{\partial t}+\nabla\times\mathbf{E}&=0\end{aligned}$$

Being in the new system, we give up the 'prime' signs, which helped to precisely describe the mathematical operations. We will call size e a **scalar induction** (by analogy to the vector potential) and it is 'the thing' which, according to the Lorenz gauge condition, does not exist in the current theory of the electric field. In Tables 7.1 and 7.2, the transformation formulas have real speeds. In the general case, where velocity is described by a complex orthogonal paravector, the scalar induction is a scalar complex function.

An energy density of the field is:

$$W=\frac{1}{2}\mathbb{E}\mathbb{E}^*=\frac{1}{2}\begin{pmatrix}e\\ \mathbf{E}+i\mathbf{B}\end{pmatrix}\begin{pmatrix}e\\ \mathbf{E}-i\mathbf{B}\end{pmatrix}=\begin{bmatrix}\frac{e^2+E^2+B^2}{2}\\ e\mathbf{E}+\mathbf{E}\times\mathbf{B}\end{bmatrix} \tag{7.6}$$

So, we can write the above equation in the following form:

$$W=\frac{1}{2}\mathbb{E}\mathbb{E}^*=\frac{V\mathbb{E}_0(V\mathbb{E}_0)^*}{2}=\frac{V\mathbb{E}_0\mathbb{E}_0^*V^*}{2}=VW_0V^* \tag{7.7}$$

We do the same with the values from Table 7.1 and the formula (7.1).

$$\partial^- \begin{pmatrix} 0 \\ -\mathbf{E}_0(\mathbb{X}'V^-) \end{pmatrix} = \begin{pmatrix} \rho(\mathbb{X}'V^-) \\ 0 \end{pmatrix} \tag{7.8}$$

Thus, by virtue of (4.14) we get

$$\begin{bmatrix} \frac{\partial}{\partial t'} \\ -\nabla' \end{bmatrix} \left(\frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix} \begin{pmatrix} 0 \\ -\mathbf{E}_0(\mathbb{X}'V^-) \end{pmatrix} \right) = \begin{pmatrix} \rho(\mathbb{X}'V^-) \\ 0 \end{pmatrix},$$

where

$$\frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix} \begin{pmatrix} 0 \\ -\mathbf{E}'_0 \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} \mathbf{v}\mathbf{E}'_0 \\ -\mathbf{E}'_0 + i\mathbf{v} \times \mathbf{E}'_0 \end{pmatrix} = \begin{pmatrix} e' \\ -\mathbf{E}' + i\mathbf{B}' \end{pmatrix}$$

In paravector form, the modified Maxwell equations are as follows:

$$\begin{bmatrix} \frac{\partial}{\partial t} \\ -\nabla \end{bmatrix} \begin{pmatrix} e \\ -\mathbf{E} + i\mathbf{B} \end{pmatrix} = \begin{pmatrix} \rho \\ 0 \end{pmatrix}, \tag{7.9}$$

which, after breaking down into ingredients, gives the same formulas as before:

$$\begin{aligned} \frac{\partial e}{\partial t} + \nabla\mathbf{E} &= \rho & \nabla\mathbf{B} &= 0 \\ \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} &= \nabla e & \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} &= 0 \end{aligned}$$

and the energy density

$$W = \frac{1}{2}\mathbb{E}^*\mathbb{E} = \frac{1}{2} \begin{pmatrix} e \\ -\mathbf{E} - i\mathbf{B} \end{pmatrix} \begin{pmatrix} e \\ -\mathbf{E} + i\mathbf{B} \end{pmatrix} = \begin{bmatrix} \frac{e^2+E^2+B^2}{2} \\ -e\mathbf{E} + \mathbf{E} \times \mathbf{B} \end{bmatrix} \tag{7.10}$$

In Chapter 3 it was showed that the equations (7.5) and (7.9) are equivalent to each other. Above, it has been confirmed once again, so in order not to complicate our reasoning, we will follow the equations described in Table 7.2. It should be noted that the conventionally obtained equations, corresponding to Maxwell's ones, do not contain current density(!). We will return to this difference from the classical theory in Chapter 9, where it will be shown that the revised Maxwell's equations also describe the electromagnetic field well. Interpretation of the obtained formulas is like wandering around blindfolded and composing in your head an image of a room after accidentally touching some objects. One has to come back to the same place many times to create an image of the room in their mind.

Let's go back to the field around stationary charges (7.2). The strength of this field, seen from the frame moving at the speed of $-\mathbf{v}$, is described by the function:

$$\mathbb{E}\left(V^-\mathbb{X}'\right) = \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} \begin{pmatrix} 0 \\ \mathbf{E}(V^-\mathbb{X}') \end{pmatrix} \tag{7.11}$$

The field's energy density is $\mathbb{E}'\mathbb{E}'^*/2$, that is

$$W\left(V^-\mathbb{X}'\right) = \frac{1}{2(1-v^2)} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} \begin{pmatrix} 0 \\ \mathbf{E}(V^-\mathbb{X}') \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{E}(V^-\mathbb{X}') \end{pmatrix} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} = \frac{\mathbf{E}'^2}{2(1-v^2)} \begin{pmatrix} 1+v^2 \\ 2\mathbf{v} \end{pmatrix} \tag{7.12}$$

The product of the electric field strength four-vectors is a scalar, so it can be factored out of the product of the velocity paravectors.

For now, the transformations CRT described by the real paravector, which was interpreted as velocity, were considered. But, which is easy to check, as a result of a composition of such paravectors (velocity composition) we obtain paravectors including the complex components which represent **paravector orthogonal transformations** (POT).

For the composed velocity represented by the orthogonal paravector Λ we have:

$$W = \frac{1}{2}\Lambda \begin{pmatrix} 0 \\ \mathbf{E} \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{E} \end{pmatrix} \Lambda^* = \frac{\mathbf{E}^2}{2}\Lambda\Lambda^* \tag{7.13}$$

Looking at the equations (7.12) and (7.13), one can find a way to reduce any orthogonal paravector representing a compound velocity to the real velocity paravector $V = \frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix}$, which we called a realisation. In the next chapter this will be explained in detail.

Chapter 8

Realisation of the orthogonal paravector to the real velocity paravector

The misleading impression that space-time is real results from two facts: 1) the information carrier is energy that is always real; 2) a space-time is real locally in any observer's frame. This chapter shows how the complex space-time phenomena can be seen by the observer in his/her real coordinate system. The concept of the realisation of the complex orthogonal paravector, which represents a compound (complex) boost, to the form of the real velocity paravector has been introduced on the basis of energy equivalence. Mathematical properties of realisation are examined here, and an attempt is made to apply realisation to describing states of physical objects. The possibility of realisation of scalar coordinates only is shown, too.

So far, the transformations described by the real paravector, interpreted as velocity, have been considered. As we know, as a result of the multiplication of such paravectors (velocity composition) we obtain complex paravectors. By acting with a complex orthogonal paravector on real coordinates, we obtain complex time. Since no interpretation for the imaginary time component could be found, the idea arose that since energy as a product of interconnected quantities is real, there should be a way to reduce the complex velocity to its real form, so that the energy remains the same regardless of the notation.

People experience the world around them through received energy stimuli. What they hear is the energy of the sound waves received by their ears. They see that their eye receptors receive the energy of electromagnetic waves. It works similarly with heat or touch. The same applies to laboratory tests. Measuring instruments record a piece of energy proportional to the tested quantity, strengthen it and transform it into information. The only information carrier is **energy**, and this **is always real**. This gives us the impression that the surrounding world is real (in the mathematical sense). As a result of the above considerations a doubt arises: is not the World perhaps more compound than we think?

8.1 Realisation, that is a projection of phenomena from the complex space-time onto the real space-time of the observer

We hypothesize that even if the description of relativistic phenomena was complex, there should be some way of 'projecting' it onto the real space available to our cognition. We have termed this 'projection' of complex paravectors onto real space-time *realisation.*

As it was shown in the previous chapter, the electric field strength seen from a car moving at speed $-\mathbf{v}$ is

$$\mathbb{E}\left(\mathrm{V}^{-}\mathbb{X}\right) = \frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix}\begin{pmatrix}0\\ \mathbf{E}(\mathrm{V}^{-}\mathbb{X})\end{pmatrix} \tag{8.1}$$

The proper energy density of this field is $\mathbb{E}\mathbb{E}^*/2$, or

$$W\left(\mathrm{V}^{-}\mathbb{X}\right) = \frac{1}{2} V \begin{pmatrix} 0 \\ \mathbf{E}(\mathrm{V}^{-}\mathbb{X}) \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{E}(\mathrm{V}^{-}\mathbb{X}) \end{pmatrix} V^* = \frac{\mathbf{E}^2(\mathrm{V}^{-}\mathbb{X})}{2(1-v^2)} \begin{pmatrix} 1+v^2 \\ 2\mathbf{v} \end{pmatrix} \tag{8.2}$$

We shifted the electric field energy density (of the stationary charges) as a scalar before the product of the velocity paravectors. For a complex velocity represented by a complex orthogonal paravector $\Lambda = \Gamma|\Gamma|^{-1}$ we have

$$W\left(\Lambda^{-}\mathbb{X}\right) = \frac{1}{2} \frac{\Gamma}{|\Gamma|} \begin{pmatrix} 0 \\ \mathbf{E}(\Lambda^{-}\mathbb{X}) \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{E}(\Lambda^{-}\mathbb{X}) \end{pmatrix} \frac{\Gamma^*}{|\Gamma|} = \frac{\mathbf{E}^2(\Lambda^{-}\mathbb{X})}{2} \frac{\Gamma\Gamma^*}{\det\Gamma} \tag{8.3}$$

As it appears from the formula (8.2) the energy density 4-vector of the field coming from the charge in-motion is the product of scalar energy density of a rest field and the VV^* paravector associated with the motion. If we inserted any other orthogonal paravector in place of V, the value of energy density would not change if

$$VV = \Lambda\Lambda^* = \frac{\Gamma\Gamma^*}{\det\Gamma} \tag{8.4}$$

We arrive at a similar result when we take the formula for mechanical energy (3.32) and instead of the velocity paravector we insert any orthogonal paravector Λ

$$\begin{pmatrix} E \\ \mathbf{p} \end{pmatrix} = \frac{m_0}{2}\Lambda\Lambda^* = \frac{m_0}{2} VV \tag{8.5}$$

Definition 8.1.1. We call **the (right) realisation of any orthogonal paravector** the transformation that assigns to it the velocity paravector $\mathrm{V} = \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix}$ according to the following relation

$$\left|\underline{\frac{\Gamma}{|\Gamma|}}\right. := \mathrm{V} \qquad \Longleftrightarrow \qquad \mathrm{VV} = \frac{\Gamma\Gamma^*}{\det\Gamma} \tag{8.6}$$

Definition 8.1.2. We call **the left realisation of any orthogonal paravector** the transformation that assigns to it the velocity paravector $\mathrm{V} = \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix}$ according to the relation

$$\left.\underline{\frac{\Gamma}{|\Gamma|}}\right| := \mathrm{V} \qquad \Longleftrightarrow \qquad \mathrm{VV} = \frac{\Gamma^*\Gamma}{\det\Gamma} \tag{8.7}$$

As it is not difficult to show for the paravector $\frac{\Gamma}{|\Gamma|} = \frac{1}{\sqrt{a^2-b^2+c^2-d^2}} \begin{bmatrix} a+id \\ \mathbf{b}+i\mathbf{c} \end{bmatrix}$ the left realisation is the transformation:

$$\left.\underline{\frac{\Gamma}{|\Gamma|}}\right| = \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} \qquad \text{where} \qquad \mathbf{v} = \frac{a\mathbf{b} + d\mathbf{c} - \mathbf{b}\times\mathbf{c}}{a^2+c^2} \quad , \tag{8.8}$$

and in accordance with the right realisation it is

$$\mathbf{v} = \frac{a\mathbf{b} + d\mathbf{c} + \mathbf{b}\times\mathbf{c}}{a^2+c^2}. \tag{8.9}$$

Then, in both cases

$$v^2 = \frac{b^2+d^2}{a^2+c^2}. \tag{8.10}$$

Proof. We start from the definition 8.1.2

$$\frac{1}{1-v^2} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} = \frac{1}{a^2-b^2+c^2-d^2} \begin{bmatrix} a-id \\ \mathbf{b}-i\mathbf{c} \end{bmatrix} \begin{bmatrix} a+id \\ \mathbf{b}+i\mathbf{c} \end{bmatrix}$$

The above equality is transformed into the form of:

$$\frac{1+v^2}{1-v^2}\begin{bmatrix}1\\ \frac{2\mathbf{v}}{1+v^2}\end{bmatrix}=\frac{1+\frac{b^2+d^2}{a^2+c^2}}{1-\frac{b^2+d^2}{a^2+c^2}}\begin{bmatrix}1\\ \frac{2\frac{a\mathbf{b}+d\mathbf{c}-\mathbf{b}\times\mathbf{c}}{a^2+c^2}}{1+\frac{b^2+d^2}{a^2+c^2}}\end{bmatrix}$$

So, if we assume that $\mathbf{v}=(a\mathbf{b}+d\mathbf{c}-\mathbf{b}\times\mathbf{c})/(a^2+c^2)$, then we can check that $v^2=(b^2+d^2)/(a^2+c^2)$. □

For formality's sake, we have to check that $v^2<1$.

Proof.

From the equation (8.10) it follows that $v^2\left(a^2+c^2\right)=b^2+d^2$

Subtracting a^2+c^2 from both sides of the above equation and dividing the result by a^2+c^2 we get

$$v^2-1=\frac{-a^2+b^2-c^2+d^2}{a^2+c^2},$$

hence

$$v^2=1-\frac{a^2-b^2+c^2-d^2}{a^2+c^2}$$

Since the paravector Γ is proper (i.e. $a^2-b^2+c^2-d^2>0$) and $a^2-b^2+c^2-d^2<a^2+c^2$, on the right of the last equality we have a real non-negative number which is less then one.

□

To facilitate remembering, in which case we write the '-' sign before the vector product, and in which case '+', we give the following rule:

- the underscore is directed towards negative numbers (left) for the left realisation and before the vector product we write the '-' sign,
- the underscore is directed towards the right (positive numbers) for the right realisation and before the vector product we write the '+' sign.

Orthogonal paravectors are equivalent if they realise to the same velocity paravector. So we can create equivalence classes in the set of orthogonal paravectors because of the realisation. We choose the velocity paravector (V) as a physically interpretable representative of each class.

Let us get back to the definition of the left realisation

$$\frac{1}{1-v^2}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix}=\frac{1}{a^2-b^2+c^2-d^2}\begin{bmatrix}a-id\\ \mathbf{b}-i\mathbf{c}\end{bmatrix}\begin{bmatrix}a+id\\ \mathbf{b}+i\mathbf{c}\end{bmatrix} \tag{8.11}$$

$$\frac{1}{1-v^2}\begin{bmatrix}1+v^2\\ 2\mathbf{v}\end{bmatrix}=\frac{1}{1-\frac{b^2+d^2}{a^2+c^2}}\begin{bmatrix}1+\frac{b^2+d^2}{a^2+c^2}\\ 2\frac{a\mathbf{b}+d\mathbf{c}-\mathbf{b}\times\mathbf{c}}{a^2+c^2}\end{bmatrix}$$

Since $v^2=(b^2+d^2)/(a^2+c^2)$, then we reduce the factors and we obtain

$$\begin{bmatrix}1+v^2\\ 2\mathbf{v}\end{bmatrix}=\begin{bmatrix}1+\frac{b^2+d^2}{a^2+c^2}\\ 2\frac{a\mathbf{b}+d\mathbf{c}-\mathbf{b}\times\mathbf{c}}{a^2+c^2}\end{bmatrix}$$

In view of the above, we can extend the definition of the realisation onto singular paravectors and we can define their left and right realisations

$$\underline{\Omega|}:=\begin{bmatrix}1\\ \frac{r\mathbf{u}+s\mathbf{w}-\mathbf{u}\times\mathbf{w}}{r^2+w^2}\end{bmatrix}\qquad\underline{|\Omega}:=\begin{bmatrix}1\\ \frac{r\mathbf{u}+s\mathbf{w}+\mathbf{u}\times\mathbf{w}}{r^2+w^2}\end{bmatrix} \tag{8.12}$$

where

$$\Omega = \begin{bmatrix} r + is \\ \mathbf{u} + i\mathbf{w} \end{bmatrix} \qquad \text{and} \qquad r^2 - u^2 + w^2 - s^2 = 0$$

As a result of the realisation of the complex paravector representing the compound speed of light, we obtain the real vector of the speed of light

$$\mathbf{c} = \frac{r\mathbf{u} + s\mathbf{w} - \mathbf{u} \times \mathbf{w}}{r^2 + w^2} = \frac{r\mathbf{u} + s\mathbf{w} - \mathbf{u} \times \mathbf{w}}{s^2 + u^2} \tag{8.13}$$

8.2 Properties of realisation

Theorem 8.2.1. Let Λ be an orthogonal paravector, that is

$$\Lambda = \frac{1}{\sqrt{a^2 - b^2 + c^2 - d^2}} \begin{bmatrix} a + id \\ \mathbf{b} + i\mathbf{c} \end{bmatrix} \quad \text{and} \quad ad = \mathbf{bc}, \tag{8.14}$$

then the realisation of the orthogonal paravector has the following properties:

1. $\underline{|\Lambda^-} = \underline{\Lambda|}^-$ or $(\underline{|\Lambda})^- = \underline{\Lambda^-|}$
2. $\underline{|\Lambda^*} = \underline{\Lambda|}$ or $\underline{\Lambda^*|} = \underline{|\Lambda}$
3. $\Lambda\underline{|\Lambda} = \underline{\Lambda|}\Lambda$ or $\underline{|\Lambda}\Lambda = \Lambda\underline{\Lambda|}$
4. $\underline{\Lambda_1\Lambda_2|} = \underline{|\Lambda_2\Lambda_1}$
5. For any orthogonal paravector $\underline{\Lambda_1\Lambda_2|} = \underline{\underline{\Lambda_1|}\Lambda_2\big|}$

   Note! The order of realisation is important. First we realise the 'earlier' paravectors (with a lower index) and then the next ones towards the 'last' one. If the indexes grow from left to right, then we use the left realisation; if they grow from right to left, then we use the right realisation [1].

   Thus for the right realisation we have $\underline{|\Lambda_2\Lambda_1} = \underline{\big|\Lambda_2\underline{|\Lambda_1}}$

6. For any orthogonal paravector
   $\underline{\Lambda_1|} = \underline{\Lambda_2|}$ if and only if $(\Lambda_1, \Lambda_2\rangle^* = (\Lambda_1, \Lambda_2\rangle^-$
   $\underline{|\Lambda_1} = \underline{|\Lambda_2}$ if and only if $\langle\Lambda_1, \Lambda_2)^* = \langle\Lambda_1, \Lambda_2)^-$
   It follows that if $\underline{\Lambda_1|} = \underline{\Lambda_2|}$, then the integrated product $(\Lambda_1, \Lambda_2\rangle$ is the unitary paravector, and in case of the right realisation, the left integrated product is a unitary paravector.

7. $\big(\Lambda, \underline{\Lambda|}\big\rangle = \frac{1}{\sqrt{a^2+c^2}} \begin{bmatrix} a \\ i\mathbf{c} \end{bmatrix}$ $\qquad \big(\underline{\Lambda|}, \Lambda\big\rangle = \frac{1}{\sqrt{a^2+c^2}} \begin{bmatrix} a \\ -i\mathbf{c} \end{bmatrix}$

   So, the integrated product of any orthogonal paravector and its left realisation gives a unitary paravector.

8. In the general case, the realisation of the orthogonal paravector does not preserve a scalar product, which means that realisation is not an orthogonal transformation, but it preserves the parallelism.

9. The realisation preserves the scalar product of the vigors of orthogonal paravectors, i.e.

$$\big(\text{vig}\,\underline{\Lambda_1|}, \text{vig}\,\underline{\Lambda_2|}\big\rangle = \big(\text{vig}\Lambda_1, \text{vig}\Lambda_2\big\rangle \tag{8.15}$$

10. $\frac{1}{a^2+c^2} \begin{bmatrix} a \\ -i\mathbf{c} \end{bmatrix} \underline{\Lambda|} \begin{bmatrix} a \\ i\mathbf{c} \end{bmatrix} = \frac{1}{a^2+c^2} \begin{bmatrix} a \\ i\mathbf{c} \end{bmatrix} \underline{\Lambda|} \begin{bmatrix} a \\ -i\mathbf{c} \end{bmatrix} = \underline{|\Lambda}$

[1] Such particular order of increasing the value of the index results from the following reasoning:
The phase difference $V^-\mathbb{X}$ after passing to the frame moving at the speed of $-\mathbf{v}_1$ has the form of $V^-V_1^-\mathbb{X}'$, where $X' = V_1\mathbb{X}$. After the realisation of compound boost, the last phase will have the form of $V_{real}^-\mathbb{X}_{real} = \underline{V^-\Lambda_1^-\Lambda_2^-\big|}\mathbb{X}_{real}$, that is $V_{real} = \underline{|\Lambda_2\Lambda_1 V}$

11. For any rotation it is true that

$$\underline{R^-\Lambda R}\Big| = R^-\underline{\Lambda}|R \qquad \text{where} \qquad R=\frac{1}{\sqrt{r^2+s^2}}\begin{bmatrix} r\\ i\mathbf{s}\end{bmatrix}$$

Proof.

1. $\left(\underline{|\Lambda^-}\right)_V = \frac{a(-\mathbf{b})+d(-\mathbf{c})+(-\mathbf{b})\times(-\mathbf{c})}{a^2+c^2} = -\frac{a\mathbf{b}+d\mathbf{c}-\mathbf{b}\times\mathbf{c}}{a^2+c^2} = \left(\underline{\Lambda|}^-\right)_V$
2. $\left(\underline{|\Lambda^*}\right)_V = \frac{a\mathbf{b}+d\mathbf{c}-\mathbf{b}\times\mathbf{c}}{a^2+c^2} = \left(\underline{\Lambda|}\right)_V$
3. Use the definition of realisation and perform calculations.
4. $\underline{\Lambda_1\Lambda_2|} = \mathrm{V} \qquad \iff \qquad (\Lambda_1\Lambda_2)^*(\Lambda_1\Lambda_2) = \mathrm{VV}$

   $\mathrm{VV} = \Lambda_2^*\Lambda_1^*\Lambda_1\Lambda_2 = \left(\Lambda_2\Lambda_1\Lambda_1^*\Lambda_2^*\right)^*$

   and since we have a real paravector in parentheses, so

   $\left(\Lambda_2\Lambda_1\Lambda_1^*\Lambda_2^*\right)^* = \Lambda_2\Lambda_1\Lambda_1^*\Lambda_2^* = \Lambda_2\Lambda_1(\Lambda_2\Lambda_1)^* \qquad \iff \qquad \mathrm{V} = \underline{|\Lambda_2\Lambda_1}$
5. $\underline{\Lambda_1\Lambda_2|} = \mathrm{V} \qquad \iff \qquad (\Lambda_1\Lambda_2)^*(\Lambda_1\Lambda_2) = \mathrm{VV} \qquad \iff \qquad \Lambda_2^*\Lambda_1^*\Lambda_1\Lambda_2 = \mathrm{VV}$

   and since $\Lambda_1^*\Lambda_1 = \mathrm{V}_1\mathrm{V}_1$, then $\Lambda_2^*\mathrm{V}_1\mathrm{V}_1\Lambda_2 = \mathrm{VV}$,

   that is $(\mathrm{V}_1\Lambda_2)^*(\mathrm{V}_1\Lambda_2) = \mathrm{VV} \qquad \iff \qquad \underline{\mathrm{V}_1\Lambda_2|} = \mathrm{V} \qquad \iff \qquad \underline{\underline{\Lambda_1}|\Lambda_2}\Big| = \mathrm{V}$
6. $\langle\Lambda_1,\Lambda_2\rangle^* = \langle\Lambda_1,\Lambda_2\rangle^- \qquad \iff \qquad \left(\Lambda_1\Lambda_2^-\right)^* = \left(\Lambda_1\Lambda_2^-\right)^- \qquad \iff$

   $\Lambda_2^{-*}\Lambda_1^* = \Lambda_2\Lambda_1^- \qquad \iff \qquad \Lambda_1^*\Lambda_1 = \Lambda_2^*\Lambda_2 \qquad$ hence $\qquad \underline{\Lambda_1|} = \underline{\Lambda_2|}$
7. The reader should substitute appropriate quantities into the definition of the integrated product and perform calculations.
8. We check what the integrated product will look like $\left\langle\underline{\Lambda_1|},\underline{\Lambda_2|}\right\rangle$.

   $\langle\Lambda_1,\Lambda_2\rangle = \Lambda_1\Lambda_2^- = \Lambda_1\underline{\Lambda_1|}^-\underline{\Lambda_1|}\,\underline{\Lambda_2|}^-\underline{\Lambda_2|}\Lambda_2^- = \left\langle\Lambda_1,\underline{\Lambda_1|}\right\rangle\left\langle\underline{\Lambda_1|},\underline{\Lambda_2|}\right\rangle\left\langle\Lambda_2,\underline{\Lambda_2|}\right\rangle^-$ which shows that realisation does not preserve integrated products.
9. Use the definition of vigor, the definition of an integrated product, and the definition of realisation.

   $\left\langle\mathrm{vig}\,\underline{\Lambda_1|},\mathrm{vig}\,\underline{\Lambda_2|}\right\rangle = (V_1V_1)(V_2V_2)^- = (\Lambda_1\Lambda_1^*)(\Lambda_2\Lambda_2^*)^- = \left\langle\mathrm{vig}\Lambda_1,\mathrm{vig}\Lambda_2\right\rangle$

   If the integrated product is preserved, then the dot product must, of course, also be preserved.
10. Use property 7.
11. If $R=\frac{1}{\sqrt{r^2+s^2}}\begin{bmatrix} r\\ i\mathbf{s}\end{bmatrix}$, then

    $\underline{R^-\Lambda R}\Big|\;\underline{R^-\Lambda R}\Big| = (R^-\Lambda R)^*(R^-\Lambda R) = R^-\Lambda^*RR^-\Lambda R = R^-\Lambda^*\Lambda R = (R^-\underline{\Lambda}|R)(R^-\underline{\Lambda}|R)$

    Since the first and the last expressions are the products of the same two real velocity paravectors, property 11 has been proved.

□

Theorem 8.2.2. Any orthogonal paravector $\Lambda = \frac{1}{\sqrt{a^2-b^2+c^2-d^2}}\begin{bmatrix} a+id\\ \mathbf{b}+i\mathbf{c}\end{bmatrix}$ can be presented unambiguously as product

$$\frac{1}{\sqrt{1^2-v^2}}\frac{1}{\sqrt{a^2+c^2}}\begin{bmatrix} 1\\ \mathbf{v}\end{bmatrix}\begin{bmatrix} a\\ i\mathbf{c}\end{bmatrix} \qquad \text{or} \qquad \frac{1}{\sqrt{1^2-w^2}}\frac{1}{\sqrt{a^2+c^2}}\begin{bmatrix} a\\ i\mathbf{c}\end{bmatrix}\begin{bmatrix} 1\\ \mathbf{w}\end{bmatrix}$$

Proof.

Suppose there are two different v vectors which satisfy the condition

$$\frac{1}{\sqrt{1-v^2}}\frac{1}{\sqrt{a^2+c^2}}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix}\begin{bmatrix}a\\ i\mathbf{c}\end{bmatrix}=\Lambda,$$

then it would have to be

$$\frac{1}{\sqrt{1-v_1^2}}\frac{1}{\sqrt{a_1^2+c_1^2}}\begin{bmatrix}1\\ \mathbf{v}_1\end{bmatrix}\begin{bmatrix}a_1\\ i\mathbf{c}_1\end{bmatrix}=\frac{1}{\sqrt{1-v_2^2}}\frac{1}{\sqrt{a_2^2+c_2^2}}\begin{bmatrix}1\\ \mathbf{v}_2\end{bmatrix}\begin{bmatrix}a_2\\ i\mathbf{c}_2\end{bmatrix}$$

Hence

$$\frac{1}{\sqrt{1-v_2^2}}\frac{1}{\sqrt{1-v_1^2}}\begin{bmatrix}1\\ -\mathbf{v}_2\end{bmatrix}\begin{bmatrix}1\\ \mathbf{v}_1\end{bmatrix}=\frac{1}{\sqrt{a_1^2+c_1^2}}\frac{1}{\sqrt{a_2^2+c_2^2}}\begin{bmatrix}a_2\\ i\mathbf{c}_2\end{bmatrix}\begin{bmatrix}a_1\\ -i\mathbf{c}_1\end{bmatrix}$$

From the real vector part, one can see that the vector $\mathbf{v}_1$ must be equal to $\mathbf{v}_2$.

The same goes for the second case.

□

The phase and space-time intervals realise as follows:

Theorem 8.2.3. For each orthogonal paravector $\Lambda=\frac{1}{\sqrt{a^2-b^2+c^2-d^2}}\begin{bmatrix}a+id\\ \mathbf{b}+i\mathbf{c}\end{bmatrix}$

$$\Lambda^{-}\mathbb{X}=\underline{\Lambda}|^{-}\mathbb{X}' \quad \text{where} \quad \mathbb{X}'=\frac{1}{\sqrt{a^2+c^2}}\begin{bmatrix}a\\ -i\mathbf{c}\end{bmatrix}\mathbb{X}$$

Proof.

$$\Lambda^{-}\mathbb{X}=\underline{\Lambda}|^{-}\underline{\Lambda}|\Lambda^{-}\mathbb{X}=\underline{\Lambda}|^{-}\left(\underline{\Lambda}|\Lambda\right\rangle\mathbb{X}$$

hence by the theorem 8.2.1.7 we obtain real coordinates

$$\mathbb{X}'=\frac{1}{\sqrt{a^2+c^2}}\begin{bmatrix}a\\ -i\mathbf{c}\end{bmatrix}\mathbb{X}$$

□

Theorem 8.2.4. Each special orthogonal paravector can be shown as a combination of velocity paravectors.

Proof.

Let $\Lambda=\frac{1}{\sqrt{a^2-b^2+c^2-d^2}}\begin{bmatrix}a+id\\ \mathbf{b}+i\mathbf{c}\end{bmatrix}=\frac{1}{\sqrt{1-v_1^2}}\begin{bmatrix}1\\ \mathbf{v}_1\end{bmatrix}\frac{1}{\sqrt{1-v_2^2}}\begin{bmatrix}1\\ \mathbf{v}_2\end{bmatrix}\frac{1}{\sqrt{1-v_3^2}}\begin{bmatrix}1\\ \mathbf{v}_3\end{bmatrix}$

It follows from theorem 8.2.1.7 that

$$\underline{\Lambda}|\frac{1}{\sqrt{a^2+c^2}}\begin{bmatrix}a\\ i\mathbf{c}\end{bmatrix}=\frac{1}{\sqrt{1-v_1^2}}\begin{bmatrix}1\\ \mathbf{v}_1\end{bmatrix}\frac{1}{\sqrt{1-v_2^2}}\begin{bmatrix}1\\ \mathbf{v}_2\end{bmatrix}\frac{1}{\sqrt{1-v_3^2}}\begin{bmatrix}1\\ \mathbf{v}_3\end{bmatrix}$$

hence

$$\frac{1}{\sqrt{a^2+c^2}}\begin{bmatrix}a\\ i\mathbf{c}\end{bmatrix}=\underline{\Lambda}|^{-}\frac{1}{\sqrt{1-v_1^2}}\begin{bmatrix}1\\ \mathbf{v}_1\end{bmatrix}\frac{1}{\sqrt{1-v_2^2}}\begin{bmatrix}1\\ \mathbf{v}_2\end{bmatrix}\frac{1}{\sqrt{1-v_3^2}}\begin{bmatrix}1\\ \mathbf{v}_3\end{bmatrix}$$

□

When examining the properties of realisation, it is worth noting that:

1. The velocity paravector realises to itself. $\lfloor \underline{\mathrm{V}} = \underline{\mathrm{V}} \rfloor = \mathrm{V}$

2. The realisation of the paravector representing the Euclidean rotation is equal to 1.
$\underline{\left\| \begin{bmatrix} \cos\alpha \\ i\mathbf{c}\sin\alpha \end{bmatrix} \right.} = \underline{\left. \begin{bmatrix} \cos\alpha \\ i\mathbf{c}\sin\alpha \end{bmatrix} \right\|} = 1$
or $\quad \underline{\big(\Lambda, \underline{\Lambda|}\big\rangle\big|} = \underline{\big|\big(\Lambda, \underline{\Lambda|}\big\rangle} = 1$

3. The Euclidean rotation can be formulated as below:
$\big(\Lambda, \underline{\Lambda|}\big\rangle X \big(\underline{\Lambda|}, \Lambda\big\rangle$

4. By writing $\quad \begin{bmatrix} id \\ \mathbf{b} \end{bmatrix} \begin{bmatrix} a \\ i\mathbf{c} \end{bmatrix}^{-1} = \begin{bmatrix} 0 \\ \frac{a\mathbf{b}+d\mathbf{c}+\mathbf{b}\times\mathbf{c}}{a^2+c^2} \end{bmatrix}$
in the scalar part we see the condition for the paravector to be proper, and in the vector part we get the realised velocity vector.

8.3 Attempts to apply realisation

The concept of realisation of the state paravector was introduced because no interpretation of imaginary scalars was found in real space-time, so only the possibility of complex multiplication of two real state paravectors was allowed for the phase. On the other hand, it seems obvious that energy, as a product of conjugated paravectors, must be a real quantity. In a complex space, if velocity were complex, the kinetic energy would have to depend on the product of mutually conjugated orthogonal paravectors. Testing this hypothesis showed promising results. However, more detailed research shows that the concept of realisation of an orthogonal paravector may only apply to a single physical object. It is impossible to describe in this way many objects that are in motion towards each other and towards the observer, because realisation is not an orthogonal transformation and it deforms space. Realisation, however, preserves the parallelism of the paravectors and their vigors, which makes it similar to projection. And, most interestingly: realisation preserves the scalar product of the paravector vigors (e.g. the energy paravector).

8.3.1 Spherical explosion

As a result of an explosion of a point material object, particles fly away evenly in all directions at a relativistic **w** velocity. In the experimenter's frame, the equation of motion of a single particle has the following form:

$$\frac{1}{\sqrt{1-w^2}} \begin{bmatrix} 1 \\ -\mathbf{w} \end{bmatrix} \begin{pmatrix} \Delta t \\ \Delta \mathbf{x} \end{pmatrix} = \begin{pmatrix} \Delta t^0 \\ 0 \end{pmatrix}$$

The same equation at the frame of the observer moving at -**v** velocity looks in the following way:

$$\frac{1}{\sqrt{1-w^2}} \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ -\mathbf{w} \end{bmatrix} \begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t' \\ \Delta \mathbf{x}' + i\mathbf{y}' \end{pmatrix} = \begin{pmatrix} \Delta t^0 \\ 0 \end{pmatrix}$$

The resultant velocity is:

$$\Lambda^- = \frac{1}{\sqrt{1-w^2}} \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1+\mathbf{wv} \\ -\mathbf{w}-\mathbf{v}+i\mathbf{w}\times\mathbf{v} \end{bmatrix} \tag{8.16}$$

The realised vector of this velocity is:

$$\underline{\Lambda^{-}|} = -\mathbf{w}' = \frac{-(1+\mathbf{wv})(\mathbf{w}+\mathbf{v})+(\mathbf{w}+\mathbf{v})\times(\mathbf{w}\times\mathbf{v})}{(1+\mathbf{wv})^2+(\mathbf{w}\times\mathbf{v})^2} \tag{8.17}$$

If the variable parameter here is the particle flight direction (direction of vector **w** relative to **v**), then on the XOY plane the real vector will have the following coordinates:

$$\begin{aligned} w'_x &= \frac{v(1+w^2)+(1+v^2)w\cos\alpha}{1+2vw\cos\alpha+v^2w^2} \\ w'_y &= \frac{(1-v^2)w\sin\alpha}{1+2vw\cos\alpha+v^2w^2} \end{aligned}$$

For the calculations, we take two cases: (w = 0.7 and v = 0.7) and (w = 0.4 and v = 0.8), while α is the angle between the **w** and **v** vectors.

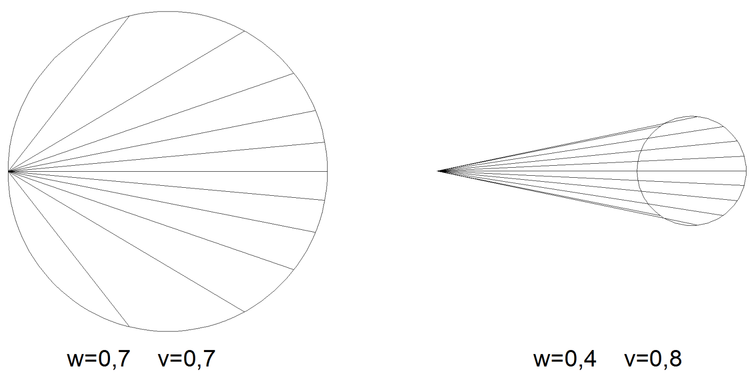

Figure 8.1: A realised image of the explosion front as seen from a rocket speeding at v velocity

As can be seen, the image of the sphere is a sphere. However, we encounter problems when we have to describe the movement of particles with an elastic rebound.

8.3.2 Elastic collision

Let us imagine an experiment: In the center of the bubble (laboratory), a spherical explosion occurs, the front of which propagates concentrically with the relativistic velocity of $w < c$, and then returns to the center after an elastic rebound from the wall. Knowing the equation of the particle motion in the laboratory frame, one should describe the particle trajectory in the observer's frame moving in relation to the laboratory.

The realised velocity of the particle towards the wall is represented by the formula (8.17). After the rebound, the realised velocity will be described by the same formula but will change the sign of the vector **w**.

$$\mathbf{w}'_2 = \frac{(1-\mathbf{wv})(-\mathbf{w}+\mathbf{v})+(-\mathbf{w}+\mathbf{v})\times(\mathbf{w}\times\mathbf{v})}{(1-\mathbf{wv})^2+(\mathbf{w}\times\mathbf{v})^2} \tag{8.18}$$

The whole way of the particle is $\Delta\mathbf{x} = \mathbf{w}'\Delta t + \mathbf{w}'_2\Delta t = (\mathbf{w}' + \mathbf{w}'_2)\Delta t$

Now, we will check what the realised movement looks like of a particle whose speed **w** is perpendicular to the speed **v**.

$$\mathbf{w}' + \mathbf{w}'_2 = 2\frac{\mathbf{v}-\mathbf{w}\times(\mathbf{w}\times\mathbf{v})}{1+w^2v^2} = 2\mathbf{v}\frac{1+w^2}{1+w^2v^2} > 2\mathbf{v}$$

Since an outside observer sees that the center has shifted on $2\mathbf{v}\Delta t$ at this time, s/he calculates that the particle will pass the center of the laboratory. Thus, the use of realisation does not give a satisfactory image of the described experiment. In fact, this could be expected because, as previously shown, realisation does not preserve the dot product. But for the sake of comfort, let us see what the same experiment looks like in the current theory.

8.3.3 Limited use of realisation

Let's imagine an experiment: In the center of a spherical laboratory there is an explosion, the front of which propagates concentrically with a relativistic velocity $w < c$, and then, after elastic reflection from the wall, returns to the center. Knowing the equation of the particle trajectory in the laboratory frame, we need to describe the real trajectories of the particles in the observer's frame moving with velocity -**v** relative to the laboratory.

The realisated velocity of the particle moving towards the wall is given by the formula (8.17). After rebound, the realisated velocity will be described by the same formula, but the vector **w** will change sign.

$$\mathbf{w}_2' = \frac{(1-\mathbf{wv})(-\mathbf{w}+\mathbf{v})+(-\mathbf{w}+\mathbf{v})\times(\mathbf{w}\times\mathbf{v})}{(1-\mathbf{wv})^2+(\mathbf{w}\times\mathbf{v})^2} \tag{8.19}$$

The entire path of the particle is $\Delta\mathbf{x} = \mathbf{w}'\Delta t_1 + \mathbf{w}_2'\Delta t_2$

Now we will check what the realistic picture of the motion of a particle whose velocity **w** is perpendicular to the velocity **v** looks like.

$$\mathbf{w}' + \mathbf{w}_2' = 2\frac{\mathbf{v}-\mathbf{w}\times(\mathbf{w}\times\mathbf{v})}{1+w^2v^2} = 2\mathbf{v}\frac{1+w^2}{1+w^2v^2} > 2\mathbf{v} \tag{8.20}$$

Since the same observer sees that the center has moved along the path $2\mathbf{v}\Delta t$ during this same time, according to his calculations, the particle will pass the center of the laboratory. Using the realisation does not provide a satisfactory picture of the described experiment. To tell the truth, this could be expected, because as previously shown, the realisation does not preserve the scalar product. After this unpromising introduction, we will try to graphically interpret the above experiment.

In the laboratory, the experiment looks the same as in the figure **??**. To make the images comparable, the same parameters were substituted into the formulas, i.e. $w = 0.4$ and $v = 0.8$. In the moving frame, the particle trajectories have a common beginning at the moment of explosion and a common end after returning to the center. The distance between the explosion and the end point is the same as in the case of the experiment discussed in Chapter 7.4, because the velocity paravector realises to itself. The velocity vectors of the particles before and after reflection are given by (8.17) and (8.19). The reflection points are located at the intersection of the corresponding lines. The above figure shows that the real phenomenon of particle reflection from the

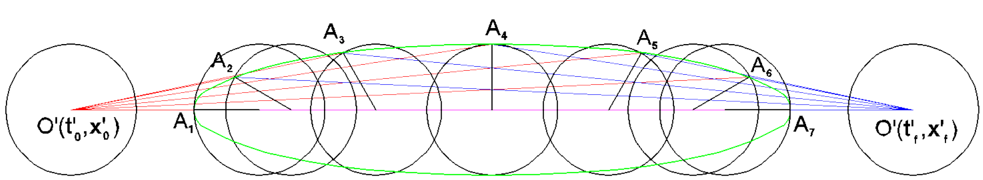

Figure 8.2: A realisated explosion front 'viewed' from a rocket traveling at speed of 0,8

laboratory wall is stretched in time and space. In addition, it is not precisely defined, which results from the fact that realisation is a non-orthogonal transformation. We conclude from this that realisation may have limited application to the physical description of individual objects and should rather not be used in confrontation with other objects in geometry. Looking at the above figure, a similarity to a photograph taken from a tripod is obvious. The figure 8.2, just like a photo of a moving vehicle taken from a tripod, is blurred and stretched along the direction of movement. When we put the figures 6.12 - 6.17 together (fig. 8.3), the image of the phenomenon is precise and it is clearly visible that the particles hit the same points of the laboratory wall in both frames. This image corresponds to a photograph taken 'hand-held', when the observer follows the photographed object with the lens.

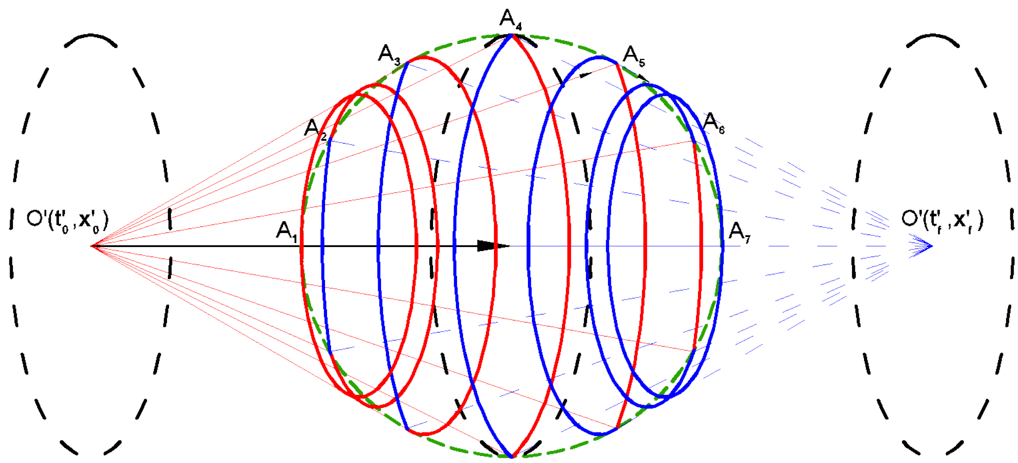

Figure 8.3: Image of the real part of the same experiment in complex space-time

8.4 Realisation of time

Space-time phenomena are inherently related to movement, i.e. a change of position in time. Time, although added as a fourth dimension to the 3-dimensional Euclidean space, does not fit into geometric concepts for one fundamental reason: Time does not stand still, but moves steadily in one direction. The properties of time are fundamentally different from those of geometric space. In order not to lose the basic property of time, which is dynamics, we assume that the time-line is not composed of points (moments), but of segments (intervals). This, in turn, implies the next assumption that space-time is not an affine space but a vector one. Such a general approach leads to the conclusion that the ordering of space-time phenomena in terms of place and time is limited in scope. So time could be a complex quantity. However, when introducing an observer (which mathematically means that we are introducing an affine space), the phenomena in his frame must be ordered in terms of his real time. Due to problems with fitting the complex theory into the real space-time of the observer, we compromise and now modernize our hypothesis. We assume that the space of motion is complex, and only the time in the observer's frame must be real.

Below we repeat the reasoning from the beginning of this chapter, but with the assumption that in the complex space velocity will be represented by a complex vector. So, we have:

$$\frac{1}{1-v^2+u^2}\begin{bmatrix}1\\ \mathbf{v}-i\mathbf{u}\end{bmatrix}\begin{bmatrix}1\\ \mathbf{v}+i\mathbf{u}\end{bmatrix}=\frac{1}{a^2-b^2+c^2-d^2}\begin{bmatrix}a-id\\ \mathbf{b}-i\mathbf{c}\end{bmatrix}\begin{bmatrix}a+id\\ \mathbf{b}+i\mathbf{c}\end{bmatrix} \tag{8.21}$$

From above we get the system of equations:

$$\frac{1+v^2+u^2}{1-v^2+u^2}=\frac{a^2+b^2+c^2+d^2}{a^2-b^2+c^2-d^2} \tag{8.22}$$

$$\frac{\mathbf{v}-\mathbf{v}\times\mathbf{u}}{1-v^2+u^2}=\frac{a\mathbf{b}+d\mathbf{c}-\mathbf{b}\times\mathbf{c}}{a^2-b^2+c^2-d^2} \tag{8.23}$$

There is also a third condition: since the new complex velocity paravector must also be orthogonal, the vectors must satisfy the following condition $\mathbf{vu}=0$. From the equation (8.22) we calculate that

$$v^2=\left(1+u^2\right)\frac{b^2+d^2}{a^2+c^2}$$

After substituting (8.23) into the vector equation, we get

$$\frac{\mathbf{v}-\mathbf{v}\times\mathbf{u}}{1+u^2}=\frac{a\mathbf{b}+d\mathbf{c}-\mathbf{b}\times\mathbf{c}}{a^2+c^2}$$

Together with the condition $\mathbf{vu}=0$, we can present the above equation in the form of a paravector equation

$$\frac{1}{1+u^2}\begin{bmatrix}0\\ \mathbf{v}\end{bmatrix}\begin{bmatrix}1\\ i\mathbf{u}\end{bmatrix}=\begin{bmatrix}0\\ \frac{a\mathbf{b}+d\mathbf{c}-\mathbf{b}\times\mathbf{c}}{a^2+c^2}\end{bmatrix}$$

Hence, it is easy to calculate the vector $\mathbf{v}$

$$\mathbf{v}=\frac{a\mathbf{b}+d\mathbf{c}-\mathbf{b}\times\mathbf{c}}{a^2+c^2}+\frac{a\mathbf{b}+d\mathbf{c}-\mathbf{b}\times\mathbf{c}}{a^2+c^2}\times\mathbf{u}\qquad\text{and}\qquad\mathbf{u}\perp\frac{a\mathbf{b}+d\mathbf{c}-\mathbf{b}\times\mathbf{c}}{a^2+c^2}\tag{8.24}$$

We have not obtained a unique solution to the equation (8.21). Let's take the vector $\mathbf{u}$ as a parameter. We can see that if we choose the zero vector $\mathbf{u}$, the problem boils down to the realisation considered at the beginning of the current chapter. However, we have some freedom here and we can choose $\mathbf{u}$ in such a way that the real velocity vector $\mathbf{v}$ takes the values that we are satisfied with.

Admittedly, the vector $\mathbf{v}$ can be longer than 1 because $\mathbf{v}^2=(\mathbf{w}+\mathbf{w}\times\mathbf{u})^2=w^2(1+u^2)$, where $\mathbf{w}=(a\mathbf{b}+d\mathbf{c}-\mathbf{b}\times\mathbf{c})/(a^2+c^2)$, but it is only a real component, so in the complex space there is no contradiction with the assumption of the maximum speed of light. The velocity vector of light, of course, also can be a complex vector.

Example: A particle is moving at a compound velocity $V=V_1V_2$.

$$V=\frac{1}{\sqrt{1-v_1^2}}\frac{1}{\sqrt{1-v_2^2}}\begin{bmatrix}1\\ \mathbf{v}_1\end{bmatrix}\begin{bmatrix}1\\ \mathbf{v}_2\end{bmatrix}=\frac{1}{\sqrt{1-v_1^2}}\frac{1}{\sqrt{1-v_2^2}}\begin{bmatrix}1+\mathbf{v}_1\mathbf{v}_2\\ \mathbf{v}_1+\mathbf{v}_2+i\mathbf{v}_1\times\mathbf{v}_2\end{bmatrix}$$

$$V^{-}\mathbb{X}'=\frac{1}{\sqrt{1-v^2+u^2}}\begin{bmatrix}1\\ -\mathbf{v}-i\mathbf{u}\end{bmatrix}\begin{pmatrix}\Delta t'\\ \Delta\mathbf{x}'\end{pmatrix}=\begin{pmatrix}\Delta t^0\\ 0\end{pmatrix},\tag{8.25}$$

where $\quad\mathbf{v}=\frac{\mathbf{v}_1+\mathbf{v}_2}{1+\mathbf{v}_1\mathbf{v}_2}\quad$ and $\quad\mathbf{u}=\frac{\mathbf{v}_1\times\mathbf{v}_2}{1+\mathbf{v}_1\mathbf{v}_2}$.

Since the proper time is a real quantity, and we want the time in the primed frame to be real as well, after extracting $\Delta t'$ we get

$$\frac{\Delta t'}{\sqrt{1-v^2+u^2}}\begin{bmatrix}1\\ -\mathbf{v}-i\mathbf{u}\end{bmatrix}\begin{bmatrix}1\\ \mathbf{v}'\end{bmatrix}=\Delta t^0\begin{bmatrix}1\\ 0\end{bmatrix}$$

From where, after transferring the velocity paravector V to the opposite side of the equality, we get

$$\Delta t'\begin{bmatrix}1\\ \mathbf{v}'\end{bmatrix}=\frac{\Delta t^0}{\sqrt{1-v^2+u^2}}\begin{bmatrix}1\\ \mathbf{v}+i\mathbf{u}\end{bmatrix}$$

The observer in the primed frame describes the object in primed coordinates in primed time, so we have to try to transform the (8.25) equation so that the way is traditionally a function of time. From the equation (8.25) one can also see that $\Delta\mathbf{x}'$ must be a complex vector.

$$\Delta\mathbf{x}'=(\mathbf{v}+i\mathbf{u})\Delta t'$$

When the observer relocates to the frame moving at the speed $-\mathbf{v}_3$ and he wants to realise the time only, then the resultant speed will be determined by the paravector

$$\underline{\frac{1}{\sqrt{1-v_1^2}}\frac{1}{\sqrt{1-v_2^2}}\begin{bmatrix}1\\ \mathbf{v}_1\end{bmatrix}\begin{bmatrix}1\\ \mathbf{v}_2\end{bmatrix}}\Bigg|\frac{1}{\sqrt{1-v_3^2}}\begin{bmatrix}1\\ \mathbf{v}_3\end{bmatrix}=\frac{1}{\sqrt{1-w^2}}\begin{bmatrix}1\\ \mathbf{w}\end{bmatrix}\frac{1}{\sqrt{1-v_3^2}}\begin{bmatrix}1\\ \mathbf{v}_3\end{bmatrix}=\frac{1}{\sqrt{1-v^2+u^2}}\begin{bmatrix}1\\ \mathbf{v}+i\mathbf{u}\end{bmatrix}$$

8.5 Discussion

In this chapter we have shown that the complex orthogonal paravector characterizing the velocity of an object in the complex space-time can always be reduced to the form of a real velocity paravector (3.3) in the observer's space-time. We can also present it in the form such that the scalar component is one and the vector is complex. This is not a disadvantage. On the contrary, it gives us a lot of possibilities for interpretation. Certain analogies arise here with the description of motion in Euclidean space. If we describe a rectilinear motion of a single object, the easiest way is to choose a coordinate system consistent with the direction of motion. We then have motion in 1-dimensional space. In the case of two objects, if we choose the frame assigned to one of the objects in the same way, then the motion of the other decomposes into components parallel and perpendicular to the motion of the first one. In the complex world, the axis to which we compare it is the observer's time. Here, to describe the motion of one or two objects, one can choose such a frame of reference that the time is real. Unfortunately, it is no longer possible to describe three or more objects moving in different directions. The time of one of the objects must have an imaginary component. It does not matter for energy paravectors (which is a vigor), because regardless of the frame of reference they are always real, but as shown in the theorem 8.2.1.9 the dot product between these paravectors is preserved. Realisation, as well as projection, maintains the relation of parallelism. Realisation, despite the disadvantage of non-orthogonality, has a great advantage: it gives positive time, which is a mathematical confirmation of the fact that a time of the physical objects never goes backwards.

Chapter 9

Electric field

In this chapter, the transformation of the electric field strength 4-vector around a point charge is analysed. After transforming Gauss's law, slightly changed Maxwell's equations are obtained. Based on the hypothesis put forward in the previous chapter, it is shown that in complex space-time the vector potential and the Lorenz gauge condition have no mathematical justification. A new quantity: a scalar induction, is introduced in their place.

In the previous chapters we showed what the idea of complex relativistic transformations is and that they do not contradict the postulates of the current theory of relativity. Since the postulate of a constant velocity of light results from the invariance of the wave equation, it is essential to check what the consequences of complex transformations are for the theory of electricity. In this chapter we return to the theory of electricity and deal with the 4-vector transformation of the electric field strength around a point charge moving in a relativistic uniform motion. To be closer to the formulas known from the classical theory of electric field, we assume that the phase is the product of real state paravectors, and velocity is represented by the paravector which we called the velocity paravector (3.3) in Chapter 3.

9.1 The field surrounding a point charge

The coordinates of an inertly moving object in the real observer frame are presented in the form of a paravector equation

$$\mathbb{X}_0 = V^{-}\mathbb{X} \qquad \text{or} \qquad \Delta t_0 = V^{-}(X - X_0) \tag{9.1}$$

The potential of an electric field with spherical symmetry around a stationary charge placed at point X_0 is described by the function

$$\varphi\,(X - X_0) = \frac{1}{r_0}q(C^{-}(X - X_0)), \tag{9.2}$$

where C is a singular paravector, e.g $C = \begin{bmatrix} 1 \\ \mathbf{c} \end{bmatrix}$. With in-phase compliance ($|\mathbb{X}| = 0$), the following is the case:

$$\mathbb{X} = X - X_0 = \begin{pmatrix} t - t_0 \\ \mathbf{x} - \mathbf{x}_0 \end{pmatrix} = \begin{pmatrix} r \\ \mathbf{r} \end{pmatrix} \qquad \text{, where } r = |\mathbf{r}|\,. \tag{9.3}$$

The expression $r^{-1}q(C^{-}(X - X_0))$ is a scalar function with the value q/r spanned on coordinates implicit in phase $C^{-}(X - X_0)$. Point X_0 is interpreted as the location in time and space of the q charge being the source of the field. From the equation (9.3) it follows that $\quad \mathbf{r} = \mathbf{x} - \mathbf{x}_0 = \mathbf{c}(t - t_0) = \mathbf{c}\Delta t^0$.

In Chapter 7, we chose the wave equation system for consideration, in which the potential is an invariant scalar function. The denominator r can only be viewed as a distance by an observer stationary with respect to the charge. The space-time distance from the payload is the argument ($\mathbb{X}$). We wrote r, but remember that the value in the denominator is the length of the vector **r** only in the payload frame, not in every other frame, because the vector **r** is not invariant. So it is better to interpret $r = \Delta t$ as the phase lag and $\mathbf{c} = \mathbf{r}/r$ as the phase direction.

The situation is different with the field strength, because it is no longer invariant. It should be remembered that the value of r in the formula describing the field strength, as in the formula for potential, remains unchanged, because it is a field parameter that can be associated with the distance from the source, but only on the condition that the source is at rest or at most it moves at a non-relativistic speed. Otherwise, r is a factor that affects the value of the field in inverse proportion to the charge.

In the frame of a field source, the electric field strength at point X from the charge placed at point X_0 is described by the formula:

$$\mathbf{E} = -\nabla\frac{1}{r}q\left(C^-(X-X_0)\right) = \frac{\mathbf{r}}{r^3}q\left(C^-(X-X_0)\right) = \frac{\mathbf{c}}{r^2}q\left(C^-(X-X_0)\right) \tag{9.4}$$

The relative position of an inertly moving object in the real observer frame is presented in the form of a paravector equation

$$\mathbb{X}^0 = V^-\mathbb{X} \qquad \text{that is} \qquad \Delta t^0 = V^-(X-X_0) \tag{9.5}$$

When passing to a moving frame, the field's strength is transformed

$$\partial^-\begin{pmatrix}\varphi(C^-(X-X_0))\\ 0\end{pmatrix} = \begin{pmatrix}0\\ \mathbf{E}(C^-(X-X_0))\end{pmatrix} \qquad \longrightarrow \tag{9.6}$$

$$\longrightarrow \qquad V^-\partial'^-\begin{pmatrix}\varphi\left((VC)^-\mathbb{X}'\right)\\ 0\end{pmatrix} = \begin{pmatrix}0\\ \mathbf{E}((VC)^-\mathbb{X}')\end{pmatrix}.$$

Hence, in a frame that moves at speed - **v**, based on the dependence (9.6), the 4-vector of the field strength takes the following form:

$$\begin{pmatrix}e'\\ \mathbf{E}'\end{pmatrix} = \frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix}\begin{pmatrix}0\\ \frac{\mathbf{c}}{r^2}q\left(C'^-\mathbb{X}'\right)\end{pmatrix}, \tag{9.7}$$

where $\mathbb{X}'$ specifies a new space-time distance from the source of the field, and we replaced the product VC with a paravector C'.

The last formula shows that:

- a scalar induction (introduced in Chapter 7, replacing the vector potential and Lorentz gauge condition) is

$$e\left(\mathbb{X}'\right) = \frac{\mathbf{vc}}{r^2\sqrt{1-v^2}}q(C'^-\mathbb{X}') \tag{9.8}$$

- an electric field strength is

$$\mathbf{E}\left(\mathbb{X}'\right) = \frac{\mathbf{c}}{r^2\sqrt{1-v^2}}q(C'^-\mathbb{X}') \tag{9.9}$$

- a magnetic induction is

$$\mathbf{B}\left(\mathbb{X}'\right) = \frac{\mathbf{v}\times\mathbf{c}}{r^2\sqrt{1-v^2}}q(C'^-\mathbb{X}') \tag{9.10}$$

where r is the phase delay and **c** is the phase direction (a unit vector). The resulting equations describe the field of a moving point charge.

Now let's have a look at the relationship between the above equations and Maxwell's equations. In the complex model, the electrostatics equations $\partial\mathbb{E} = \rho$ transformed under the relativistic transformation have the following form:

$$\begin{aligned}
\frac{\partial \mathbf{vE}}{\partial t}+\nabla\mathbf{E} &= \rho\sqrt{1-v^2} && (9.11)\\
\nabla(\mathbf{v}\times\mathbf{E}) &= 0 && (9.12)\\
\frac{\partial(\mathbf{v}\times\mathbf{E})}{\partial t} &= -\nabla\times\mathbf{E} && (9.13)\\
\frac{\partial \mathbf{E}}{\partial t}+\nabla(\mathbf{vE}) &= \nabla\times(\mathbf{v}\times\mathbf{E}) && (9.14)
\end{aligned}$$

Introducing $\mathbf{B}=\mathbf{v}\times\mathbf{E}$ we can see that the equations (9.12) and (9.13) do not differ from the known Maxwell equations. For the speed $v \ll c$ the equation (9.11) also takes a familiar form. However, the dependency (9.14) does not follow the valid theory. Let's discuss this equation.

9.2 The field surrounding a wire

At first, we'll calculate the value of the product **vE** at any point in space in case of a field resulting from the flow of current in a closed circuit, because we deal with such currents in practice.

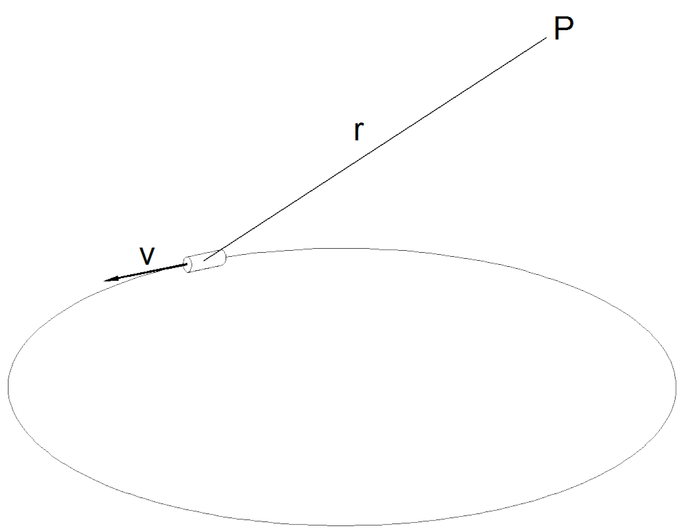

Figure 9.1:

The scalar induction from the specified slice is:

$$\begin{aligned}
de(P) &= \mathbf{v}d\mathbf{E}(P)=\rho\left(\mathbf{vr}/r^3\right)(\mathbf{s}d\mathbf{l}),\\
\text{which gives}\qquad de(P) &= \rho\left(\mathbf{vs}/r^3\right)(\mathbf{r}d\mathbf{l}),
\end{aligned}$$

because $\mathbf{v}\parallel d\mathbf{l}$, which is the cosine of the angle between vectors **s** and $d\mathbf{l}$, is the same as between vectors **s** and **v** and the same goes for the vector **r**. Since $\mathbf{s}\mathbf{v}\rho = J$ is the circuit current, the scalar induction e(P) at any point P from the entire circuit is:

$$e(P)=J\oint\frac{\mathbf{r}}{r^3}d\mathbf{l}=J\iint\nabla\times\frac{\mathbf{r}}{r^3}d\mathbf{s}=0 \tag{9.15}$$

Conclusion 9.2.1. In the case of a field resulting from the current flowing in a closed circuit, the scalar field component is equal to 0 at any place of the space.

The above result is important because it shows and confirms that in practice we do not encounter scalar induction, because in macroscopic systems we always deal with closed current circuits. So, the equations (9.11) - (9.14) take the following form:

$$\nabla \mathbf{E} = \rho\sqrt{1-v^2} \tag{9.16}$$

$$\nabla \mathbf{B} = 0 \tag{9.17}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} \tag{9.18}$$

$$\frac{\partial \mathbf{E}}{\partial t} = \nabla\times\mathbf{B} \tag{9.19}$$

Biot-Savart law.

We assume that the conductor has a constant cross-section and that the charge density is constant along the length of the conductor.

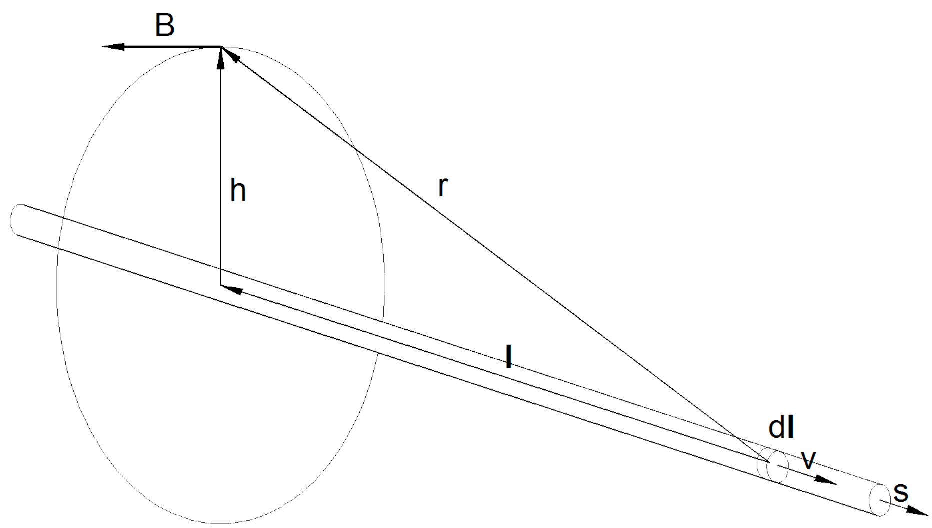

Figure 9.2:

At any point in space, we integrate the fields of charges flowing at a relativistic velocity **v** along the conductor L.

$$\begin{pmatrix} e \\ \mathbf{E}+i\mathbf{B} \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \int_{-\infty}^{+\infty} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} \begin{pmatrix} 0 \\ \frac{(\mathbf{l}+\mathbf{h})}{r^3}\rho\,(\mathbf{s}d\mathbf{l}) \end{pmatrix} \tag{9.20}$$

We get three integrals:

1. $e = \frac{\rho}{\sqrt{1-v^2}} \int_{-\infty}^{+\infty} \frac{\mathbf{lv}}{r^3}\mathbf{s}d\mathbf{l}$

2. $\mathbf{E} = \frac{\rho}{\sqrt{1-v^2}} \int_{-\infty}^{+\infty} \frac{\mathbf{r}}{r^3}\mathbf{s}d\mathbf{l}$

3. $\mathbf{B} = \frac{\rho}{\sqrt{1-v^2}} \int_{-\infty}^{+\infty} \frac{\mathbf{v}\times\mathbf{h}}{r^3}\mathbf{s}d\mathbf{l}$

Since $\mathbf{s} \parallel \mathbf{v} \parallel \mathbf{l} \perp \mathbf{h}$ we receive following conclusions

1. $e=\frac{\rho}{\sqrt{1-v^2}}\int\limits_{-\infty}^{+\infty}\frac{\mathbf{lv}}{r^3}\mathbf{s}d\mathbf{l}=\frac{\rho}{\sqrt{1-v^2}}\int\limits_{-\infty}^{+\infty}\frac{lv}{r^3}sdl=\frac{J}{\sqrt{1-v^2}}\int\limits_{-\infty}^{+\infty}\frac{l}{\left(\sqrt{l^2+h^2}\right)^3}dl=0$

2. There are as many negative charges in the wire as positive ones, only negative charges move and their field is $\mathbf{E}_-=\frac{\rho_-}{\sqrt{1-v^2}}\int\limits_{-\infty}^{+\infty}\frac{\mathbf{r}}{r^3}\mathbf{s}d\mathbf{l}$. The field produced by the positive charges differs by the dilation factor $\mathbf{E}_+=\rho_+\int\limits_{-\infty}^{+\infty}\frac{\mathbf{r}}{r^3}\mathbf{s}d\mathbf{l}$.
   In practice, the resultant electric field around the conductor is zero because the electrons in the conductors move at non-relativistic velocities.

3. $\mathbf{B}=\frac{\rho}{\sqrt{1-v^2}}\int\limits_{-\infty}^{+\infty}\frac{\mathbf{v}\times\mathbf{h}}{r^3}\mathbf{s}d\mathbf{l}=\frac{\mathbf{j}\times\mathbf{h}}{\sqrt{1-v^2}}\int\limits_{-\infty}^{+\infty}\frac{1}{\left(\sqrt{l^2+h^2}\right)^3}dl=\frac{2\mathbf{j}\times\mathbf{h}}{h^2\sqrt{1-v^2}}\lim_{l\to\infty}\frac{l}{\sqrt{l^2+h^2}}=\frac{2\mathbf{j}\times\mathbf{h}}{h^2\sqrt{1-v^2}}$

The last equation is the Biot-Savart law.

9.3 General field equations

Starting from the known Gauss, Stokes and Maxwell integral equations (no current, because they should be consistent with (9.16) - (9.19)) we can derive their integral complex equivalents.

Gauss equation:	$\oiint(\mathbf{f}+i\mathbf{g})d\mathbf{s}$	$=$	$\iiint\nabla(\mathbf{f}+i\mathbf{g})d\Omega$
Stokes equation:	$\oint(\mathbf{f}+i\mathbf{g})d\mathbf{l}$	$=$	$\iint\nabla\times(\mathbf{f}+i\mathbf{g})d\mathbf{s}$
Maxwell equations:	$\oiint(\mathbf{E}+i\mathbf{B})d\mathbf{s}$	$=$	$\iiint\rho\, d\Omega$
(without current):	$\oint(\mathbf{E}+i\mathbf{B})d\mathbf{l}$	$=$	$\iint\left(\frac{\partial\mathbf{E}}{\partial t}+i\frac{\partial\mathbf{B}}{\partial t}\right)id\mathbf{s}$

From the above equations we get the following dependencies:

$$\iint\left[\frac{\partial}{\partial t}(\mathbf{E}+i\mathbf{B})+i\nabla\times(\mathbf{E}+i\mathbf{B})\right]id\mathbf{s}=0 \tag{9.21}$$

$$\iiint\left[\nabla(\mathbf{E}+i\mathbf{B})-\rho\right]d\Omega=0 \tag{9.22}$$

For compliance with our equations, we need the gradient of the scalar field ∇e in the equation (9.21), and in the equation (9.22) the differential of this field over time:

$$\iint\left[\frac{\partial}{\partial t}(\mathbf{E}+i\mathbf{B})+\nabla e+i\nabla\times(\mathbf{E}+i\mathbf{B})\right]id\mathbf{s}=0 \tag{9.23}$$

$$\iiint\left[\nabla(\mathbf{E}+i\mathbf{B})+\frac{\partial e}{\partial t}-\rho\right]d\Omega=0 \tag{9.24}$$

The conditions

$$\iint\nabla e\,ds=0 \qquad \text{and} \qquad \iiint\frac{\partial e}{\partial t}d\Omega=0$$

are certainly met in the case of fields around stationary charges and those moving in closed circuits. Based on these results, we can conclude that the Gauss and Stokes theorems are also invariant when integrations take place on a time-independent surface or contour. It should be noted here that the integral $\iiint\rho(\mathbb{X})d\Omega$ is invariant, and since the volume and the scalar charge density are invariant, the charge must be invariant too.

9.4 Ampere's law

Let's get back to the formula (9.10) again. The magnetic field at X' from charges distributed throughout the entire space can be described by the equation

$$\mathbf{B}\left(X'-X_0'\right)=\int \frac{\mathbf{v}\times\mathbf{c}}{|V|\,r^2} q\left((VC)^{-}\left(X'-X_0'\right)\right)d^3x_0',$$

and since $\frac{\mathbf{v}}{|V|}q\left(C'^{-}\left(X'-X_0'\right)\right)=\mathbf{j}\left(C'^{-}\left(X'-X_0'\right)\right)$ is the current that flowed at the t_0 moment at the point $\mathbf{x}_0$, then we obtain the dependence

$$\mathbf{B}\left(\mathbb{X}'\right)=\int \frac{\mathbf{j}(C'^{-}(\mathbb{X}'))\times\mathbf{c}}{r^2}d^3x_0'$$

Given the above formula, following Jackson [12] (Chapter 5.3) one can derive the Ampere's law:

$$\nabla\times\mathbf{B}(\mathbb{X})=\mathbf{j}(\mathbb{X})$$

Hence it follows that Ampere's law needs not necessarily appear explicitly in Maxwell's equations. Using a complex relativistic transformation, we can derive it from the complex electric field equations.

9.5 Potential energy

Since in our model the charge density and the potential are always real invariant scalar fields, the potential energy density should also be an invariant real scalar field:

$$\begin{aligned} w(\mathbb{X}) &= \frac{1}{2}\rho(\mathbb{X})\varphi(\mathbb{X}) \\ w(\mathrm{X}-X_0) &= \frac{1}{2}\rho(\mathrm{X}-X_0)\varphi(X-\mathrm{X}_0), \end{aligned} \tag{9.25}$$

The first equation is always true, but the second one is an equation of electrostatics, where $\mathbb{X}=X-X_0$, that is: $\varphi\,(\mathrm{X}-X_0)$ is the electric field potential at X from the charge located at X_0.

9.6 Discussion

Summarizing the above chapter, we conclude that it is possible to create such a theory of electricity and magnetism, in which starting from the equations of electrostatics

$$\begin{bmatrix}\frac{\partial}{\partial t}\\ \nabla\end{bmatrix}\begin{pmatrix}0\\ \mathbf{E}\end{pmatrix}=\begin{pmatrix}\rho\\ 0\end{pmatrix}$$

and transforming them according to the principles of complex relativistic transformations, we obtain the laws of electrodynamics (Maxwell's equations) that do not contain the current density component and do not require the Lorenz gauge condition:

$$\begin{aligned} \frac{\partial e}{\partial t}+\nabla\mathbf{E} &= \rho \\ \nabla\mathbf{B} &= 0 \\ \frac{\partial\mathbf{B}}{\partial t} &= -\nabla\times\mathbf{E} \\ \frac{\partial\mathbf{E}}{\partial t}+\nabla e &= \nabla\times\mathbf{B}, \end{aligned}$$

where e is the scalar induction of the electric field introduced by us.

Below is a set of electromagnetic field equations in paravector notation with equations for potentials:

$$\partial \begin{pmatrix} e(\mathbb{X}) \\ (\mathbf{E}+i\mathbf{B})(\mathbb{X}) \end{pmatrix} = \begin{pmatrix} \rho(\mathbb{X}) \\ 0 \end{pmatrix} \qquad \text{and} \qquad \partial^- \begin{pmatrix} \varphi(\mathbb{X}) \\ 0 \end{pmatrix} = \begin{pmatrix} e(\mathbb{X}) \\ (\mathbf{E}+i\mathbf{B})(\mathbb{X}) \end{pmatrix}$$

It should be noted that although there is no Ampere's law in the paravector equivalents of Maxwell's equations, they do not contradict the classical theory. We also pay attention to the aesthetic side of such modified equations. Although in the real form they are not fully symmetrical, in the paravector form their elegance is unquestionable. It should be remembered that, in general, the velocity paravector is complex. This means that the scalar induction we introduced can also be a complex number.

It is important to check whether the principle of the conservation of the charge is respected

$$\frac{\partial \rho}{\partial t} + \nabla \mathbf{j} = 0 \tag{9.26}$$

Please pay attention to the meaning of the above formula. In the rest frame, the sense of the vector current density $\mathbf{j}$ has no direct relation to the movement of the ρ charge, because this is the current that flows from the source in any direction, and ρ is the charge density in the source region. The current may or may not be the movement of charges that flow through the source region, so it does not need to be a charge 'lift' current resulting from the relative movement of the observer and the ρ charges. The (9.26) formula is a quantitative scalar relationship between the varying amount of charges present in the source and the charges flowing from the source. Therefore, in this case, one must be careful deciding whether the transformational identities derived in Chapter 4 can be applied directly. Nevertheless, a formula can be constructed that confirms the relativistic invariance of the charge conservation principle. The formula is mathematically correct, but in order to be able to say that it captures the essence of the phenomenon, a deeper analysis is needed. The (9.26) formula can be represented in a symbolic form as:

$$(\partial^- J^-)_S = 0 \tag{9.27}$$

where J is the 4-dimensional charge-current density function. The above formula in its explicit form is as follows

$$\begin{bmatrix} \frac{\partial}{\partial t} \\ -\nabla \end{bmatrix} \begin{pmatrix} \rho \\ -\mathbf{j} \end{pmatrix} = \begin{pmatrix} 0 \\ ? \end{pmatrix} \tag{9.28}$$

The question mark means that we are not interested in the vector part of the above formula. According to the theorem 4.1.1, after switching to a frame that moves at -$\mathbf{v}$ velocity, the above formula transforms into

$$\frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial t'} \\ -\nabla' \end{bmatrix} \begin{pmatrix} \rho' \\ -\mathbf{j}\prime \end{pmatrix} = \begin{pmatrix} 0 \\ ? \end{pmatrix} \tag{9.29}$$

After moving the velocity paravector to the other side and the right-hand multiplication by the paravector $V^- = \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix}$ we get the dependence

$$\begin{bmatrix} \frac{\partial}{\partial t'} \\ -\nabla' \end{bmatrix} (\begin{pmatrix} \rho' \\ -\mathbf{j}\prime \end{pmatrix} V^-) = V \begin{pmatrix} 0 \\ ? \end{pmatrix} V^- \tag{9.30}$$

Since on the right side of the equation we have a rotation that does not change the scalar, in the symbolic notation the scalar part of the above equation is

$$[\partial'^-(V J')^-]_S = 0 \tag{9.31}$$

Thus, it has been shown that the principle of conservation of charge is invariant.

The previous chapter showed that realisation as a transformation has a serious drawback, namely the non-orthogonality. However, above we derived the formulas of the theory of electricity using the transformation described by the real velocity paravector, as if the realisation could be applicable. In Chapter 10, devoted to selected issues of Special Relativity, we will present the interpretation of realisation and will make a hypothesis explaining when and why realisation can be used.

Chapter 10

Selected issues of Special Relativity in complex space-time

Analysis of some formulas describing the inertial motion of particles allows us to show the possibility of various interpretations of the image of this motion. In complex space-time the Cauchy and the triangle inequalities have opposite directions and are not as general as their counterparts in Euclidean space. The Hamilton-Jacobi free-particle equation of motion and the Klein-Gordon equation are invariant with respect to paravector orthogonal transformations. We present the paravector formulas for the kinetic energy of a physical object, which differ from their classical relativistic counterparts. Finally, it is shown that in non-relativistic case a composition of velocities as well as energy and momentum formulas assume the form known from classical physics.

Chapter 6 presented the first interpretations of basic geometric concepts in as yet undefined complex space-time, where we only dealt with the real parts of the paravectors describing the physical concepts. In the meantime, we expanded our knowledge to field theory issues where it was easier to interpret imaginary vectors. The time has come to tackle complex space-time again, but already taking into account all the imaginary 4-vector components.

In a rest frame, we observe an object moving with a uniform motion. During the time Δt it traverses the way $\Delta\mathbf{x}$, which we write down in the form of a 4-vector $\begin{pmatrix} \Delta t \\ \Delta\mathbf{x} \end{pmatrix}$, where Δt is the observer's elapsed time interval, so it is a positive real number. Likewise, $\Delta\mathbf{x}$ is a real vector. Pulling out Δt in front of the 4-vector we get a paravector describing the velocity of the object $\begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix}$. Since it was not possible to observe that the objects with energy were moving faster than the speed of light, we assume that $0 \leqslant \det\begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} \leqslant 1$. When examining the invariance of the wave equation where the observer passed between two inertial frames, we used the paravector $\frac{1}{\sqrt{1-v^2}}\begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix}$, where the vector $\mathbf{v}$ was the relative speed of the frames.

We will now deal with the transformation equation in more detail. After the observer enters a frame moving at speed $\mathbf{v}$, the time interval Δt^0 of the observed object will change as follows:

$$\frac{1}{\sqrt{1-v^2}}\begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix}\begin{pmatrix} \Delta t \\ \Delta\mathbf{x} \end{pmatrix} = \begin{pmatrix} \Delta t^0 \\ 0 \end{pmatrix}, \tag{10.1}$$

hence

$$\frac{1}{\sqrt{1-v^2}}\begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix} = \frac{1}{\Delta t^2 - \Delta x^2}\begin{pmatrix} \Delta t^0 \\ 0 \end{pmatrix}\begin{pmatrix} \Delta t \\ -\Delta\mathbf{x} \end{pmatrix} = \frac{1}{\Delta t^2 - \Delta x^2}\begin{pmatrix} \Delta t^0 \Delta t \\ -\Delta t^0 \Delta\mathbf{x} \end{pmatrix}.$$

From the scalar part it follows that

$$\frac{1}{\sqrt{1-v^2}}=\frac{\Delta t^0\Delta t}{\Delta t^2-\Delta x^2}$$

If we change the dilation factor in the vector part according to the above equality, we get the classical definition of velocity

$$\mathbf{v}=\frac{\Delta\mathbf{x}}{\Delta t} \tag{10.2}$$

So, the velocity is the ratio of the spatial component to the time component of the four-vector position of the described object in the observer's frame.

Looking at the equation representing the real 4-vector transformation

$$\begin{pmatrix}\Delta t'\\ \Delta\mathbf{x}'+i\mathbf{y}'\end{pmatrix}=\frac{1}{\sqrt{1-v_1^2}}\begin{bmatrix}1\\ \mathbf{v}_1\end{bmatrix}\begin{pmatrix}\Delta t\\ \Delta\mathbf{x}\end{pmatrix}$$

it can be seen that the 4-vectors on the right and left represent the motion of the object in two mutually moving frames. So it is enough to draw the times before the 4-vectors to get the object's velocities in these frames.

$$\Delta t'\begin{bmatrix}1\\ \mathbf{v}'\end{bmatrix}=\frac{\Delta t}{\sqrt{1-v_1^2}}\begin{bmatrix}1\\ \mathbf{v}_1\end{bmatrix}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix}=\frac{\Delta t(1+\mathbf{v}_1\mathbf{v})}{\sqrt{1-v_1^2}}\begin{bmatrix}1\\ \frac{\mathbf{v}+\mathbf{v}_1+i\mathbf{v}_1\times\mathbf{v}}{1+\mathbf{v}\mathbf{v}_1}\end{bmatrix} \tag{10.3}$$

where we get the well-known formula for the syntax of velocity and the formula for the relationship between times in frames moving in relation to each other at the speed $\mathbf{v}'$. The second formula is a bit strange, because the dilation depends not only on the Lorentz factor, but also on the direction, which we have already interpreted as a spatial 'crossing' of simultaneity, in other words, a spatial desynchronization.

$$\Delta t'=\frac{(1+\mathbf{v}\mathbf{v}_1)}{\sqrt{1-v_1^2}}\Delta t \qquad \text{and} \qquad \mathbf{v}'=\frac{\mathbf{v}+\mathbf{v}_1+i\mathbf{v}_1\times\mathbf{v}}{1+\mathbf{v}\mathbf{v}_1} \tag{10.4}$$

Let us return to the formula (10.1) from which it follows

$$\frac{1}{\sqrt{1-v^2}}\begin{pmatrix}\Delta t-\mathbf{v}\Delta\mathbf{x}\\ \Delta\mathbf{x}-\mathbf{v}\Delta t-i\mathbf{v}\times\Delta\mathbf{x}\end{pmatrix}=\begin{pmatrix}\Delta t^0\\ 0\end{pmatrix},$$

which can be written as a system of equations

$$\begin{aligned}\Delta t-\mathbf{v}\Delta\mathbf{x} &= \Delta t^0\sqrt{1-v^2}\\ \Delta\mathbf{x} &= \mathbf{v}\Delta t\\ \mathbf{v}\times\Delta\mathbf{x} &= 0\end{aligned}$$

After placing the $\Delta\mathbf{x}$ from the second equation into the first one, we obtain

$$\frac{\Delta t^0}{\Delta t}=\sqrt{1-v^2}$$

Thus, the dilation factor determines the proportion of the time flowing in the observed moving frame to the time in the observer's frame. Of course, the so-called twin paradox does not result from this, because this factor only works when both frames are in relative motion. After the velocities are equalized, the dilation factor becomes one and it may turn out that the same time has elapsed in both frames. The difference of both times occurs only and exclusively in the 4-vector simultaneously with the spatial component, and thus simultaneously with the path covered by the moving object. This phenomenon should therefore be understood as an illusion that accompanies movement at a great speed, just as an illusion is the shortening of the mast of a sailing ship moving towards the horizon. Incidentally, the same is true of the applicable theory. Someone once interpreted the results very unfortunately and compared the times in different frames, disregarding the fact that NO ONE CAN DO IT! Only invariant quantities can be compared, i.e. space-time intervals, not times.

10.1 A spherical explosion

We will now consider what the image of the explosion front looks like after Δt time from the moment of the explosion. The parameter here is the direction of the particle velocity vector $\mathbf{w}$, and its length is a constant velocity w of the explosion front (Fig. 10.1).

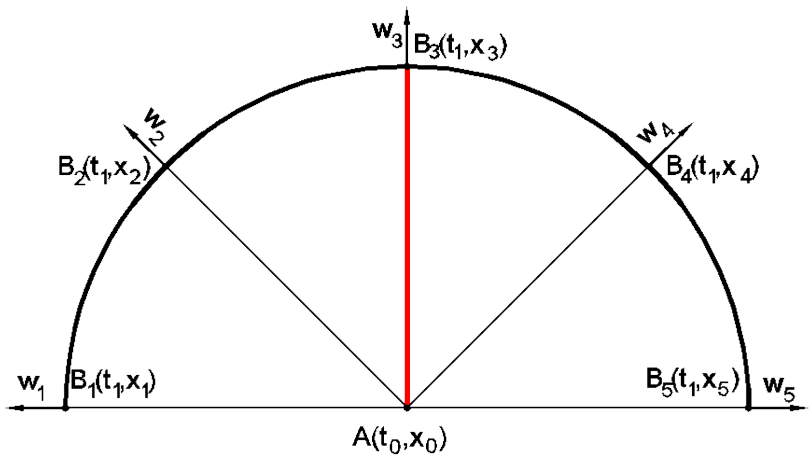

Figure 10.1:

$$\frac{1}{\sqrt{1-v_0^2}}\left[\begin{array}{c}1\\-\mathbf{w}\end{array}\right]\left(\begin{array}{c}\Delta t\\ \Delta\mathbf{x}\end{array}\right)=\left(\begin{array}{c}\Delta t^0\\0\end{array}\right) \tag{10.5}$$

Since the dilation factor only makes sense when moving from a resting to a moving frame, and we as observers are not interested in the proper time (Δt^0) of the observed particle, we unite them and replace them with the parameter s. In the rest OX frame the explosion front is described by the equations

$$\begin{array}{l}\Delta t-\mathbf{w}\Delta\mathbf{x}=s\\ \Delta\mathbf{x}=\mathbf{w}\Delta t\\ \mathbf{w}\times\Delta\mathbf{x}=0\end{array} \tag{10.6}$$

The number s is some parameter that is proportional to the time Δt because if we substitute $\Delta\mathbf{x}$, from the second equation we get $s=\Delta t(1-w^2)$. For the observer, the first equation is irrelevant because he is interested in observing the motion of particles ($\Delta\mathbf{x}$) per his time (Δt). From the second equation of the above system, we can see that the blast front is a sphere expanding with time. The $\Delta\mathbf{x}$ vectors are real.

Now, we pass to the OX' frame moving at velocity of $-\mathbf{v}$.

$$\frac{1}{\sqrt{1-v^2}}\left[\begin{array}{c}1\\-\mathbf{w}\end{array}\right]\left[\begin{array}{c}1\\-\mathbf{v}\end{array}\right]\left(\begin{array}{c}\Delta t'\\ \Delta\mathbf{x}'+i\mathbf{y}'\end{array}\right)=\left(\begin{array}{c}\Delta t^0\\0\end{array}\right) \tag{10.7}$$

We can transform the above formula in two ways:

1. $$\left[\begin{array}{c}1\\-\frac{\mathbf{v}+\mathbf{w}}{(1+\mathbf{vw})}+i\frac{\mathbf{w}\times\mathbf{v}}{(1+\mathbf{vw})}\end{array}\right]\left(\begin{array}{c}\Delta t'\\ \Delta\mathbf{x}'+i\mathbf{y}'\end{array}\right)=\left(\begin{array}{c}k\Delta t^0/(1+\mathbf{vw})\\0\end{array}\right) \tag{10.8}$$

2. $$\left(\begin{array}{c}\Delta t'\\ \Delta\mathbf{x}'+i\mathbf{y}'\end{array}\right)=\frac{\Delta t^0}{k}\left[\begin{array}{c}1\\ \mathbf{v}\end{array}\right]\left[\begin{array}{c}1\\ \mathbf{w}\end{array}\right]=\frac{\Delta t^0}{k}\left[\begin{array}{c}1+\mathbf{vw}\\ \mathbf{v}+\mathbf{w}+i\mathbf{v}\times\mathbf{w}\end{array}\right] \tag{10.9}$$

The value of $k=\sqrt{1-v^2}\sqrt{1-w^2}$ is a real number. Time in the OX' frame is real, and the vector of the object's position change is complex.

In the formula (10.8) the OX' time is the 'deformed' OX frame time. Time 'deformation' is a spatial desynchronization of $\Delta t' = k\Delta t^0/(1+\mathbf{vw})$. When observing an explosion in a primed frame, we are not interested in the mutual relation of the coordinates of both systems. We want to describe the observed experiment in primed coordinates, i.e. the change of spatial primed coordinates from primed time. To avoid confusing real and imaginary parts, let's write complex quantities in the equation (10.8) using Greek letters:

$$\begin{bmatrix} 1 \\ -\boldsymbol{\vartheta} \end{bmatrix} \begin{pmatrix} \Delta t' \\ \Delta \boldsymbol{\chi} \end{pmatrix} = \begin{pmatrix} k\Delta t/(1+\mathbf{vw}) \\ 0 \end{pmatrix} \tag{10.10}$$

Let's just deal with the vector equation:

$$\Delta\boldsymbol{\chi} - \Delta t' \boldsymbol{\vartheta} - i\boldsymbol{\vartheta} \times \Delta\boldsymbol{\chi} = 0$$

Since complex vectors have the same properties as real ones, it must be:

$$\Delta\boldsymbol{\chi} - \Delta t' \boldsymbol{\vartheta} = 0 \qquad \text{and} \qquad \boldsymbol{\vartheta} \times \Delta\boldsymbol{\chi} = 0$$

from where it follows that

$$\Delta\mathbf{x}' = \frac{\mathbf{v}+\mathbf{w}}{1+\mathbf{vw}}\Delta t' \qquad \text{and} \qquad \mathbf{y}' = \frac{\mathbf{v}\times\mathbf{w}}{1+\mathbf{vw}}\Delta t' \tag{10.11}$$

In the vector part of the equation (10.8) we have a velocity whose real vectors for selected directions are plotted in Figure 10.2. The real equation (10.11) shows that particles spread simultaneously in the time of the rest frame (Δt) form a flattened sphere in primed time. Now let us do a small swap that mathematically changes nothing, but gives a different physical interpretation: Let us move the expression $1+\mathbf{vw}$ from under the velocity vector against time.

$$\Delta\mathbf{x}' = (\mathbf{v}+\mathbf{w})\frac{\Delta t'}{1+\mathbf{vw}} \tag{10.12}$$

The above equation shows the same particles but at different times. It seems to be more in line with the spirit of the equation (10.8), since from its equivalent (10.9) it can be seen that the relativistic transformation does not preserve spatial simultaneity. We must remember that the formulas (10.11) and (10.12) follow from the formula (10.8), but are not equivalent to it. The formulas (10.8) and (10.9) connect coordinates from different frames. The formulas (10.11) and (10.12) show how the coordinates in one frame must depend on each other. Interpretations of the latter two formulas are presented in the figure below. A flattened sphere resulting from the equation (10.11) is drawn with a continuous black line, while the broken line is a sphere not spatially deformed, but deformed in time, resulting from the formula (10.12). Blue color means that the particles were there before B_3 was in the place $(t'_3, \mathbf{x}'_3)$, and in green are the places of the particles that will get there later. In order for the vector $(\mathbf{v}+\mathbf{w})\Delta t'/(1+\mathbf{vw})$ to circle the ball, it must be such that for $\mathbf{vw} < 0$ time $\Delta t'$ is shorter than $\Delta t'_{B_3}$ of particle B_3, that is $\Delta t' < \Delta t'_{B_3}$. It is difficult to present on a flat, stationary sheet of paper what is happening in space and time. The reader need to use their imagination for this.

For the calculations, $v = 0.4$ and $w = 0.8$ are assumed.

Since all paravectors in (10.7) are proper, the vector $\Delta\mathbf{x}'$ must be perpendicular to the vector $\mathbf{y}'$. The value of k is a real number and does not depend on the mutual directions of the vectors $\mathbf{w}$ and $\mathbf{v}$, so we can conclude from the scalar equation that at the same moment in different places of the explosion front there are particles that correspond to the same particles but at a different moment in the OX frame, which results from the component $1+\mathbf{vw}$. This means that simultaneity in a rest frame does not have to correspond to simultaneity in a moving frame. Otherwise, time would flow differently in different directions in the OX' frame (time in the OX' frame would depend on the angle between vectors $\mathbf{v}$ and $\mathbf{w}$), which for an observer in the OX' frame does not make sense.

Simultaneous particles in the OX frame non-simultaneously outline an ellipsoid in the OX' frame, while the simultaneous particles in the OX' frame form a sphere. We choose the time of the particle (B_3) as the reference time in the primed frame, because the dot product of the compounded velocity vectors is equal to 0. If we measure the time from the explosion, then after the corresponding time in both frames we will only 'see' the red point (B_3). Blue points will be earlier and green points will be later. We used the word 'deformation' in

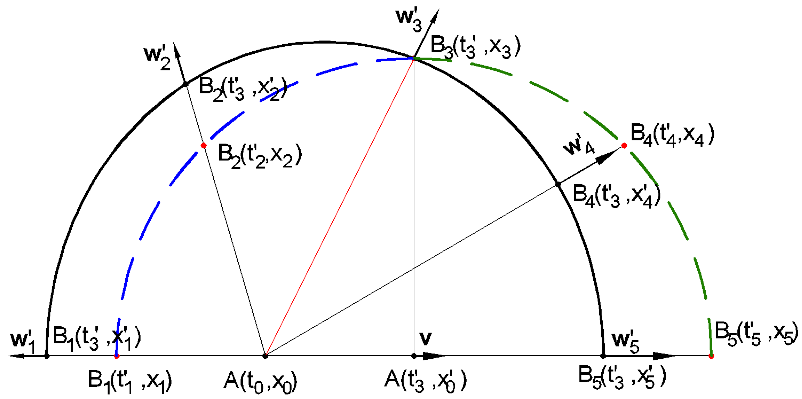

Figure 10.2:

apostrophes because the real image is subject to deformation. In a space-time sense, no deformation has occurred because the relativistic transformation preserves the dot product. It is because of this 'deformation' that we see part of the electric field of a moving charge as a magnetic field.

The problem of understanding relativistic phenomena lies not so much in considering the dilation factor, as emphasized in the explanations of the applicable STR, as in the scalar and vector products occurring in transformation formulas and in getting used to imaginary components.

We can see that by describing many particles, we could interpret an imaginary vector as a parameter of spatial deformation, but what about the imaginary component of a single particle motion? We hypothesized that living in a stationary frame, we should be able to 'project' the coordinates of the complex vectors onto our real space-time. After all, we perceive time which flows at the same pace here as it does in a moving frame, as if it flew slower. For the description of a single object (or many objects, but in a situation where their mutual relation is not important), we can use the orthogonal velocity paravector realisation, which also realises the position 4-vector. Below Figure 10.3 we add a purple sphere resulting from the realisation of the velocity paravector.

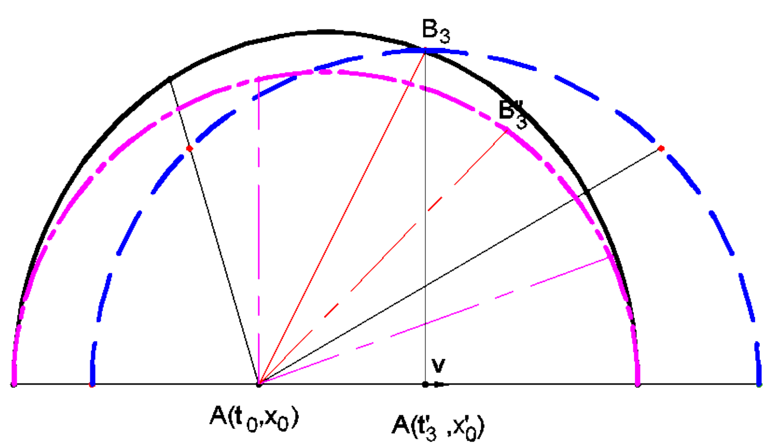

Figure 10.3:

Explanation for Figure 10.3:

Dashed line - $\mathbf{w}' = \mathbf{w} + \mathbf{v}$

Black line - $\mathbf{w}' = \dfrac{\mathbf{v} + \mathbf{w}}{(1 + \mathbf{vw})}$

Violet line - $\mathbf{w}' = \dfrac{(1 + \mathbf{wv})(\mathbf{w} + \mathbf{v}) + (\mathbf{w} + \mathbf{v}) \times (\mathbf{w} \times \mathbf{v})}{(1 + \mathbf{wv})^2 + (\mathbf{w} \times \mathbf{v})^2}$

In Figure 10.3 everything is done in the primed frame, so prime here means spatial coordinates simultaneous in time t_3. It should be noted that in the first two interpretations, the direction of motion of any selected particle is the same and results from a simple Euclidean sum of the vectors $\mathbf{w} + \mathbf{v}$, while if it is made real, the direction changes. The lengths of the resultant vector are also different. In the first two cases, the actual resultant velocity may be higher than the velocity of light, but there is an imaginary component which makes the resultant complex velocity lower than the velocity of light, which is consistent with the 2nd postulate of STR. In the first case, we included the dot product of velocity in the coordinates of the position, so there is an extension of time in relation to the observer's time (directional desynchronization).

10.2 Particle motion with elastic rebound

The material point starts from the center of the sphere, for the time Δt_1 it flies at a uniform velocity $\mathbf{w}$ and returns to the center after an elastic rebound from the wall (Fig. 10.2).

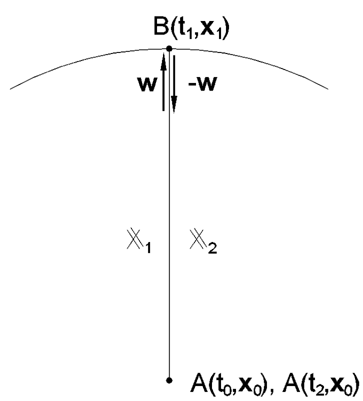

Figure 10.4:

$$\mathbb{X}_1 + \mathbb{X}_2 = \Delta t, \tag{10.13}$$

where the coordinates satisfy the equations:

before the rebound: $\frac{1}{\sqrt{1-w^2}} \begin{bmatrix} 1 \\ -\mathbf{w} \end{bmatrix} \begin{pmatrix} \Delta t_1 \\ \Delta \mathbf{x}_1 \end{pmatrix} = \begin{pmatrix} \Delta t^0 \\ 0 \end{pmatrix}$

and after the rebound: $\frac{1}{\sqrt{1-w^2}} \begin{bmatrix} 1 \\ \mathbf{w} \end{bmatrix} \begin{pmatrix} \Delta t_2 \\ \Delta \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \Delta t^0 \\ 0 \end{pmatrix}$

From the above we get the obvious conclusions that $\Delta t_1 = \Delta t_2$ and $\Delta \mathbf{x}_1 = -\Delta \mathbf{x}_2$. Since the 4-vectors $\mathbb{X}_1$ and $\mathbb{X}_2$ are inverted to each other, the equation (10.13) can be written as follows

$$\mathbb{X}_1 + \mathbb{X}_1^- = \Delta t \tag{10.14}$$

The determinant of the left side of the equation (10.13) is

$$\det(\mathbb{X}_1 + \mathbb{X}_2) = \det \mathbb{X}_1 + \det \mathbb{X}_2 + 2\langle \mathbb{X}_1, \mathbb{X}_2 \rangle = \tag{10.15}$$

$$= \Delta t_1^2 - \Delta \mathbf{x}^2 + \Delta t_2^2 - \Delta \mathbf{x}^2 + 2(\Delta t_1 \Delta t_2 + \Delta \mathbf{x}^2) = (\Delta t_1 + \Delta t_2)^2 = \det \mathbb{X}$$

We pass to the moving frame

$$V^- V \mathbb{X}_1 + V^- V \mathbb{X}_2 = V^- V \Delta t,$$

which, in a moving frame, is a sum of four-vectors

$$\mathbb{X}_1' + \mathbb{X}_2' = \mathbb{X}_3', \tag{10.16}$$

because

$$\begin{cases} \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t_1' \\ \Delta \mathbf{x}_1' + i \mathbf{y}_1' \end{pmatrix} = \begin{pmatrix} \Delta t_1 \\ \Delta \mathbf{x}_1 \end{pmatrix} \\ \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t_2' \\ \Delta \mathbf{x}_2' + i \mathbf{y}_2' \end{pmatrix} = \begin{pmatrix} \Delta t_1 \\ -\Delta \mathbf{x}_1 \end{pmatrix} \\ \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t_3' \\ \Delta \mathbf{x}_3' \end{pmatrix} = \begin{pmatrix} \Delta t \\ 0 \end{pmatrix} \end{cases} \tag{10.17}$$

Let us describe the equation (10.16) in the paravector form without the dilation factor, which will be reduced in the equation, and since we have the coordinates of the four-vectors from one frame of reference, it does not matter anyway. In other words, we omit the dilation factor when we conduct observations in one frame (e.g. primed).

$$\begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t_3' \\ \Delta \mathbf{x}_3' \end{pmatrix} = \begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t_1' \\ \Delta \mathbf{x}_1' + i \mathbf{y}_1' \end{pmatrix} + \begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t_2' \\ \Delta \mathbf{x}_2' + i \mathbf{y}_2' \end{pmatrix} = \tag{10.18}$$

$$= \begin{pmatrix} \Delta t_1' - \mathbf{v} \Delta \mathbf{x}_1' \\ \Delta \mathbf{x}_1' - \mathbf{v} \Delta t_1' - i \mathbf{v} \times \Delta \mathbf{x}_1 \end{pmatrix} + \begin{pmatrix} \Delta t_2' - \mathbf{v} \Delta \mathbf{x}_2' \\ \Delta \mathbf{x}_2' - \mathbf{v} \Delta t_2' - i \mathbf{v} \times \Delta \mathbf{x}_2 \end{pmatrix}$$

$$\begin{pmatrix} \Delta t_3' - \mathbf{v} \Delta \mathbf{x}_3' \\ \Delta \mathbf{x}_3' - \mathbf{v} \Delta t_3' \end{pmatrix} = \begin{pmatrix} \Delta t_1' + \Delta t_2' - \mathbf{v} \left(\Delta \mathbf{x}_1' + \Delta \mathbf{x}_2' \right) \\ \Delta \mathbf{x}_1' + \Delta \mathbf{x}_2' - \mathbf{v} \left(\Delta t_1' + \Delta t_2' \right) - i \mathbf{v} \times \left(\Delta \mathbf{x}_1' + \Delta \mathbf{x}_2' \right) \end{pmatrix} \tag{10.19}$$

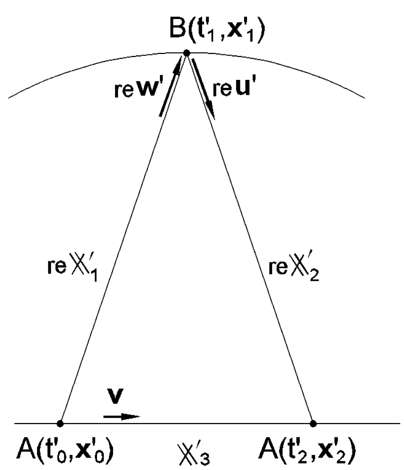

Figure 10.5:

The $\mathbf{v} \times \Delta \mathbf{x}_3'$ component in the last equation was omitted because it is equal to 0. For particles moving perpendicularly to velocity $\mathbf{v}$ ($\mathbf{w} \perp \mathbf{v}$), the total time is equal to the sum of the times before and after the rebound $\Delta t_3' = \Delta t_1' + \Delta t_2'$, and on the left we have a real vector, so the sum of the vectors $\Delta \mathbf{x}_1' + \Delta \mathbf{x}_2'$ must be parallel to $\mathbf{v}$. There is a shift in time for the remaining particles. We can see that although we started from the formulas of the relativistic transformation in space-time, real vectors are composed as in Euclidean geometry. The above results do not depend on the direction of the composite velocity vectors as long as we do not pay attention to simultaneity, which is our greatest strain.

We still need to check the determinant of the equation (10.16), because we will soon define the metric based on the determinant. The formulas (10.17) show that

$$\left\{\begin{array}{rcl}\Delta t_1'+\Delta t_2' & = & \Delta t_3'\\ \Delta\mathbf{x}_1'+\Delta\mathbf{x}_2' & = & \Delta\mathbf{x}_3'\\ \mathbf{y}_1'+\mathbf{y}_2' & = & 0\end{array}\right. \tag{10.20}$$

so

$$\det(\mathbb{X}_1'+\mathbb{X}_2')=\det\mathbb{X}_1'+\det\mathbb{X}_2'+2\langle\mathbb{X}_1',\mathbb{X}_2'\rangle= \tag{10.21}$$

$$=\Delta t_1'^2-\Delta\mathbf{x}_1'^2+\mathbf{y}_1'^2+\Delta t_2'^2-\Delta\mathbf{x}_2'^2+\mathbf{y}_2'^2+2[\Delta t_1'\Delta t_2'-\Delta\mathbf{x}_1'\Delta\mathbf{x}_2'+\mathbf{y}_1'\mathbf{y}_2'-i(\Delta\mathbf{x}_1'\mathbf{y}_2'+\Delta\mathbf{x}_2'\mathbf{y}_1')]=$$

$$=(\Delta t_1'+\Delta t_2')^2-(\Delta\mathbf{x}_1'+\Delta\mathbf{x}_2')^2=\Delta t_3'^2-\Delta\mathbf{x}_3'^2=\det\mathbb{X}_3'$$

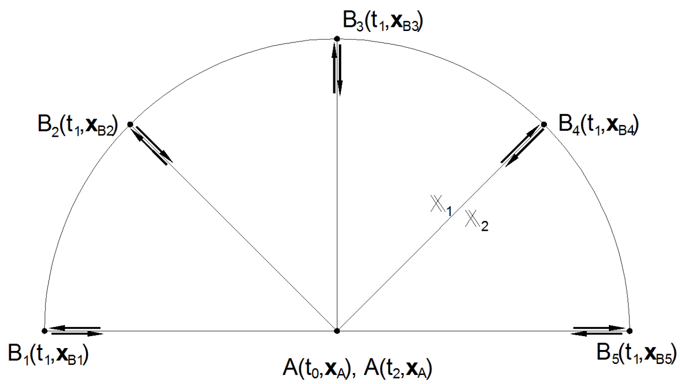

Figure 10.6:

In the rest frame, the b particle moves uniformly from the point $A(t_0,\mathbf{x}_A)$ to the point $B(t_1,\mathbf{x}_B)$, then returns at the same speed to the initial point, where it arrives at t_2, that is $A(t_2,\mathbf{x}_A)$. The second particle a waits for the first particle at $\mathbf{x}_A$. We 'observe' at the same time several particles scattering in different directions. It is presented in Fig. 10.6. When an observer passes to a frame that moves at the speed $-\mathbf{v}$, he 'sees' 4-vectors as in Figure 10.7.

In the rest frame, we describe the motion of a and b particles in the following way:

The b particle before rebound $\quad \mathbb{X}_1=\begin{pmatrix}\Delta t_1\\ \Delta\mathbf{x}_1\end{pmatrix}=\begin{pmatrix}t_1-t_0\\ \mathbf{x}_B-\mathbf{x}_A\end{pmatrix},$

the b particle after rebound $\quad \mathbb{X}_2=\begin{pmatrix}\Delta t_2\\ \Delta\mathbf{x}_2\end{pmatrix}=\begin{pmatrix}t_2-t_1\\ \mathbf{x}_A-\mathbf{x}_B\end{pmatrix}$

and the rest a particle $\quad \mathbb{X}_3=\begin{pmatrix}\Delta t_3\\ 0\end{pmatrix}=\begin{pmatrix}t_2-t_0\\ \mathbf{x}_A-\mathbf{x}_A\end{pmatrix}.$

After the observer passes to the frame moving at real speed $-\mathbf{v}$, the above 4-vectors are transformed:

the b particle before rebound $\quad \mathbb{X}_1'=\frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix}\begin{pmatrix}\Delta t_1\\ \Delta\mathbf{x}_1\end{pmatrix},$

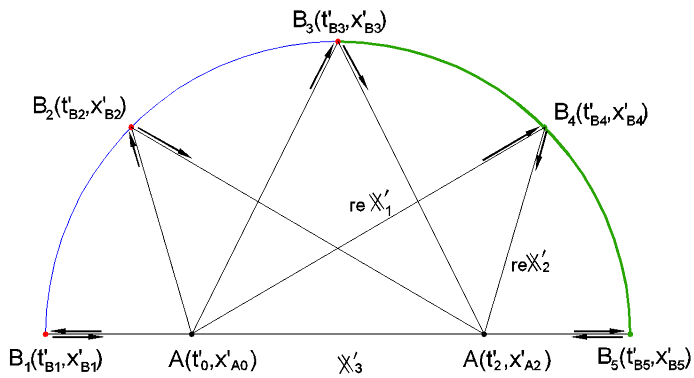

Figure 10.7:

the b particle after rebound $\quad \mathbb{X}_2' = \frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix}\begin{pmatrix}\Delta t_2\\ \Delta \mathbf{x}_2\end{pmatrix}$

and the a particle $\quad \mathbb{X}_3' = \frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix}\begin{pmatrix}\Delta t_3\\ 0\end{pmatrix},$

which gives the following sum

$$\begin{pmatrix}\Delta t_3'\\ \Delta \mathbf{x}_3'\end{pmatrix} = \begin{pmatrix}\Delta t_1'\\ \Delta \mathbf{x}_1' + i\mathbf{y}_1'\end{pmatrix} + \begin{pmatrix}\Delta t_2'\\ \Delta \mathbf{x}_2' + i\mathbf{y}_2'\end{pmatrix} \tag{10.22}$$

The times $\Delta t_1'$ and $\Delta t_2'$, although they correspond to the same times in the rest frame, do not have to be the same in the moving frame, because they were transformed with values depending on the direction ($\mathbf{v}\Delta\mathbf{x}$). Therefore, the simultaneous rebounds in the rest frame correspond to the collision points shifting in time in the moving frame. In the OX' frame it is not visible, unlike the spherical mark left by the rebound. We ask the question: Has a deformation occurred or not? In real space - yes, in time too, but in complex space-time there is no deformation, because the relativistic transformation is orthogonal.

The imaginary component of the path $\mathbf{y}$ is not the path to go, but it is closely related to the path $\Delta\mathbf{x}$, the time Δt, and the speed that causes it to get out of sync. For the imaginary component of the road $\mathbf{y}_1' = -\mathbf{y}_2'$, dilation factors are omitted because, as said numerous times, they do matter only when passing from frame to frame. For describing the position in the coordinates of one frame (this time a primed ones), they are not needed for anything and only introduce complications. It should be noted that the position vectors of the b particle are complex and their imaginary components shorten. So we can write the sum without the imaginary part and the equation will be correct (this is how we had seen the particles before the introduction of our theory), but we leave these imaginary vectors because they make sense in complex space-time. Thanks to them, we can seemingly exceed the speed of light (the length of the real velocity vector may be greater than 1), remaining in accordance with the fact that the highest speed at which physical objects can move is c.

10.3 Metric of movement of physical objects

We will now follow the same experiment but with a loss of energy on rebounding. In the experimenter's frame inside the spherical laboratory, the material point starts from the center of the sphere, travels at velocity $\mathbf{v}_1$, then bounces off the lab wall and returns to the center of the sphere at velocity $\mathbf{v}_2$ (collision with energy loss).

In the frame of a moving point, the entire experiment lasted $\Delta t^0 = \Delta t_1^0 + \Delta t_2^0$. In the experimenter's frame, the motion of a point can be described by the vector part from the paravector equations:

$$\frac{1}{\sqrt{1-v_1^2}}\begin{bmatrix}1\\ -\mathbf{v}_1\end{bmatrix}\begin{pmatrix}\Delta t_1\\ \Delta\mathbf{x}_1\end{pmatrix} = \begin{pmatrix}\Delta t_1^0\\ 0\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{1-v_2^2}}\begin{bmatrix}1\\ -\mathbf{v}_2\end{bmatrix}\begin{pmatrix}\Delta t_2\\ \Delta\mathbf{x}_2\end{pmatrix} = \begin{pmatrix}\Delta t_2^0\\ 0\end{pmatrix}$$

Since the entire experiment takes $\Delta t_1 + \Delta t_2$, and the observed point at the end will return to the initial point, the coordinates must satisfy the relationship:

$$\begin{pmatrix} \Delta t_1 \\ \Delta \mathbf{x}_1 \end{pmatrix} + \begin{pmatrix} \Delta t_2 \\ \Delta \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \Delta t_3 \\ 0 \end{pmatrix}, \qquad \text{that is} \qquad \mathbb{X}_1 + \mathbb{X}_2 = \mathbb{X}_3, \tag{10.23}$$

$$\text{where} \qquad \begin{pmatrix} \Delta t_1 \\ \Delta \mathbf{x}_1 \end{pmatrix} = \frac{\Delta t_1^0}{\sqrt{1-v_1^2}} \begin{bmatrix} 1 \\ \mathbf{v}_1 \end{bmatrix} \qquad \text{and} \qquad \begin{pmatrix} \Delta t_2 \\ \Delta \mathbf{x}_2 \end{pmatrix} = \frac{\Delta t_2^0}{\sqrt{1-v_2^2}} \begin{bmatrix} 1 \\ \mathbf{v}_2 \end{bmatrix}.$$

We are talking about the coordinates of a physical object, that is, all four vectors above are proper.

Another observer sits in a vehicle moving at $-\mathbf{v}$ and describes the point with the following formulas:

- first segment $\quad V^- \mathbb{X}_1' = \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t_1' \\ \Delta \mathbf{x}_1' + i\mathbf{y}_1' \end{pmatrix} = \begin{pmatrix} \Delta t_1 \\ \Delta \mathbf{x}_1 \end{pmatrix}$
- second segment $\quad V^- \mathbb{X}_2' = \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t_2' \\ \Delta \mathbf{x}_2' + i\mathbf{y}_2' \end{pmatrix} = \begin{pmatrix} \Delta t_2 \\ \Delta \mathbf{x}_2 \end{pmatrix}$
- path travelled through the center of the laboratory $\quad V^- \mathbb{X}_3' = \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t' \\ \Delta \mathbf{x}' \end{pmatrix} = \begin{pmatrix} \Delta t_3 \\ 0 \end{pmatrix}$

We return to the equation (10.23) and write down $V\mathbb{X}_1 + V\mathbb{X}_2 = V\mathbb{X}_3$, so $\mathbb{X}_1' + \mathbb{X}_2' = \mathbb{X}_3'$. The result is obvious: In both frames the corresponding four-vectors are additive. Since the velocity paravector represents the quotient:

$$V = \frac{\mathbb{X}}{|\mathbb{X}|} = \frac{1}{\sqrt{\Delta^2 t - \Delta^2 x}} \begin{pmatrix} \Delta t \\ \Delta \mathbf{x} \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix},$$

otherwise the sum of four-vectors $\mathbb{X}_1 + \mathbb{X}_2 = \mathbb{X}_3$ can also be written by extracting times.

$$\Delta t_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \Delta t_1 \begin{bmatrix} 1 \\ \mathbf{v}_1 \end{bmatrix} + \Delta t_2 \begin{bmatrix} 1 \\ \mathbf{v}_2 \end{bmatrix}$$

In a frame moving at velocity $-\mathbf{v}$ we have:

$$\frac{\Delta t_3}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} = \frac{\Delta t_1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v}_1 \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} + \frac{\Delta t_2}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} \tag{10.24}$$

In the above equation, we can reduce the factor $\sqrt{1-v^2}$ because it is not important for the relationship between the variables in the primed frame, hence

$$\Delta t_3' \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} = \Delta t_1' \begin{bmatrix} 1 \\ \mathbf{v}_1' \end{bmatrix} + \Delta t_2' \begin{bmatrix} 1 \\ \mathbf{v}_2' \end{bmatrix},$$

where $\quad \Delta t_i' = (1 + \mathbf{v}\mathbf{v}_i)\Delta t_i \quad$ and $\quad \mathbf{v}_i' = (\mathbf{v}_i + \mathbf{v} + i\mathbf{v}_i \times \mathbf{v})/(1 + \mathbf{v}\mathbf{v}_i)$

or otherwise $\quad \begin{pmatrix} \Delta t' \\ \Delta t' \mathbf{v} \end{pmatrix} = \begin{pmatrix} \Delta t_1' \\ \Delta t_1' \mathbf{v}_1' \end{pmatrix} + \begin{pmatrix} \Delta t_2' \\ \Delta t_2' \mathbf{v}_2' \end{pmatrix},$

which means

$$\begin{pmatrix} \Delta t' \\ \Delta \mathbf{x}' \end{pmatrix} = \begin{pmatrix} \Delta t_1' \\ \Delta \mathbf{x}_1' \end{pmatrix} + \begin{pmatrix} \Delta t_2' \\ \Delta \mathbf{x}_2' \end{pmatrix} \tag{10.25}$$

At this point, we bring to the readers' attention one more very important conclusion that will allow us to understand the metric of physical space. Since the relativistic transformation preserves the scalar product, in order for the triangle to close (i.e. for the particles to meet at one place and time), the 4-vectors describing the position of the physical particles in the triangle must satisfy the following theorem:

Theorem 10.3.1. If $\mathbb{X}'_1, \mathbb{X}'_2, \mathbb{X}'_3$ are proper 4-vectors and $\mathbb{X}'_1 + \mathbb{X}'_2 = \mathbb{X}'_3$, then

1. $\left\langle \mathbb{X}'_1, \mathbb{X}'_2 \right\rangle \in R_+$
2. $\langle \mathbb{X}'_1, \mathbb{X}'_2 \rangle \geqslant \left|\mathbb{X}'_1\right| \left|\mathbb{X}'_2\right|$

Proof.

1. From the sum $\mathbb{X}'_1 + \mathbb{X}'_2 = \mathbb{X}'_3$ it follows that $\quad \det(\mathbb{X}'_1 + \mathbb{X}'_2) = \det \mathbb{X}'_3$
   Based on the theorem 2.3.4 we can write
   $\det \mathbb{X}'_1 + \det \mathbb{X}'_2 + 2\langle \mathbb{X}'_1, \mathbb{X}'_2 \rangle = \det \mathbb{X}'_3$
   Since the four-vectors $\mathbb{X}'_1, \mathbb{X}'_2$ and $\mathbb{X}'_3$ are proper, the scalar product of the four-vectors $\mathbb{X}'_1$ and $\mathbb{X}'_2$ is a real number.
   From the example above, we know that $\mathbb{X}'_1 = \Lambda \mathbb{X}_1$, $\mathbb{X}'_2 = \Lambda \mathbb{X}_2$ and $\mathbb{X}'_3 = \Lambda \mathbb{X}_3$, where Λ is an orthogonal paravector. In the frame of a particle moving along the way $\mathbb{X}'_3$ we have
   $\mathbb{X}_1 = \begin{pmatrix} \Delta t_1 \\ \Delta \mathbf{x}_1 \end{pmatrix} \quad , \quad \mathbb{X}_2 = \begin{pmatrix} \Delta t_2 \\ -\Delta \mathbf{x}_1 \end{pmatrix} \quad \text{and} \quad \mathbb{X}_3 = \begin{pmatrix} \Delta t_1 + \Delta t_2 \\ 0 \end{pmatrix}$
   Hence $(\Delta t_1)^2 - (\Delta \mathbf{x}_1)^2 + (\Delta t_2)^2 - (\Delta \mathbf{x}_1)^2 + 2\left\langle \mathbb{X}'_1, \mathbb{X}'_2 \right\rangle = (\Delta t_1 + \Delta t_2)^2$,
   which gives
   $$\left\langle \mathbb{X}'_1, \mathbb{X}'_2 \right\rangle = \Delta t_1 \Delta t_2 + ((\Delta \mathbf{x}_1)^2 + (\Delta \mathbf{x}_1)^2)/2 > 0$$
2. Based on the formula (2.5), we know that $\quad \langle \mathbb{X}_1, \mathbb{X}_2 \rangle^2 - (\mathbb{X}_1, \mathbb{X}_2\}^2 = \det \mathbb{X}_1 \det \mathbb{X}_2$
   Let us take an example as above, where $\mathbb{X}_2 = \begin{pmatrix} \Delta t_2 \\ -\Delta \mathbf{x}_1 \end{pmatrix}$.
   For proper paravectors, the scalar product must have a real value. A relativistic transformation preserves the determinants and the scalar product. The vector product in this case is also a real vector, so $\langle \mathbb{X}'_1, \mathbb{X}'_2 \rangle^2 \geqslant \det \mathbb{X}'_1 \det \mathbb{X}'_2$.
   Since the 4-vectors $\mathbb{X}_1$, $\mathbb{X}_2$ are proper or singular and the dot product $\langle \mathbb{X}'_1, \mathbb{X}'_2 \rangle \geqslant 0$, so
   $$\langle \mathbb{X}'_1, \mathbb{X}'_2 \rangle \geqslant \left|\mathbb{X}'_1\right| \left|\mathbb{X}'_2\right|$$

□

It is seen from the above theorem that the Cauchy-Buniakowski-Schwarz (CBS) inequality does not apply in the physical space we construct. What is more, for physical objects moving at speeds not greater than light, the inequality is opposite! So after introducing scalar coordinates, the CBS inequality changes its direction, but in fact it plays the same role for the constructed geometry.

This is the best moment to reflect on the basic geometrical property of space, which is the triangle inequality. Does it also apply in space-time? If not, is there any other relationship between the four-vectors that corresponds to this relationship? We can guess that, unlike Euclidean geometry, in space-time the triangle inequality does not have the same generality, because not every 4-vector has a module. Only proper and singular paravectors have their modules. So we guess that the triangle relation can only be formulated for the physical objects we talked about above. The experiments described above correspond to space-time triangles. Physical objects start simultaneously from one place (point P_0) to meet again simultaneously elsewhere (point P_2). One object rushes directly to the target, and the other rushes the longer route - through P_1. Using the paravector formalism, we write down times and ways in the following way

$$\mathbb{X} = \mathbb{X}_1 + \mathbb{X}_2 \tag{10.26}$$
$$\begin{pmatrix} t_2 - t_0 \\ \mathbf{x}_2 - \mathbf{x}_0 \end{pmatrix} = \begin{pmatrix} t_1 - t_0 \\ \mathbf{x}_1 - \mathbf{x}_0 \end{pmatrix} + \begin{pmatrix} t_2 - t_1 \\ \mathbf{x}_2 - \mathbf{x}_1 \end{pmatrix} \tag{10.27}$$

Since the above 4-vectors describe the position of physical objects, they must be proper or singular. From this assumption it follows that the values Δt and $\Delta\mathbf{x}$ are not arbitrary. Also note that they do not have to be real. To investigate the relationship between the above 4-vectors, we will use the following reasoning:

In the rest frame, we describe the movement of a physical object from point 0 to point 1 and back to point 0. All coordinates are real. Then we move on to a frame that moves at any speed lower than c. As a result of transforming 4-vectors from the rest frame, we get 4-vectors forming the triangle we want.

Four-vectors in a rest frame can be written down as

$$\begin{pmatrix} \Delta t \\ \Delta\mathbf{x} \end{pmatrix} + \begin{pmatrix} \Delta t \\ -\Delta\mathbf{x} \end{pmatrix} = \begin{pmatrix} 2\Delta t \\ 0 \end{pmatrix}, \tag{10.28}$$

and the relation R between the modules of its sides is the following

$$|\mathbb{X}| + |\mathbb{X}^{-}| \qquad R \qquad |\mathbb{X} + \mathbb{X}^{-}| \tag{10.29}$$

$$\text{or} \qquad \sqrt{\Delta^2 t - \Delta^2\mathbf{x}} + \sqrt{\Delta^2 t - \Delta^2\mathbf{x}} \qquad R \qquad 2\Delta t$$

Since we have non-negative real values on both sides of the relation, we can square both sides and the relation will not change.

$$\Delta^2 t - \Delta^2\mathbf{x} \qquad R \qquad \Delta^2 t \tag{10.30}$$

The left side of the relation is therefore less than or equal to the right side. We get any 4-vector triangle we are interested in by passing to the frame moving in relation to the previous one. This means that we are multiplying the equation (10.28) by any orthogonal paravector. To bring the result closer to our actual perception of motion, we choose the velocity paravector. So we have

$$\mathbb{X}_1 = V\mathbb{X}$$

$$\mathbb{X}_2 = V\mathbb{X}^{-}$$

$$\mathbb{X}_1 + \mathbb{X}_2 = V(\mathbb{X} + \mathbb{X}^{-})$$

Since $|V| = 1$, so for any orthogonal V (or Λ in general) the relation will be the same as in the equation (10.30). Thus, we have shown that for space-time physical objects the triangle inequality has a form of

$$|\mathbb{X}_1| + |\mathbb{X}_2| \quad \leqslant \quad |\mathbb{X}_1 + \mathbb{X}_2| \tag{10.31}$$

that is, it is opposite to the Euclidean geometry. This inequality can also be proved from the polarization identity and from the theorem 10.3.1.

In the frame moving at speed -**v** a sum of 4-vectors (10.28) is

$$\frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t \\ \mathbf{x} \end{pmatrix} + \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t \\ -\mathbf{x} \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} \begin{pmatrix} 2\Delta t \\ 0 \end{pmatrix} \tag{10.32}$$

which in this frame gives

$$\begin{pmatrix} \Delta t_1' \\ \mathbf{x}_1' + i\mathbf{y}_1' \end{pmatrix} + \begin{pmatrix} \Delta t_2' \\ \mathbf{x}_2' + i\mathbf{y}_2' \end{pmatrix} = \begin{pmatrix} \Delta t_3' \\ \mathbf{x}_3' \end{pmatrix} \tag{10.33}$$

Below, the system of equations follows from the above equation

$$\begin{aligned} \Delta t_1' + \Delta t_2' &= \Delta t_3' \\ \mathbf{x}_1' + \mathbf{x}_2' &= \mathbf{x}_3' \end{aligned} \tag{10.34}$$

As it is easy to check, the (10.31) inequality is preserved. In the above reasoning a physical triangle was considered, the sides of which were the paths of motion of particles with energy, so their velocity was lower than c and they left a certain point at the same moment and later met one another in the same way. The situation in which the particles never meet again after they start moving, even if their paths intersect, but they are at different times at the point of intersection, is no longer a closed triangle.

From the computational point of view, everything looks simple, but the above equations describe the motion of physical objects. Then, how to interpret the imaginary components? The answer is simple: **one has to get used to them**. The imaginary distance has no physical significance in terms of the way to travel. It is a spatial effect of desynchronization and it should be understood as a quantity characterizing a certain deformation of real coordinates, or simply, as a dependent variable needed to balance the calculations. An imaginary vector never exists by itself, but it is always related to a real one. The above reasoning shows the simplicity and flexibility of interpretation of complex space-time as opposed to a real one.

10.4 Equations of motion of a charged particle in electric field

In this monograph, mechanics is treated very briefly, but it is our duty to check whether the studied complex model does not contradict the laws of these branches of physics. Therefore, we will perform a few mathematical transformations of selected formulas to show that the applicable basic equations of theoretical mechanics and quantum mechanics are invariant to complex relativistic transformation. This shows that there is a point in working on translating mechanics into the language of paravectors and interpreting it in complex space-time.

The total energy of the body in the potential field is described by the Hamilton equation

$$\frac{d\mathbf{p}}{dt}+\nabla H=0 \quad , \tag{10.35}$$

where $H=T+V$ is the Hamiltonian of the body. T is kinetic energy, V is potential energy, and $\mathbf{p}$ is its momentum. If the particle has an electric charge q and the electric field is given by the function $\mathbf{E}$, then the above formula takes the form

$$\frac{d\mathbf{p}}{dt}+\nabla T=q\mathbf{E} \quad . \tag{10.36}$$

This equation is the real vector component of the next equation (without the dilation factor)

$$\begin{bmatrix}\frac{\partial}{\partial t}\\ \nabla\end{bmatrix}\begin{pmatrix}T\\ \mathbf{p}\end{pmatrix}=\begin{pmatrix}0\\ \mathbf{E}\end{pmatrix}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix}\frac{q}{\sqrt{1-v^2}} \tag{10.37}$$

In the above equation, the observer is in a field frame and the q charge with mass m is moving of $\mathbf{v}$ relativistic velocity in this frame. The above equation will be true when we add any constant to the variable kinetic energy. We add half the mass of the object whose motion is described by this equation. Instead of T we will write K, where $K=T+m/2$. Since the field intensity in the rest frame is $\mathbb{E}=\partial^-\varphi$, we have

$$\begin{bmatrix}\frac{\partial}{\partial t}\\ \nabla\end{bmatrix}\begin{pmatrix}K\\ \mathbf{p}\end{pmatrix}=\begin{bmatrix}\frac{\partial}{\partial t}\\ -\nabla\end{bmatrix}\begin{pmatrix}\varphi\\ 0\end{pmatrix}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix}\frac{q}{\sqrt{1-v^2}} \tag{10.38}$$

If we pass to a frame that moves at a constant relativistic velocity -$\mathbf{w}$, the above equation transforms according to the identities proved in Chapter 4

$$\begin{bmatrix}\frac{\partial}{\partial t'}\\ \nabla'\end{bmatrix}\frac{1}{\sqrt{1-w^2}}\begin{bmatrix}1\\ \mathbf{w}\end{bmatrix}\begin{pmatrix}K'\\ \mathbf{p}'\end{pmatrix}=\frac{1}{\sqrt{1-w^2}}\begin{bmatrix}1\\ -\mathbf{w}\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial t'}\\ -\nabla'\end{bmatrix}\begin{pmatrix}\varphi'\\ 0\end{pmatrix}\frac{q}{\sqrt{1-v^2}}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix} \quad , \tag{10.39}$$

where prime on a function means the primed arguments, and not some other value of the function. On the right, in brackets, we have a new 4-vector of field intensity/induction, so let us write the above equation as

$$\begin{bmatrix}\frac{\partial}{\partial t'}\\ \nabla'\end{bmatrix}\frac{1}{\sqrt{1-w^2}}\begin{bmatrix}1\\ \mathbf{w}\end{bmatrix}\begin{pmatrix}K'\\ \mathbf{p}'\end{pmatrix}=q\begin{pmatrix}e'\\ \mathbf{E}'+i\mathbf{B}'\end{pmatrix}\frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix} \quad , \tag{10.40}$$

If we multiply both sides of the above equation by the velocity paravector W, we get

$$\begin{bmatrix}\frac{\partial}{\partial t'}\\ \nabla'\end{bmatrix}[\frac{1}{\sqrt{1-w^2}}\begin{bmatrix}1\\ \mathbf{w}\end{bmatrix}\begin{pmatrix}K'\\ \mathbf{p}'\end{pmatrix}\frac{1}{\sqrt{1-w^2}}\begin{bmatrix}1\\ \mathbf{w}\end{bmatrix}]=q\begin{pmatrix}e'\\ \mathbf{E}'+i\mathbf{B}'\end{pmatrix}\frac{1}{\sqrt{1-v^2}\sqrt{1-w^2}}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix}\begin{bmatrix}1\\ \mathbf{w}\end{bmatrix} \tag{10.41}$$

On the left side, it was possible to include the paravector W under the derivative operator, because the **w** vector is constant (inertial motion). To simplify the calculations, let us write the above formula in a symbolic form

$$\partial'\mathbb{K}' = q\mathbb{E}'V'^{*} \tag{10.42}$$

where

$$\mathbb{K}' = \frac{1}{\sqrt{1-w^2}}\begin{bmatrix}1\\ \mathbf{w}\end{bmatrix}\begin{pmatrix}K'\\ \mathbf{p}'\end{pmatrix}\frac{1}{\sqrt{1-w^2}}\begin{bmatrix}1\\ \mathbf{w}\end{bmatrix}, \quad \mathbb{E}' = \begin{pmatrix}e'\\ \mathbf{E}'+i\mathbf{B}'\end{pmatrix}, \quad V' = \frac{1}{\sqrt{1-w^2}\sqrt{1-v^2}}\begin{bmatrix}1\\ \mathbf{w}\end{bmatrix}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix}$$

From the above, it can be seen that the equation of motion (10.37) is covariate with POT, and the energy-momentum 4-vector is consistent with the equation (3.32). In general, the equation of motion of a charged particle with m mass and q charge in the $\mathbb{E}$ field has the following form

$$\partial\mathbb{K} = q\mathbb{E}\Lambda^{*} \tag{10.43}$$

where $\mathbb{K} = m\Lambda\Lambda^{*}/2$, Λ is an orthogonal paravector, and on the right side is the complex force
$\mathbb{F} = q\mathbb{E}\Lambda^{*}$.

Note that in order for the above equation for a single particle to be interpretable, it needs to be realised (as we did in Chapter 8) and then it becomes the equation (10.37).

We still have to check what the above equation of motion will look like at low velocity. In the SI system, the equation (10.37) has the following form

$$\begin{bmatrix}\frac{\partial}{c\partial t}\\ \nabla\end{bmatrix}\begin{pmatrix}T\\ c\mathbf{p}\end{pmatrix} = \begin{pmatrix}0\\ \mathbf{E}\sqrt{\epsilon_0}\end{pmatrix}\begin{bmatrix}1\\ \mathbf{v}/c\end{bmatrix}\frac{q}{\sqrt{\epsilon_0[1-(v/c)^2]}} \tag{10.44}$$

which with $v/c \to 0$ gives a system of equations

$$\frac{dT}{dt} + c^2\nabla\mathbf{p} = q\mathbf{E}\mathbf{v} \tag{10.45}$$

$$\nabla T + \frac{d\mathbf{p}}{dt} = q\mathbf{E} \tag{10.46}$$

$$\nabla\times\mathbf{p} = \frac{\mathbf{E}\times\mathbf{v}}{c^2} = 0 \tag{10.47}$$

In the first equation there is a component $c^2\nabla\mathbf{p}$ which is equal to 0 because

$$c^2\nabla\mathbf{p} = mc^2\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \tag{10.48}$$

The sum of the differentials is

$$\frac{\partial^2 x}{\partial x\partial t} + \frac{\partial^2 y}{\partial y\partial t} + \frac{\partial^2 z}{\partial z\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 0, \tag{10.49}$$

because the partial differentiation is commutative, and we differentiate along the path of the particle's motion. So we get the classical equations of motion of a charged particle in an electric field

$$\frac{dT}{dt} = q\mathbf{E}\mathbf{v} \tag{10.50}$$

$$\nabla T + \frac{d\mathbf{p}}{dt} = q\mathbf{E} \tag{10.51}$$

10.5 On the compliance of paravector orthogonal transformations with the relativistic equations of theoretical and quantum mechanics

The relativistic Hamilton-Jacobi equation for a free particle with the mass m_0 in the Cartesian system denoted in SI units has the following form:

$$\sqrt{(\nabla S)^2 + K_0^2} + \frac{\partial S}{c\partial t} = 0, \tag{10.52}$$

where $K_0 = m_0 c^2$, which is the current SR, but might as well be half of that value. This does not affect the proof of invariance of the equation under POT. After transformation of the above equation, we obtain:

$$\left(\frac{\partial S}{c\partial t}\right)^2 - (\nabla S)^2 = K_0^2. \tag{10.53}$$

This equation written in a natural system of units, using the differentiation operators, is:

$$\begin{bmatrix} \frac{\partial S}{\partial t} \\ \nabla S \end{bmatrix}\begin{bmatrix} \frac{\partial S}{\partial t} \\ -\nabla S \end{bmatrix} = (\partial S)\partial^- S = E_0^2, \tag{10.54}$$

where S is an action and therefore it is a scalar function. From the formulas derived in Chapter 4 it follows that the above equality transforms in following way:

$$(\partial S)\partial^- S = \left(\partial' \Lambda S'\right)\Lambda^- \partial'^- S' = \left(\partial' S'\right)\Lambda\Lambda^- \partial'^- S' = \left(\partial' S'\right)\partial'^- S' = E_0^2 \tag{10.55}$$

Therefore, the equation (10.52), as equivalent to (10.53), is invariant with respect to the orthogonal paravector transformation. A prim on an action function means that its arguments change, but its value is left unchanged.

The **Klein-Gordon equation**, also known as the relativistic Schroedinger equation (in natural unit system)

$$-\frac{\partial^2 \Psi(t,\mathbf{x})}{\partial^2 t} + \nabla^2 \Psi(t,\mathbf{x}) = m^2 \Psi(t,\mathbf{x}) \tag{10.56}$$

transforms under the boost in the same way as the wave equation (1.3), so we do not repeat the proof of its invariance. The proof can also be made using the formulas derived in chapter 4. It also follows that the function Ψ is invariant.

From the above cursory considerations, it can be seen that complex relativistic transformations should not contradict the existing relativistic theoretical mechanics or quantum mechanics.

10.6 The definitions of momentum and kinetic energy are different than in the classical SR

It was shown above that the equations of motion covariate with POT. However, if we look closely at the energy-momentum 4-vector, we can see that ours is different from its classical counterpart. In the classical SR, a energy-momentum 4-vector has the same properties as a any space-time 4-vector. In complex space-time, although energy is covariate with POT, it transforms so that it is always real. For a moving object, it is proportional to the product of mutually conjugated paravectors of its velocities (3.32), (7.13). Hence, the energy dilation factor is the square of the Lorentz factor in SR. The definition of an object's energy in complex spacetime differs significantly from the classical SR. This results in the following differences:

Table 10.1:

	Momentum	Kinetic energy	Total energy
Classic SR	$\mathbf{p}_r = \frac{m\mathbf{v}}{\sqrt{1-v^2}}$	$T_r = m(\frac{1}{\sqrt{1-v^2}} - 1)$	$K_r = \frac{m}{\sqrt{1-v^2}}$
Complex model	$\mathbf{p}_c = \frac{m\mathbf{v}}{1-v^2}$	$T_c = \frac{mv^2}{2(1-v^2)}$	$K_c = \frac{m(1+v^2)}{2(1-v^2)}$

The table below shows the differences in the momentum and energy of a moving electron depending on its velocity. The mass of the electron is $m = 0.510998946$ MeV.

The above results in graphic form

As can be seen, for very high speeds (> 0.6c) the difference becomes large. Since both energy and speed can be measured, the both models can be tested and compared experimentally.

Table 10.2: The difference in electron energy and momentum between the SR (r) and the complex model (c) depending on the velocity.

v	p_r [MeV]	p_c [MeV]	T_r [MeV]	T_c [MeV]	K_r [MeV]	K_c [MeV]
0.1	0.051357	0.051616	0.002574	0.002581	0.513573	0.260661
0.2	0.104307	0.106458	0.010537	0.010646	0.521536	0.276791
0.3	0.160702	0.168461	0.024674	0.025269	0.535672	0.306038
0.4	0.223018	0.243333	0.046547	0.048667	0.557546	0.352833
0.5	0.295025	0.340666	0.079052	0.085166	0.590051	0.425832
0.6	0.383249	0.479062	0.12775	0.143718	0.638749	0.542936
0.7	0.500879	0.701371	0.204543	0.24548	0.715542	0.746459
0.8	0.681332	1.135553	0.340666	0.454211	0.851665	1.163942
0.9	1.055081	2.420521	0.661313	1.089235	1.172312	2.433696

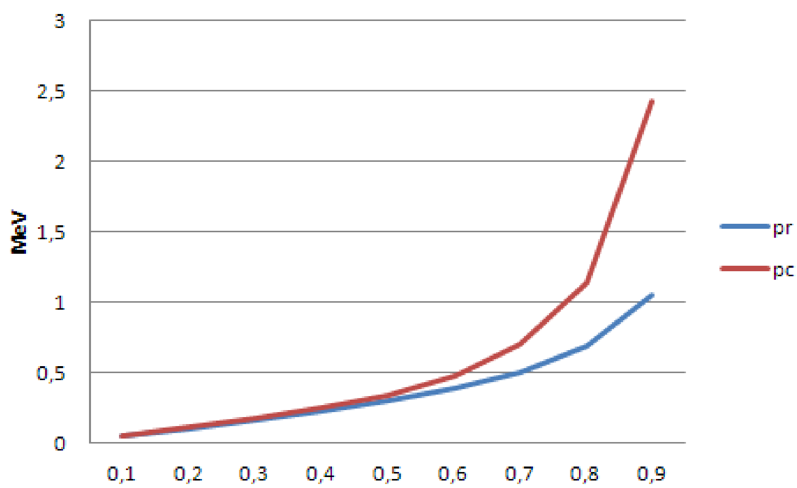

Figure 10.8: Momentum

In laboratory practice, the above differences do not matter, because the velocity of the particles is not measured, but the energy balance is made, which is the same in both theories, as we will show on the example of muon decay. However, we will show earlier that

The law of mass conservation is valid only for low velocities.

Let us analyse the decay of a particle with mass m_0, without energy exchange, into two particles with masses m_1 and m_2 in the particle m_0 system

$$m_0 = m_1 V_1 V_1 + m_2 V_2 V_2 \tag{10.57}$$

According to our theory, all components should be divided by 2, which does not affect the correctness of the above formula, which results the following system of equations

$$m_0 = m_1 \frac{1+v_1^2}{1-v_1^2} + m_2 \frac{1+v_2^2}{1-v_2^2} \tag{10.58}$$

$$\mathbf{0} = \frac{m_1 \mathbf{v}_1}{1-v_1^2} + \frac{m_2 \mathbf{v}_2}{1-v_2^2} \tag{10.59}$$

The determinant of paravectors in the equation (10.57) is

$$m_0^2 = \det(m_1 V_1 V_1 + m_2 V_2 V_2) = m_1^2 + m_2^2 + 2 m_1 m_2 \langle V_1 V_1, V_2 V_2 \rangle \tag{10.60}$$

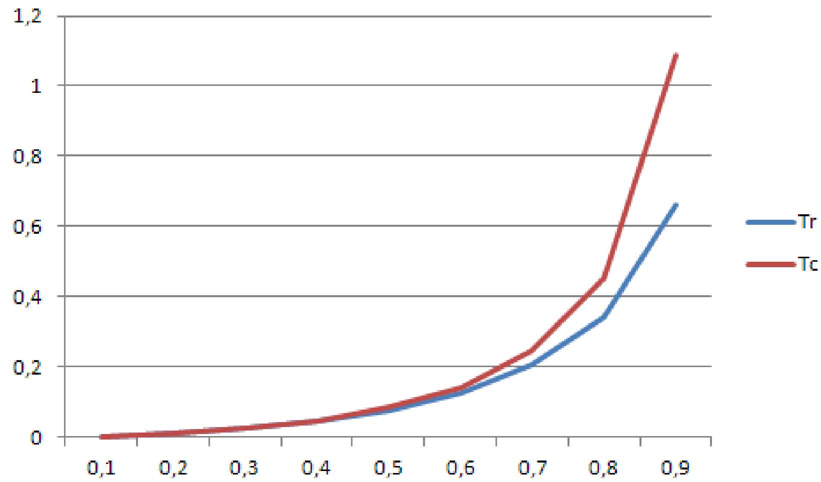

Figure 10.9: Kinetic energy

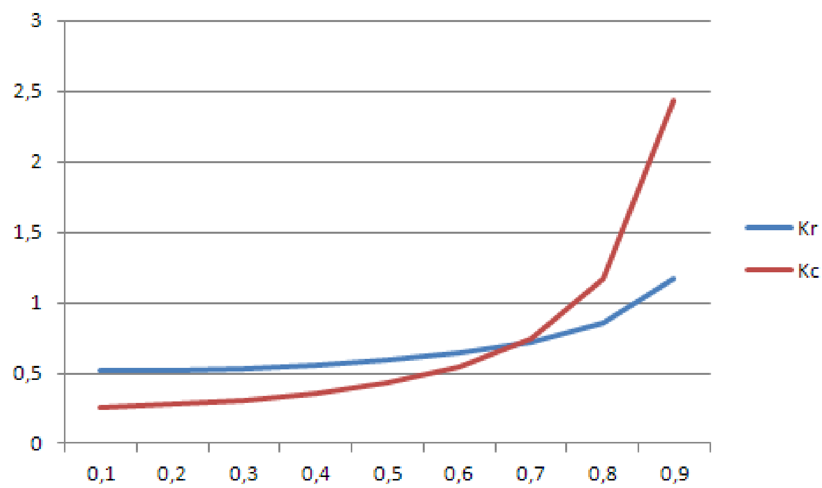

Figure 10.10: Total energy

The last term of the equation above is

$$\frac{2m_1 m_2}{(1-v_1^2)(1-v_2^2)}(1+v_1^2+v_2^2+v_1^2v_2^2-4\mathbf{v}_1\mathbf{v}_2)= \tag{10.61}$$

$$=\frac{2m_1 m_2}{(1-v_1^2)(1-v_2^2)}\left[(\mathbf{v}_1-\mathbf{v}_2)^2+(1-\mathbf{v}_1\mathbf{v}_2)^2+(\mathbf{v}_1\times\mathbf{v}_2)^2\right]$$

We conclude from above that the law of mass conservation is valid only for non-relativistic velocities, i.e. when v_1 and v_2 are close to 0. Then $\langle V_1V_1, V_2V_2\rangle = 1$, so

$$m_0 = m_1 + m_2 \tag{10.62}$$

In a frame that moves at the speed of **v** with respect to the m_0 particle, the formula (10.57) takes the form

$$m_0VV = m_1VV_1V_1V + m_2VV_2V_2V \qquad \text{where} \qquad V = \frac{1}{\sqrt{1-v^2}}\begin{bmatrix}1\\ \mathbf{v}\end{bmatrix} \tag{10.63}$$

or

$$m_0 V V = m_1 \Lambda_1 \Lambda_1^* + m_2 \Lambda_2 \Lambda_2^*, \tag{10.64}$$

which gives after realisation

$$m_0 V V = m_1 V_1' V_1' + m_2 V_2' V_2' \qquad \text{where} \qquad V_x' = \underline{\Lambda_x|} \tag{10.65}$$

Example: The muon decay $M^* \to M^- + \pi^0$

On the example taken from the publication [4] we will write in the paravector form the decay of a muon with a rest mass of $m_{M^*} = 0.8917$ GeV into two particles with a rest mass of $m_{M^-} = 0.4937$ GeV and $m_{\pi^0} = 0.1350$ GeV. In the M^* frame of reference, which is the center of mass system, the energy balance of such a phenomenon is:

$$\frac{m_{M^*}}{2} = \frac{m_{M^-} V_1 V_1}{2} + \frac{m_{\pi^0} V_2 V_2}{2} \tag{10.66}$$

$$\begin{pmatrix} K_{M^*} \\ 0 \end{pmatrix} = \begin{pmatrix} K_{M^-} \\ \mathbf{p}_{M^-} \end{pmatrix} + \begin{pmatrix} K_{\pi^0} \\ \mathbf{p}_{\pi^0} \end{pmatrix} \tag{10.67}$$

where

$$\det\begin{pmatrix} K_{M^-} \\ \mathbf{p}_{M^-} \end{pmatrix} = \frac{m_{M^-}^2}{4}, \qquad \det\begin{pmatrix} K_{\pi^0} \\ \mathbf{p}_{\pi^0} \end{pmatrix} = \frac{m_{\pi^0}^2}{4}, \qquad \det\begin{pmatrix} K_{M^*} \\ 0 \end{pmatrix} = \frac{m_{M^*}^2}{4} \quad \text{or} \quad K_{M^*} = \frac{m_{M^*}}{2}.$$

From (10.67) obtains that

$$\det\left(\begin{pmatrix} K_{M^*} \\ 0 \end{pmatrix} - \begin{pmatrix} K_{M^-} \\ \mathbf{p}_{M^-} \end{pmatrix}\right) = \det\begin{pmatrix} K_{\pi^0} \\ \mathbf{p}_{\pi^0} \end{pmatrix} \qquad \text{and} \qquad \mathbf{p}_{\pi^0} = -\mathbf{p}_{M^-} \tag{10.68}$$

hence

$$\frac{m_{\pi^0}^2}{4} = \frac{m_{M^*}^2}{4} - m_{M^*} K_{M^-} + \frac{m_{M^-}^2}{4} \tag{10.69}$$

From above we calculate the total energy of the muon M^-

$$K_{M^-} = \frac{m_{M^*}^2 + m_{M^-}^2 - m_{\pi^0}^2}{4 m_{M^*}} = 0.28615 \text{ GeV} \tag{10.70}$$

The result is half of the result obtained classically by P. Avery. We showed that despite different definitions of the momentum, the kinetic, total and proper energy, the energy balance is consistent with the current theory of the scale 1/2, which results from the covariate equations of motion in complex space-time.

10.7 Non-relativistic approximations of the transformation formulas

It is high time to check what the transformation formulas for non-relativistic velocities look like ($v \ll c$). For this purpose, it is better to change the units so as to explicitly write c where the value of the speed of light is present, i.e. go to the SI units. This means that we will write $\mathbf{v}/c$ instead of $\mathbf{v}$ and $t \to c\,t$ instead of time. Relativistic transformation formulas are as follows:

$$\begin{pmatrix} c\Delta t \\ \Delta\mathbf{x} \end{pmatrix} = \frac{1}{\sqrt{1-(v/c)^2}} \begin{bmatrix} 1 \\ -\mathbf{v}/c \end{bmatrix} \begin{pmatrix} c\Delta t' \\ \Delta\mathbf{x}' + i\mathbf{y}' \end{pmatrix} = \tag{10.71}$$

$$= \frac{1}{\sqrt{1-(v/c)^2}} \begin{pmatrix} c\Delta t' - \mathbf{v}\Delta\mathbf{x}'/c \\ \Delta\mathbf{x}' - \mathbf{v}\Delta t' - i\mathbf{v}\times\Delta\mathbf{x}'/c \end{pmatrix}$$

For non-relativistic velocities ($v/c \to 0$), we obtain $\begin{pmatrix} c\Delta t \\ \Delta\mathbf{x} \end{pmatrix} = \begin{pmatrix} c\Delta t' \\ \Delta\mathbf{x}' - \mathbf{v}\Delta t' \end{pmatrix}$ which are the formulas of Galilean transformation.

$$\Delta t = \Delta t' \qquad \text{and} \qquad \Delta\mathbf{x} = \Delta\mathbf{x}' - \mathbf{v}\Delta t' \tag{10.72}$$

The velocities composition was described by the equation

$$\frac{1}{\sqrt{1-(v/c)^2+(w/c)^2}}\begin{bmatrix} 1 \\ \mathbf{v}/c+i\mathbf{w}/c \end{bmatrix}=\frac{1}{\sqrt{1-(v_1/c)^2}}\frac{1}{\sqrt{1-(v_2/c)^2}}\begin{bmatrix} 1 \\ \mathbf{v}_1/c \end{bmatrix}\begin{bmatrix} 1 \\ \mathbf{v}_2/c \end{bmatrix} \tag{10.73}$$

$$\text{or}\qquad \frac{\sqrt{1-(v_1/c)^2}\sqrt{1-(v_2/c)^2}}{\sqrt{1-(v/c)^2+(w/c)^2}}\begin{bmatrix} 1 \\ \mathbf{v}/c+i\mathbf{w}/c \end{bmatrix}=\begin{bmatrix} 1+\mathbf{v}_1\mathbf{v}_2/c^2 \\ \mathbf{v}_1/c+\mathbf{v}_2/c+i\mathbf{v}_1\times\mathbf{v}_2/c^2 \end{bmatrix}$$

Since the fraction in front of the paravector on the left is equal to 1 for low velocities, we get the system of equations

$$\begin{aligned} 1 &= 1+\mathbf{v}_1\mathbf{v}_2/c^2 \\ \mathbf{v}/c &= \mathbf{v}_1/c+\mathbf{v}_2/c \\ \mathbf{w}/c &= \mathbf{v}_1\times\mathbf{v}_2/c^2 \end{aligned} \tag{10.74}$$

which for very small velocities compared to light, reduces to the second equation, that is to a simple sum of vectors $\mathbf{v}=\mathbf{v}_1+\mathbf{v}_2$.

Momentum and kinetic energy

In the SI system, the formulas for momentum and kinetic energy have the form

$$\mathbf{p}c=\frac{mc^2\mathbf{v}/c}{1-(v/c)^2} \qquad T=\frac{mc^2(v/c)^2}{2(1-(v/c)^2)}, \tag{10.75}$$

which after simplification gives

$$\mathbf{p}=\frac{m\mathbf{v}}{1-(v/c)^2} \qquad T=\frac{mv^2}{2(1-(v/c)^2)} \tag{10.76}$$

hence it is seen that for $v \ll c$ the formulas take the form

$$\mathbf{p}=m\mathbf{v} \qquad T=\frac{mv^2}{2} \tag{10.77}$$

It is clearly visible here that when passing to non-relativistic velocities, the complex relativistic transformation turns into a Galilean transformation, and the paravector formula of velocities composition changes into a vector sum, relativistic momentum turns into Newtonian momentum, and so is kinetic energy.

10.8 Discussion

We have mentioned the compliance of our theory with the postulates of the classical STR numerous time, but the presented calculations show that the model we propose differs from the classical theory. However, there is no mistake here!

- 1st postulate says that *the laws of physics do not depend on the choice of the translational physical system as a whole*. In our case, the postulate is assumed to be fulfilled because we started from the wave equation, and all the reasoning was based on a transformation that maintains its form.
- 2nd postulate is: *The speed of light does not depend on the motion of its source*. It is clear that it is an absolute value of the speed of light, not a vector. The compliance of our model with this postulate has also been checked. It cannot be otherwise, because this postulate results from a 1st postulate. The speed of light depends on the electric permittivity and magnetic permeability, which are constants of the vacuum, and results from the solution of the homogeneous wave equation. So, if this equation is invariant under the boost, it must have the same solution in every inertial frame.

So what is the difference, since the results of our theory differ from the classical results? The authors of the classic STR implicitly made one more assumption: **Lorentz factor** ($\sqrt{1-v^2}$) **works only in the direction of the relativistic movement**. In the classical STR, the description of the phenomena is divided into components in line with the direction of motion and perpendicular to it, i.e. we deal with relativistic one-dimensional (in the spatial sense) phenomena. This assumption results from the seemingly obvious fact that space-time is real. The adoption of such assumptions leads in consequence to paradoxes such as, for example, a change of shape. By examining mathematically a very simple linear transformation that preserves the invariance of the wave equation, we looked for consequences for the theory of the electric field and built the foundations of the geometry of the space in which this field could reasonably exist. The results were surprising. We found that the theory works best when we don't try to confine it to real space-time. From the mathematical point of view, the adopted definition of the scalar product naturally extends the definition known from Euclidean geometry and carries the invariability of the shape of the sphere distinguished in nature. Consequently, this means that shapes are invariant, which is more intuitive and simplifies many problems. Maxwell's equations are also great for complex space.

We have shown that, despite the differently defined momentum and kinetic energy, the energy balance of the complex theory is completely consistent with the balance made in accordance with the formulas of the applicable theory. The obtained theoretical results make it possible to confront both theories, because the same kinetic energy and momentum correspond to different velocities in both theories.

A very important conclusion from the considerations so far concerns the dilation factor, which in the classic theory is called the Lorentz factor. One should pay attention to a very important detail: **when we transferred the description of the phenomenon from a rest frame to a moving one, then in the moving frame this factor was reduced** ((6.29), (6.34), (10.12), (10.19), (10.22), (10.25)). In the classical SR, this factor did not concern the coordinates perpendicular to the direction of motion and therefore it did not disappear when describing space-time phenomena. In complex space-time, it applies to all spatial directions, therefore it can be treated as a transient scaling factor that disappears in the observer's coordinate system, regardless of the inertial frame of the observer. In cases where we refer to the values from the rest frame, the dilation factor remains there. It should be noted, however, that what happens to the factor $1/\sqrt{1-v^2}$ also happens to the magnetic field: it occurs only in the case of relative motion of the charge and the observer. If the charge is stationary relative to the observer, the magnetic field will disappear. When one twin brother returns after light years from a long trip and sits down over cup of coffee with his twin brother, it may turn out that both are equally old, because in this joyful moment, the dilation factor is equal to 1.

In addition to the advantages of the complex model, there are also disadvantages that mainly indicate the need for a deeper understanding of the mathematical structure of space-time. It seems that geometric space-time must be distinguished from the space-time of physical objects. The geometric four-vectors can take any values and can be any para-vector, while the coordinates of four-vectors of the objects endowed with energy must meet the conditions of proper or singular paravectors. This means that geometric space-time does not have a metric, while the space-time of physical objects would be a classical unitary space, if it was not for the other properties of the metric. We'll show the outline of a space-time structure in the next chapter.

Chapter 11

Draft of the mathematical structure of complex space-time and the physical objects contained in it

In this chapter, information about paravectors is gathered and organized in terms of its application to the description of physical problems, which gives us an outline of the mathematical structure of complex space-time. Space-time is divided into a geometric and a physical layer. The elements of the geometrical layer are any four vectors, therefore, although it is an orthogonal space, it does not have a metric. The physical layer is made up of objects with energy, the state coordinates of which are proper or singular paravectors. Therefore a metric can be defined for these objects. Since this metric does not have classical properties, but is also not Minkowski's pseudo-metric, it is called a para-metric. In the complex physical space-time the triangle and Cauchy inequalities are valid, but they have opposite directions to their equivalents known from Euclidean geometry. Finally, there are a few tips given on how to help readers imagine a complex space-time.

Chapter 2 presents the algebra of paravectors. As was shown later, some of the paravectors are additive and others are not. We call those of them that can be summed up four-vectors. We have learned about the geometric interpretations of the components of the phase interval and the orthogonal transformation. We have also distinguished between geometric and physical concepts. The relativistic transformation corresponds to the shift of an observer to another frame, moving at the relativistic speed. Every physical object is in motion, both in space and in time. Movement is an inherent quality of nature. In order to say that time is passing, at least two clock ticks are needed. The interval between these ticks can be very short, but it must be greater than null. At the moment of the tick, there is no time, as in the a photo picture there is no movement. Therefore, we assumed that time consists of whiles, i.e. arbitrarily short intervals. In Euclidean geometry, a vector is defined as an ordered pair of points, where the point is a basic, undefined concept. When assuming that time is nonzero, we must assume that the basic concept of space-time is a nonzero 4-vector. A point in space can be defined as a vector by which a rest object has moved at any time. For an observer moving in time, a point is a fleeting concept. We define a point as a place in the observer's real space-time, which is the beginning or the end of the 4-vector. The space of places is an affine space, but time belongs to a semi-group, hence the complex space-time is a vector space (not an affine one). There is a physical relationship between the and the beginning of a 4-vector, if the 4-vector is proper or singular. Besides, 4-vectors having a physical meaning are so directed that the beginning must always be earlier than the end.

It was an intuitive description of space-time. In other words, but more formally:

1. Any single object in the observer's space-time is described by two parameters: its position relative to the observer and his time. The observer's proper time and the coordinates of his frame of reference are real. We have assumed that time passes in steps, that is, the structure of time is 'granular', with an infinitesimal step.

The behaviour of a physical object in the observer's space-time is described by the following coordinates: time ones $\Delta t \in R_+ \setminus \{0\}$ and spatial ones $\Delta\mathbf{x} \in R^3$.

2. The state of an inertial physical object in the observer's real space-time is determined by the phase interval

$$\begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix}\begin{pmatrix} \Delta t \\ \Delta\mathbf{x} \end{pmatrix} = \begin{pmatrix} \Delta t - \mathbf{v}\Delta\mathbf{x} \\ \Delta\mathbf{x} - \mathbf{v}\Delta t - i\mathbf{v}\times\Delta\mathbf{x} \end{pmatrix} = k \in R_+.$$

where the vector $\mathbf{v}$ is interpreted as the velocity of the object relative to the observer. Time lapse can be interpreted as the movement of the observer in one direction along the time axis. The time of the observer Δt is therefore targeted. It is the same with any physical object. The above formula shows that $\mathbf{v} \parallel \Delta\mathbf{x}$. Physical phenomena are those in which energy is involved. Energy has not been found to travel faster than the speed of light. Mathematically, this means that for a physical object, individual elements of the phase interval must meet the conditions $\Delta^2 t - \Delta^2\mathbf{x} \geqslant 0$ and $0 \leqslant v^2 \leqslant 1$. In the case of $v^2 = 1$, the object moves relative to the observer at the speed of light. The phase interval of objects slower than light can be depicted as

$$V^{-}\mathbb{X} = \frac{1}{\sqrt{1-v^2}}\begin{bmatrix} 1 \\ -\mathbf{v} \end{bmatrix}\begin{pmatrix} \Delta t \\ \Delta\mathbf{x} \end{pmatrix} = \frac{1}{\sqrt{1-v^2}}\begin{pmatrix} \Delta t - \mathbf{v}\Delta\mathbf{x} \\ \Delta\mathbf{x} - \mathbf{v}\Delta t - i\mathbf{v}\times\Delta\mathbf{x} \end{pmatrix} = \Delta t^0 \in R_+ \setminus \{0\}$$

In the formula above, the phase interval is equivalent to the proper time interval of observed object. The phase interval cannot be a negative number because time does not run backwards.

3. When an observer describes two objects (or more), their mutual relations require one of them to move in a complex motion, so the coordinates of his state paravectors must be complex

$$\Lambda^{-}\mathbb{X} = \frac{1}{\sqrt{a^2 - b^2 + c^2 - d^2}}\begin{bmatrix} a + id \\ -\boldsymbol{b} - ic \end{bmatrix}\begin{pmatrix} \Delta t + is \\ \Delta\mathbf{x} + i\mathbf{y} \end{pmatrix} = \Delta t^0 \in R_+ \setminus \{0\}$$

where $ad = \mathbf{bc}$ and $s\Delta t = \mathbf{y}\Delta\mathbf{x}$. The imaginary components s and $\mathbf{y}$ are auxiliary quantities and are not independent. The real vector $\Delta\mathbf{x}$ is the vector between the start and end points of the event in the affine local observer space. Likewise, Δt is the duration of the event. The imaginary scalar s and the imaginary vector $\mathbf{y}$ are dependent parts of the corresponding real variables.

4. The transition from a rest frame to an inertly moving frame is determined by the relativistic transformation described by the complex orthogonal paravector Λ, and defined as the transformation mapping the four-vector $\mathbb{X}$ into the $\mathbb{X}'$ one such that the phase interval is an invariant of this transformation.

$$\mathbb{X} \xrightarrow{\Lambda} \mathbb{X}' \qquad \text{and} \qquad \Gamma \xrightarrow{\Lambda} \Gamma' \qquad \text{so that} \qquad \Gamma^{-}\mathbb{X} = \Gamma'^{-}\mathbb{X}'$$

5. The observer describes the state of a single object with real state paravectors, where the velocity is determined by the parameter in the form $V = \frac{1}{\sqrt{1-v^2}}\begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix}$, and change of its position is determined by the real proper four-vector $\mathbb{X} \in (R_+ \setminus \{0\}) \times R^3$. When the observer describes an object moving in relation to another object, which is also moving, the state of the first object is described by complex paravectors. The observer can reduce these complex state paravectors to the real form by realisation of state paravectors, which is defined on the basis of the value of the energy paravector, which is always the real, as the following transformation:

$$\Lambda^{-}\mathbb{X} \rightarrow V^{-}\mathbb{X}' = \underline{\Lambda^{-}|}\mathbb{X}' \qquad \text{such that} \qquad \Lambda\Lambda^{*} = VV$$

where Λ is a complex orthogonal paravector, $\mathbb{X}$ is the proper complex four-vector, $\mathbb{X}'$ is the proper real four-vector, and V is the velocity paravector. Realisation distorts the mutual space-time relationship between these objects because it is not an orthogonal transformation. However, it preserves the scalar product between the real energy paravectors of these objects.

Since the realisation does not maintain the scalar product, we can only apply it to the description of the motion of a single physical object (or space-time parallel objects) and it is used to choose a frame related to this object(s) which is the most convenient for the observer. The benefit of using realisation is that the coordinates of the object are real. When describing two inertial objects whose paths intersect (motion on a plane), we have less freedom and by choosing real coordinates for one of them and wanting to maintain their mutual relations, we can only choose a system in which time will be real for the other one. Realisation cannot be applied to the description of the movement of spatially moving objects, as it introduces deformations (it is not an orthogonal transformation). Since the realisation preserves the parallelism of the paravectors, we interpret them as a kind of projection on the real space-time of the (rest) observer.

11.1 Mathematical structure of space-time

For an even more precise ordering of the obtained results, we should define the outline of the mathematical structure of the space in which we build our model. In the current state of knowledge, the following construction should be treated only as a working mathematical model for imagining the complex space-time. As a symbol of the structure, we use the designation of its element in curly brackets.

GEOMETRICAL LEVEL

1. We define the geometric space-time as a tangled ring of paravectors $\mathsf{GST} = (\{\mathbb{X}\} \odot \{\Gamma\}, (C, +, \cdot))$ over the field of complex numbers, where $\{\mathbb{X}\} = (C \times C^3, +)$ is an abelian group of four-vectors with a summation operation, and $\{\Gamma\} = (C \times C^3, \cdot)$ is a semigroup of paravectors with the operation of multiplication.

   The elements of the $\{\mathbb{X}\}$ group are four-vectors

   $$\mathbb{X} = \begin{pmatrix} \Delta t + is \\ \Delta \mathbf{x} + i\mathbf{y} \end{pmatrix}$$

   On four-vectors we define the external multiplication operation $\odot$ such that the 4-vector product is a paravector from the $\{\Gamma\}$ semigroup. The product of the paravectors from the $\{\Gamma\}$ semigroup and the four-vector is a four-vector. This operation is an internal operation of the semigroup $\{\Gamma\}$

   $$\mathbb{X}_1\mathbb{X}_2 \in \{\Gamma\}, \qquad \Gamma_1\Gamma_2 \in \{\Gamma\} \qquad \text{and} \qquad (\Gamma\mathbb{X} \in \{\mathbb{X}\} \qquad \text{or} \qquad \mathbb{X}\Gamma \in \{\mathbb{X}\})$$

   The space-time defined in this way corresponds to the vector space, therefore the term coordinates is understood as the coordinates of 4 vectors. It is a geometrical structure - an empty space in which there are no objects with energy as yet, and it only serves to define the mathematical operations that can be performed.

2. In space-time, we define the relationships:

   - Integrated products:
     right one $\quad (\mathbb{X}_1, \mathbb{X}_2\rangle = \mathbb{X}_1\mathbb{X}_2^-$ $\quad$ and left one $\quad \langle\mathbb{X}_1, \mathbb{X}_2) = \mathbb{X}_1^-\mathbb{X}_2$
   - scalar product $\quad \langle\mathbb{X}_1, \mathbb{X}_2\rangle = (\mathbb{X}_1, \mathbb{X}_2\rangle_S = \langle\mathbb{X}_1, \mathbb{X}_2)_S \in C$
   - determinant $\quad \det\mathbb{X} = \mathbb{X}\mathbb{X}^-$

   These relations are valid for all elements of the ring, both 4-vectors and paravectors.

3. The determinant has the following properties:

   - $\det\mathbb{X} \in C$
   - parallelogram identity $\quad \det(\mathbb{X}+\mathbb{Y}) + \det(\mathbb{X}-\mathbb{Y}) = 2(\det\mathbb{X} + \det\mathbb{Y})$
   - polarization identity $\quad \det(\mathbb{X}+\mathbb{Y}) = \det\mathbb{X} + 2\langle\mathbb{X},\mathbb{Y}\rangle + \det\mathbb{Y}$

which introduces into space-time the skeleton of geometric space on which the metric of physical objects is based.

PHYSICAL LEVEL

4. We create physical space-time by placing physical objects in geometric space-time, the state paravectors of which cannot assume any value but are complex proper or singular paravectors ($\det\mathbb{X}, \det\Gamma \in R_+$). A physical object is distinguished by two attributes:

   a The proper time of a physical object is discrete and takes values from the set of non-negative rational numbers.

   b A physical object has energy. Energy is the ability to do work, and work is the act of transferring energy to another physical object. Energy is a discrete quantity and its scalar value is a positive number.

5. Among physical objects, we distinguish material objects whose proper time is a rational positive number ($\Delta t > 0$). A force field is also a physical object.

6. We create the inertial boost group $(\Lambda, \cdot)$, where Λ is the set of complex orthogonal paravectors and the operation is a multiplication of paravectors.

7. In the $(\Lambda, \cdot)$ group we introduce the unary relation $\Lambda\Lambda^*$, the result of which is called vigor, and it has to do with the kinetic energy of a physical object.

8. We create the phase space of physical objects $\{\Theta\} = ([\Lambda^-\mathbb{X}], +) = ([\Delta t], +)$, where $[\Lambda^-\mathbb{X}]$ is a set of pairs of elements consisting of a position 4-vector of a physical object and an ortogonal (boost) paravector (a pair of state paravectors) connected by the multiplication operation $\Lambda^-\mathbb{X}$. The value of this product is a real non-negative number equal to the proper time interval, i.e. $\Lambda^-\mathbb{X} \in R_+$ (or Q_+, if time is a discrete quantity). Therefore, the $\mathbb{X}$ and Λ paravectors are not independent of each other.

9. For these 4-vectors, we define the $|\mathbb{X}| = \sqrt{\det\mathbb{X}}$ module. This means that we are introducing a function that acts as a metric that is valid only for physical objects(!). This function has slightly different properties than the known metrics, namely:

   - $|\mathbb{X}| \in R_+$
   - $\mathbb{X} = 0 \Rightarrow |\mathbb{X}| = 0$
     $\mathbb{X} \neq 0 \quad \text{and} \quad |\mathbb{X}| = 0 \Leftrightarrow$ when an object is moving at the light velocity
   - $|\mathbb{X}| = |\mathbb{X}^-|$
   - If $\mathbb{X}_1 + \mathbb{X}_2$ is a proper or singular paravector, then $|\mathbb{X}_1 + \mathbb{X}_2| \geqslant |\mathbb{X}_1| + |\mathbb{X}_2|$. This is the inverse triangle condition in a space-time.

   We call the above function *para-metrics*.

10. Function fields are spread over this space.

On transition to a moving frame, the 4-vectors of physical objects in the old frame of reference $\mathbb{X}$ are replaced with new complex 4-vectors $\mathbb{X}'$ so that the phase interval is invariant $\Lambda^-\mathbb{X} = \Lambda'^-\mathbb{X}'$. In the case of a single physical object movement description, the observer in his real frame can always present the state of this object using real paravectors, i.e. in such a way that the object moves along real coordinates and does it by realisation.

A while (minimal proper time of the material object) is the real infinitesimal interval ($\delta t > 0$). Space-time defined in this way is not an affine space, that is, there is no coordinate system in it. Each observer builds his affine space by determining their real coordinate system. In this system, he can only describe a non-relativistic movement of individual object that is relativistic in relation to him. However, he is not able to accurately describe the relationship between objects in relative relativistic motion using real coordinates. In other words: **space-time is real locally**. The word *local* primarily means a slow velocity relative to the observer. Frames of various observers are 'immersed' in a complex vector space-time, in which any physical phenomenon can be described by means of paravectors. In the observer frame, these paravectors (implicit in phase intervals) are real when the observer describes a single object. If he wants to describe the motion of the same object, but with respect to another moving object, real paravectors are not enough. The coordinates of the same object described in relation to different other objects will be different. Thus, it is not possible to objectively determine the position of objects in the real coordinate system of the observer, but any phenomenon can be objectively described by means of complex paravectors.

11.2 Few remarks on the nature of time and space

We already know enough to think about the nature of time and space:

- Is time a physical or mathematical concept?
- Does the space have physical properties? Or to put it another way: Does space have the property of transmitting energy, that is the property of ether?

Physical objects have the property that their determinant is a non-negative real number[1]. This feature distinguishes physical objects in the mathematical structure of space-time. Physical objects are therefore located in a specific substructure of the more general space-time that we built in Chapter 2. We want space-time to be a mathematical environment for physical objects and not to have properties specific only to physical objects. Of course, the properties of space-time and physical objects must be complementary.

Any physical objects has two specific properties:

1. The arrow of its proper time is pointing in one direction, it towards future. But, there are physical objects that do not have proper time - timeless objects. These are objects moving at the speed of light.
2. They have energy, a property that physical objects can transfer to each other. Energy is described by a real paravector whose scalar component has a real positive value. This explains why we see the world as real, because energy is the only carrier of information.

The physical quantities in physical formulas, written in SI system, include so-called material constants, e.g. c, μ_0, ε_0 (see table 1). It is assumed that they describe the properties of the medium, which in this case is vacuum. We tend to claim that these are properties of physical objects, such as the electric field or, for example, a charge. This assumption gives complete freedom to mathematical operations and imposes restrictions on physical quantities. We wrote time as ct. The letter c stands for the speed of light, a property characteristic of certain physical objects. Time cannot be transferred from object to object. Time is one of the dimensions of space-time, but directed time ct is shared by physical objects. This dimension is definitely different from spatial dimensions. Time is a dynamic dimension of a physical object. Thanks to this dimension, physical objects are always in motion and can transfer energy to each other. Saint Augustine stated that '*Time does not exist without the movement of successive changes*'[5]. So, he linked time with the movement of matter. This explains why time as an oriented dimension of physical objects has a c factor.

When it comes to space, the formulas of classical physics do not give it any physical properties. In SI system, the magnetic permeability and electric permettivity factors are attached to a field or charge. In the formulas the speed of light occurs with mass or momentum, but never with spatial coordinates. Here we are in agreement with the classical SR and so we can say that there is no aether. Force fields are specific physical objects that fill space-time and they have physical properties in opposite to a space-time.

An attention should be paid to the significant advantage of complex space-time: Thanks to complex paravectors, it is possible to exceed the speed of light in real coordinates, which means that we do not get paradoxes such as deformation of moving objects. Despite the apparent exceedance of the c speed, we cannot determine it by measurements, because the measurement is always made using an electromagnetic wave in its source frame. At this point, there is a contradiction in the current theory: If we assume that the speed of light is

[1]This is due to the fact that physical objects are subject to entropy - they age, which mathematically means that their proper time is a semigroup

constant, then we must accept the deformation of space, and if we want space to be rigid, then the speed of light cannot be constant. In complex space-time there is no such contradiction, because real coordinates change into imaginary ones and vice versa.

Chapter 12

On the classical Special Theory of Relativity based on William Baylis' publications

Based on the 'Algebra of Physical Space' (APS) by Professor William Baylis, in this chapter the classical Lorentz transformation is presented. The advantages of the 'Lorentz rotation', such as consistency with the current theory of electricity and limitation of space-time dimensions to the real domain, are shown. Finally, the APS flaw is shown, consisting in the conclusion that the observer can turn around in place by assembling rectilinear inertial movements, which in our opinion proves to the disadvantage of the concept developed by W. Baylis, and thus the current SR.

12.1 Lorentz rotation

Using the tool of the paravector calculus we will now look at the classical theories of SR and EM as proposed by William Baylis. In his works, W. Baylis describes the Lorentz transformation with the below formula and calls it the *Lorentz rotation*:

$$\mathbb{X}' = \Lambda \mathbb{X} \Lambda^* \tag{12.1}$$

where Λ is an orthogonal paravector, and $\mathbb{X}$ belongs to real space-time.

$$\Lambda \mathbb{X} \Lambda^* = \frac{1}{\alpha^2 - \beta^2} \begin{bmatrix} \alpha \\ \boldsymbol{\beta} \end{bmatrix} \begin{pmatrix} \Delta t \\ \Delta \mathbf{x} \end{pmatrix} \begin{bmatrix} \alpha^* \\ \boldsymbol{\beta}^* \end{bmatrix} =$$

$$= \frac{1}{a^2 - b^2 + c^2 - d^2} \begin{bmatrix} a + id \\ \mathbf{b} + i\mathbf{c} \end{bmatrix} \begin{pmatrix} \Delta t \\ \Delta \mathbf{x} \end{pmatrix} \begin{bmatrix} a - id \\ \mathbf{b} - i\mathbf{c} \end{bmatrix} = \tag{12.2}$$

$$= \frac{1}{\det \Lambda} \begin{pmatrix} (a^2 + b^2 + c^2 + d^2)\Delta t + 2(a\mathbf{b} + \mathbf{c}d - \mathbf{b} \times \mathbf{c})\Delta \mathbf{x} \\ 2(a\mathbf{b} + \mathbf{c}d + \mathbf{b} \times \mathbf{c})\Delta t + (a^2 - b^2 - c^2 + d^2)\Delta \mathbf{x} + 2(a\mathbf{c} + d\mathbf{b}) \times \Delta \mathbf{x} + 2[\mathbf{b}(\mathbf{b}\Delta \mathbf{x}) + \mathbf{c}(\mathbf{c}\Delta \mathbf{x})] \end{pmatrix}$$

From the above it follows that if $\mathbb{X} \in R^4$, then $\mathbb{X}' \in R^4$. Assuming that $\mathbf{b} = 0$ and $d = 0$, the Lorentz rotation is a Galilean rotation

$$\mathbb{X}' = \Lambda \mathbb{X} \Lambda^* = \begin{pmatrix} \Delta t \\ \frac{a^2 - c^2}{a^2 + c^2} \Delta \mathbf{x} - 2\frac{a\mathbf{c} \times \Delta \mathbf{x}}{a^2 + c^2} + 2\frac{\mathbf{c}(\mathbf{c}\Delta \mathbf{x})}{a^2 + c^2} \end{pmatrix} \quad , \tag{12.3}$$

which is easy to check when we replace

$$\cos\phi = \frac{a}{\sqrt{a^2 + c^2}} \qquad \sin\phi = \frac{c}{\sqrt{a^2 + c^2}}$$

and decompose the $\Delta\mathbf{x}$ vector into the components parallel and perpendicular to the $\mathbf{c}$ vector. If we assume that $c = 0$ and $d = 0$ we obtain

$$\mathbb{X}' = \Lambda\mathbb{X}\Lambda^* = \frac{1}{a^2 - b^2}\begin{pmatrix} (a^2+b^2)\Delta t + 2(a\mathbf{b})\Delta\mathbf{x} \\ 2a\mathbf{b}\Delta t + (a^2-b^2)\Delta\mathbf{x} + 2\mathbf{b}(\mathbf{b}\Delta\mathbf{x}) \end{pmatrix}, \tag{12.4}$$

which is a classic Lorentz transformation, which we can find out assuming that $\mathbf{b} \perp \Delta\mathbf{x}$

$$\Lambda\mathbb{X}\Lambda^* = \frac{1}{a^2-b^2}\begin{pmatrix} (a^2+b^2)\Delta t \\ 2a\mathbf{b}\Delta t + (a^2-b^2)\Delta\mathbf{x} \end{pmatrix} = \begin{pmatrix} \frac{a^2+b^2}{a^2-b^2}\Delta t \\ \frac{2a\mathbf{b}}{a^2-b^2}\Delta t + \Delta\mathbf{x} \end{pmatrix} = \begin{pmatrix} \Delta t' \\ \Delta\mathbf{x}' \end{pmatrix}. \tag{12.5}$$

As can be seen, the Lorentz factor does not affect the spatial components perpendicular to the direction of motion. So it is a classic Lorentz transformation. The image of the time interval (rest object) in the Lorentz transformation is the space-time interval (moving object):

$$\frac{1}{\alpha^2-\beta^2}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\begin{pmatrix} \Delta t \\ 0 \end{pmatrix}\begin{bmatrix} \alpha^* \\ \beta^* \end{bmatrix} = \begin{pmatrix} \Delta t' \\ \Delta\mathbf{x}' \end{pmatrix} \tag{12.6}$$

which means that the observed point covered the distance $\Delta\mathbf{x}'$ in the time $\Delta t'$ at a velocity of $\mathbf{v}$ such that:

$$\frac{(a^2+b^2+c^2+d^2)\Delta t}{a^2-b^2+c^2-d^2}\begin{bmatrix} 1 \\ \frac{2(a\mathbf{b}+\mathbf{c}d+\mathbf{b}\times\mathbf{c})}{a^2+b^2+c^2+d^2} \end{bmatrix} = \Delta t'\begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix}, \tag{12.7}$$

hence the velocity is

$$\mathbf{v} = \frac{2(a\mathbf{b}+\mathbf{c}d+\mathbf{b}\times\mathbf{c})}{a^2+b^2+c^2+d^2} \tag{12.8}$$

Value $\frac{a^2+b^2+c^2+d^2}{a^2-b^2+c^2-d^2}$ is the Lorentz factor $\gamma = 1/\sqrt{1-v^2}$, which is shown below

$$\frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-\left(\frac{2(a\mathbf{b}+\mathbf{c}d+\mathbf{b}\times\mathbf{c})}{a^2+b^2+c^2+d^2}\right)^2}} = \frac{a^2+b^2+c^2+d^2}{a^2-b^2+c^2-d^2} \tag{12.9}$$

Again, for formality's sake, we check that the resulting velocity is always less than the speed of light.

$$\mathbf{v}^2 = \frac{4(a\mathbf{b}+\mathbf{c}d+\mathbf{b}\times\mathbf{c})^2}{(a^2+b^2+c^2+d^2)^2} = 4\frac{a^2\mathbf{b}^2+\mathbf{c}^2d^2+(\mathbf{b}\times\mathbf{c})^2+2a\mathbf{bc}d}{(a^2+b^2+c^2+d^2)^2} \tag{12.10}$$

Since $(\mathbf{b}\times\mathbf{c})^2 = b^2c^2 - (\mathbf{bc})^2$ and $\mathbf{bc} = ad$ (the assumption is that the paravector Λ is s proper one) we obtain

$$v^2 = 4\frac{a^2b^2+c^2d^2+b^2c^2+a^2d^2}{(a^2+b^2+c^2+d^2)^2} = 4\frac{(a^2+c^2)(b^2+d^2)}{(a^2+b^2+c^2+d^2)^2} = \tag{12.11}$$

$$= \frac{[(a^2+c^2)+(b^2+d^2)]^2-[(a^2+c^2)-(b^2+d^2)]^2}{(a^2+b^2+c^2+d^2)^2} = 1 - \frac{(a^2-b^2+c^2-d^2)^2}{(a^2+b^2+c^2+d^2)^2} < 1$$

The transformation proposed by W. Baylis is an orthogonal transformation, i.e. it preserves the scalar product (def.2.2.3).

Proof. Under the transformation (12.1) the scalar product of paravectors has the following form

$$\langle A_1', A_2'\rangle = \left[\Lambda A_1\Lambda^*(\Lambda A_2\Lambda^*)^-\right]_S = \left[\Lambda A_1 A_2^- \Lambda^-\right]_S = \langle A_1, A_2\rangle \tag{12.12}$$

Since Λ is an orthogonal paravector, $\Lambda(A_1A_2^-)\Lambda^-$ is the rotation of the integrated product, and as we know from Theorem 2.3.6, the rotation does not change the scalar. □

The last request is equivalent with the statement that the Lorentz transformation does not change the shape of objects/phenomena in space-time. It is necessary to check what non-relativistic approximation of Lorentz transformation looks like. To simplify the calculation, in the same way as before, we assume that the paravector Λ is a real one ($c = 0$ and $d = 0$).

$$\begin{pmatrix} \Delta t' \\ \Delta \mathbf{x}' \end{pmatrix} = \frac{1}{a^2 - b^2} \begin{bmatrix} a \\ \mathbf{b} \end{bmatrix} \begin{pmatrix} \Delta t \\ 0 \end{pmatrix} \begin{bmatrix} a \\ \mathbf{b} \end{bmatrix} \tag{12.13}$$

which gives

$$\begin{pmatrix} \Delta t' \\ \Delta \mathbf{x}' \end{pmatrix} = \frac{1}{a^2 - b^2} \begin{bmatrix} a \\ \mathbf{b} \end{bmatrix} \begin{bmatrix} a \\ \mathbf{b} \end{bmatrix} \begin{pmatrix} \Delta t \\ 0 \end{pmatrix} = \frac{a^2 + b^2}{a^2 - b^2} \begin{bmatrix} 1 \\ \frac{2a\mathbf{b}}{a^2+b^2} \end{bmatrix} \begin{pmatrix} \Delta t \\ 0 \end{pmatrix} \tag{12.14}$$

If we divide the numerator and denominator of the vector fraction by a^2 and substitute

$$\mathbf{v} = 2\frac{\mathbf{b}/a}{1 + (\mathbf{b}/a)^2} \tag{12.15}$$

then we obtain

$$\begin{pmatrix} \Delta t' \\ \Delta \mathbf{x}' \end{pmatrix} = \frac{1}{1 - v^2} \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} \begin{pmatrix} \Delta t \\ 0 \end{pmatrix} \tag{12.16}$$

If we change the system of units to SI, then instead of Δt we write $c\Delta t$, and instead of $\mathbf{v} \rightarrow \mathbf{v}/c$, from where we obtain

$$\begin{pmatrix} c\Delta t' \\ \Delta \mathbf{x}' \end{pmatrix} = \begin{bmatrix} 1 \\ \mathbf{v}/c \end{bmatrix} \begin{pmatrix} c\Delta t \\ 0 \end{pmatrix}, \tag{12.17}$$

which is the Galilean transformation

$$\begin{aligned} \Delta t' &= \Delta t \\ \mathbf{x}' &= \mathbf{v}\Delta t + \mathbf{x}'_0 \end{aligned}$$

All of that confirms the compliance of our calculations with the William Baylisean Algebra of Physical Space.

12.2 Electromagnetic field

Now, we will have a closer look at the electric field equations. Based on the Chapter 4, differentiation operators (4-gradient ∂^- and 4-divergence ∂) under transformation $\mathbb{X}' = \Lambda \mathbb{X} \Lambda^*$ change to:

$$\partial A(\mathbb{X}) = \Lambda^* \partial' \Lambda A(\Lambda^- \mathbb{X}' \Lambda^{*-}) \tag{12.18}$$

$$\partial^- A(\mathbb{X}) = \Lambda^- \partial'^- \Lambda^{*-} A(\Lambda^- \mathbb{X}' \Lambda^{*-}) \tag{12.19}$$

We transform equations of electrostatics according to the first identity

$$\begin{bmatrix} \frac{\partial}{\partial t} \\ \nabla \end{bmatrix} \begin{pmatrix} 0 \\ \mathbf{E}(\mathbb{X}) \end{pmatrix} = \rho(\mathbb{X}) \quad \longrightarrow \quad \begin{bmatrix} \frac{\partial}{\partial t'} \\ \nabla' \end{bmatrix} \Lambda \begin{pmatrix} 0 \\ \mathbf{E}(\Lambda^- \mathbb{X}' \Lambda^{*-}) \end{pmatrix} = \Lambda^{*-} \rho \left(\Lambda^- \mathbb{X}' \Lambda^{*-}\right) \tag{12.20}$$

Based on formula 4.10 we can multiply the resulting equation on the right by any orthogonal paravector, for example Λ^-

$$\begin{bmatrix} \frac{\partial}{\partial t'} \\ \nabla' \end{bmatrix} (\Lambda \begin{pmatrix} 0 \\ \mathbf{E}(\Lambda^- \mathbb{X}' \Lambda^{*-}) \end{pmatrix} \Lambda^-) = \Lambda^{*-} [\rho \left(\Lambda^- \mathbb{X}' \Lambda^{*-}\right)] \Lambda^- \tag{12.21}$$

hence, on the left side of the above equation we have

$$\begin{pmatrix} 0 \\ \mathbf{E}' + i\mathbf{B}' \end{pmatrix} = \frac{1}{a^2 - b^2 + c^2 - d^2} \begin{bmatrix} a + id \\ \mathbf{b} + i\mathbf{c} \end{bmatrix} \begin{pmatrix} 0 \\ \mathbf{E} \end{pmatrix} \begin{bmatrix} a + id \\ -\mathbf{b} - i\mathbf{c} \end{bmatrix}, \tag{12.22}$$

and on the right side

$$\begin{pmatrix} \rho' \\ -\mathbf{j}' \end{pmatrix} = \frac{\rho}{a^2-b^2+c^2-d^2}\begin{bmatrix} a-id \\ -\mathbf{b}+i\mathbf{c} \end{bmatrix}\begin{bmatrix} a+id \\ -\mathbf{b}-i\mathbf{c} \end{bmatrix} \tag{12.23}$$

The equation (12.21) is a system of Maxwell's equations in the primed frame.

$$\begin{bmatrix} \frac{\partial}{\partial t'} \\ \nabla' \end{bmatrix}\begin{pmatrix} 0 \\ \mathbf{E}'+i\mathbf{B}' \end{pmatrix} = \begin{pmatrix} \rho' \\ -\mathbf{j}' \end{pmatrix} \tag{12.24}$$

or

$$\begin{array}{ll} \nabla'\mathbf{E}' = \rho' & \nabla'\mathbf{B}' = 0 \\ \nabla'\times\mathbf{B}' = \frac{\partial \mathbf{E}'}{\partial t'} + \mathbf{j}' & \nabla'\times\mathbf{E}' = -\frac{\partial \mathbf{B}'}{\partial t'} \end{array} \tag{12.25}$$

By identity (12.19) we obtain the conditions for the field to meet the wave equation.

$$\begin{bmatrix} \frac{\partial}{\partial t} \\ -\nabla \end{bmatrix}\begin{pmatrix} \varphi(\mathbb{X}) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{E}(\mathbb{X}) \end{pmatrix} \tag{12.26}$$

$$\Updownarrow$$

$$\begin{bmatrix} \frac{\partial}{\partial t'} \\ -\nabla' \end{bmatrix}\Lambda^{*-}\begin{pmatrix} \varphi(\Lambda^{-}\mathbb{X}'\Lambda^{*-}) \\ 0 \end{pmatrix} = \Lambda\begin{pmatrix} 0 \\ \mathbf{E}(\Lambda^{-}\mathbb{X}'\Lambda^{*-}) \end{pmatrix} \tag{12.27}$$

Just as before we multiply the right side of the received equation by Λ^{-}, and we obtain

$$\begin{bmatrix} \frac{\partial}{\partial t'} \\ -\nabla' \end{bmatrix}(\Lambda^{*-}\begin{pmatrix} \varphi(\Lambda^{-}\mathbb{X}'\Lambda^{*-}) \\ 0 \end{pmatrix}\Lambda^{-}) = \Lambda\begin{pmatrix} 0 \\ \mathbf{E}(\Lambda^{-}\mathbb{X}'\Lambda^{*-}) \end{pmatrix}\Lambda^{-}. \tag{12.28}$$

which gives

$$\begin{pmatrix} \varphi' \\ \mathbf{A}' \end{pmatrix} = \Lambda^{*-}\begin{pmatrix} \varphi \\ 0 \end{pmatrix}\Lambda^{-} \quad \text{and} \quad \begin{pmatrix} 0 \\ \mathbf{E}'+i\mathbf{B}' \end{pmatrix} = \Lambda\begin{pmatrix} 0 \\ \mathbf{E} \end{pmatrix}\Lambda^{-} \tag{12.29}$$

So, the equation (12.28) can be denoted as

$$\begin{bmatrix} \frac{\partial}{\partial t'} \\ -\nabla' \end{bmatrix}\begin{pmatrix} \varphi' \\ -\mathbf{A}' \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{E}'+i\mathbf{B}' \end{pmatrix} \tag{12.30}$$

or

$$\frac{\partial \varphi'}{\partial t'} + \nabla'\mathbf{A}' = 0 \tag{12.31}$$

$$-\frac{\partial \mathbf{A}'}{\partial t'} - \nabla'\varphi' = \mathbf{E}' \tag{12.32}$$

$$\nabla'\times\mathbf{A}' = \mathbf{B}' \tag{12.33}$$

The results confirm the compatibility of our considerations and the theory that has been applied for over a century, because the field is transformed in such a way that in Maxwell's equations we obtain the density of current and the Lorenz gauge condition is maintained. The field is transformed by rotation, and based on 2.3.6 we know that the rotation does not change the scalar component of the rotated paravector, which is equivalent to the Lorenz gauge invariance. Since this is not a Euclidean rotation, the real vector of the electric field becomes a complex vector, whose imaginary component is interpreted as the magnetic field. Assuming that paravector Λ is the real one (for the sake of simplicity), by equations (12.22) (12.23) we get:

$$\mathbf{E}' = \frac{a^2+b^2}{a^2-b^2}\mathbf{E} - 2\frac{\mathbf{b}(\mathbf{bE})}{a^2-b^2} + 2ia\frac{\mathbf{b}\times\mathbf{E}}{a^2-b^2} = \frac{a^2+b^2}{a^2-b^2}(\mathbf{E}+i\mathbf{v}\times\mathbf{E}) - 2\frac{\mathbf{b}(\mathbf{bE})}{a^2-b^2} \tag{12.34}$$

$$\rho' = \frac{a^2+b^2}{a^2-b^2}\rho \quad \text{and} \quad \mathbf{j}' = \mathbf{v}\rho, \quad \text{where} \quad \mathbf{v} = \frac{2a\mathbf{b}}{a^2+b^2} \tag{12.35}$$

For unrelativistic velocities we have $b^2 \ll a^2$, so the time-dilation factor is equal to 1 and the last component of the equation (12.34) disappears.

Now we calculate the fields that appear around an infinite straight wire in which a constant current has always flown. The intensity of the electric field of the load placed at a distance of **r** is

$$\mathbf{E} = \frac{Q\mathbf{r}}{r^3} = \frac{\mathbf{r}}{r^3}\rho(\mathbf{s}d\mathbf{l}) \tag{12.36}$$

Since the electrical charges are always distributed equally along an infinite straight wire, so the electric field of negative charges at a point of h of the wire equals:

$$\mathbf{E} = \rho s \int_{-\infty}^{+\infty} \frac{\mathbf{l}+\mathbf{h}}{\left(\sqrt{l^2+h^2}\right)^3} dl, \qquad \text{since} \quad \mathbf{r} = \mathbf{l}+\mathbf{h}, \mathbf{l} \perp \mathbf{h} \text{ and } \mathbf{s} \parallel \mathbf{l} \tag{12.37}$$

If electrons move along the wire at the velocity of $\mathbf{v} = 2\mathbf{b}/(a^2+b^2)$, that is parallel to **l**, then by equation (12.29) the vectors of the electric field and magnetic induction are the spatial components of 4-vector:

$$E = \frac{\rho s}{a^2-b^2} \int_{-\infty}^{+\infty} \left[\begin{array}{c} a \\ \mathbf{b} \end{array}\right] \left(\begin{array}{c} 0 \\ \frac{\mathbf{l}+\mathbf{h}}{\left(\sqrt{l^2+h^2}\right)^3} dl \end{array}\right) \left[\begin{array}{c} a \\ \mathbf{b} \end{array}\right] = \left(\begin{array}{c} 0 \\ \mathbf{E}+i\mathbf{B} \end{array}\right) \tag{12.38}$$

$$\mathbf{E}+i\mathbf{B} = \frac{\rho s}{a^2-b^2} \int_{-\infty}^{+\infty} \frac{(a^2+b^2)(\mathbf{l}+\mathbf{h}) - 2\mathbf{b}(\mathbf{bl}) + 2i\mathbf{b}\times\mathbf{h}}{\left(\sqrt{l^2+h^2}\right)^3} dl = \tag{12.39}$$

$$\frac{a^2+b^2}{a^2-b^2}\left(- \frac{\rho s \mathbf{l}_1}{\sqrt{l^2+h^2}}\Bigg|_{-\infty}^{+\infty} + \frac{\rho s \mathbf{h} l}{h^2\sqrt{l^2+h^2}}\Bigg|_{-\infty}^{+\infty} + \frac{\mathbf{b}b}{a^2+b^2}\,\frac{2\rho s}{\sqrt{l^2+h^2}}\Bigg|_{-\infty}^{+\infty} + 2i\frac{\mathbf{b}\times\mathbf{h}}{a^2+b^2}\,\frac{\rho s l}{h^2\sqrt{l^2+h^2}}\Bigg|_{-\infty}^{+\infty}\right)$$

where before the parenthesis we have the dilation factor. Vector $\mathbf{l}_1$ is a unitary vector, whose direction is in line with the direction of wire, so $\mathbf{l}_1 = \mathbf{b}/b$. The first and the third integrals reset, and since $2\rho s\mathbf{b}/(a^2+b^2) = \mathbf{j}$ is the current density, we get

$$\mathbf{E}+i\mathbf{B} = \frac{a^2+b^2}{a^2-b^2}\left(\frac{2\rho s \mathbf{h} l}{h^2\sqrt{l^2+h^2}}\Bigg|_{-\infty}^{+\infty} + i\,\frac{\mathbf{j}\times\mathbf{h} l}{h^2\sqrt{l^2+h^2}}\Bigg|_{-\infty}^{+\infty}\right) \tag{12.40}$$

The real component is the electric field of moving electrons, while the imaginary component is the magnetic field proportional to the current flowing in the wire.

$$\mathbf{E}_{-} = \frac{a^2+b^2}{a^2-b^2}\,\frac{2\rho s \mathbf{h} l}{h^2\sqrt{l^2+h^2}}\Bigg|_{-\infty}^{+\infty} = \frac{a^2+b^2}{a^2-b^2}\,\frac{2\rho s \mathbf{h}}{h^2} \tag{12.41}$$

$$\mathbf{B} = \frac{a^2+b^2}{a^2-b^2}\,\frac{\mathbf{j}\times\mathbf{h} l}{h^2\sqrt{l^2+h^2}}\Bigg|_{-\infty}^{+\infty} = \frac{a^2+b^2}{a^2-b^2}\,\frac{2\mathbf{j}\times\mathbf{h}}{h^2} \tag{12.42}$$

The second formula shows the Biot-Savart law, so it is another confirmation of the correctness of the theory created by Professor Baylis. In the wire there is the same number of positive and negative charges, but only negative ones move. Integrating the formula (12.34) by the wire, but getting positive charges only, we get

$$\mathbf{E}_{+} = \frac{\rho_{+} s l \mathbf{h}}{h^2\sqrt{l^2+h^2}}\Bigg|_{-\infty}^{+\infty} = \frac{2\rho_{+} s \mathbf{h}}{h^2} \tag{12.43}$$

Since the electrons in the wire move at the non-relativistic velocity, there is no resultant electric field. Using the formulas from Chapter 4 we have to prove that the wave equation

$$\left[\begin{array}{c} \frac{\partial}{\partial t} \\ \nabla \end{array}\right] \left[\begin{array}{c} \frac{\partial}{\partial t} \\ -\nabla \end{array}\right] \left(\begin{array}{c} \varphi(\mathbb{X}) \\ 0 \end{array}\right) = \rho(\mathbb{X}) \tag{12.44}$$

transforms into

$$\left[\begin{array}{c}\frac{\partial}{\partial t'}\\ \nabla'\end{array}\right]\left[\begin{array}{c}\frac{\partial}{\partial t'}\\ -\nabla'\end{array}\right]\Lambda^{*-}\left(\begin{array}{c}\varphi\left(\Lambda^{-}\mathbb{X}'\Lambda^{*-}\right)\\ 0\end{array}\right)\Lambda^{-}=\Lambda^{*-}\rho\left(\mathbb{X}'\right)\Lambda^{-} \tag{12.45}$$

What follows from the considerations above is that the Baylis' version of the Lorentz transformation does not look bad:

- It is compatible with the Lorentz transformation
- It is compatible with the valid rules of the theory of electricity and magnetism, that is:
  - as a result of transformation of the scalar potential there appears the vector potential
  - as a result of transformation of the charge density there appears the current density
  - the Lorentz gauge condition is invariant
- Although the Lorentz rotation is a complex transformation, it transforms the real space-time into itself.

12.3 Doubts

Unfortunately, some details cause concern:

1. In the formula (12.21) there is a discrepancy between the direction of charges movement (coordinates of charges $\Lambda^{-}\mathbb{X}\Lambda^{*-}$), and the direction of electric current ($\Lambda^{*-}\rho\Lambda^{-}$).
2. As a result of the composition of inertial rectilinear movements, one can get a rotation in place.

The second point is the most serious objection against 'Lorentz rotation'. In Chapter 10.4 we showed that the Lorentz transformation for a non-relativistic approximation becomes a Galilean transformation, but we did so by simplifying the transformational formulas. First, the paravectors representing the transformation were real, not complex, and second, the transformed interval was time, not space-time. By composing the velocities, we obtain complex paravectors and we have to check what the Lorentz transformations will look like for them.

Although the idea of realisation did not give fully satisfactory results, while working on it, many very interesting mathematical properties of orthogonal paravectors were found. We will now use them to show that if we accept the Baylisean definition of the Lorentz transformation, we conclude that we can compose the boosts so that we end up turning on the spot. In other words **one can choose rectilinear translational movements so that as a result of their putting together we get a rotation in place** (not to be confused with rotational movement). This seems to be a serious error of Baylis' theory, and thus also of the classical theory (as far as both theories are equivalent?).

A Lorentz rotation is defined by:

$$\Lambda\mathbb{X}\Lambda^{*}=\frac{1}{a^{2}-b^{2}+c^{2}-d^{2}}\left[\begin{array}{c}a+id\\ \mathbf{b}+i\mathbf{c}\end{array}\right]\left(\begin{array}{c}\Delta t\\ \Delta\mathbf{x}\end{array}\right)\left[\begin{array}{c}a-id\\ \mathbf{b}-i\mathbf{c}\end{array}\right] \tag{12.46}$$

The complex Λ paravector is the product of the multiplication of real orthogonal paravectors. We will now prove that for each complex paravector Γ we can find such a real paravector that their product will be a special paravector. We already know that it is possible from the properties of the theorem 8.2.1.7.

A special paravector has a form of $R=\left[\begin{array}{c}r\\ i\mathbf{s}\end{array}\right]$, and a real one $B=\left[\begin{array}{c}k\\ \mathbf{l}\end{array}\right]$. We need to find a B paravector that

$$\left[\begin{array}{c}r\\ i\mathbf{s}\end{array}\right]=\left[\begin{array}{c}k\\ \mathbf{l}\end{array}\right]\left[\begin{array}{c}a+id\\ \mathbf{b}+i\mathbf{c}\end{array}\right].$$

As a result of the multiplication, we have to obtain the zero imaginary component of a scalar and a zero real vector, which gives the following conditions:

$$\left\{\begin{array}{rcl} kd+\mathbf{cl} &=& 0 \\ k\mathbf{b}+a\mathbf{l} &=& \mathbf{l}\times\mathbf{c} \end{array}\right.$$

The above system of equations can be represented by non-singular paravectors

$$\begin{bmatrix} 0 \\ -\mathbf{l} \end{bmatrix}\begin{bmatrix} a \\ i\mathbf{c} \end{bmatrix}=k\begin{bmatrix} id \\ \mathbf{b} \end{bmatrix}$$

Since paravector $\begin{bmatrix} a \\ i\mathbf{c} \end{bmatrix}$ is proper, then choosing k as a parameter, we compute the vector $\mathbf{l}$

$$\mathbf{l}=-k\frac{a\mathbf{b}+d\mathbf{c}+\mathbf{b}\times\mathbf{c}}{a^2+c^2},$$

hence the paravector

$$B=k\begin{bmatrix} 1 \\ \frac{a\mathbf{b}+d\mathbf{c}+\mathbf{b}\times\mathbf{c}}{a^2+c^2} \end{bmatrix}^{-}=k\sqrt{\frac{a^2+c^2}{a^2-b^2+c^2-d^2}}\left(\frac{\Gamma}{\underline{|\Gamma|}}\right)^{-}.$$

We calculate a special paravector

$$R=k\begin{bmatrix} 1 \\ \frac{a\mathbf{b}+d\mathbf{c}+\mathbf{b}\times\mathbf{c}}{a^2+c^2} \end{bmatrix}^{-}\begin{bmatrix} a+id \\ \mathbf{b}+i\mathbf{c} \end{bmatrix}=k\frac{a^2-b^2+c^2-d^2}{a^2+c^2}\begin{bmatrix} a \\ i\mathbf{c} \end{bmatrix}$$

After dividing both sides by $k\frac{a^2-b^2+c^2-d^2}{\sqrt{a^2+c^2}}$ and performing some transformations, we get the equation for orthogonal paravectors

$$\frac{1}{\sqrt{a^2+c^2}}\begin{bmatrix} a \\ i\mathbf{c} \end{bmatrix}=\frac{1}{\sqrt{1-\frac{b^2+d^2}{a^2+c^2}}}\begin{bmatrix} 1 \\ \frac{a\mathbf{b}+d\mathbf{c}+\mathbf{b}\times\mathbf{c}}{a^2+c^2} \end{bmatrix}^{-}\frac{1}{\sqrt{a^2-b^2+c^2-d^2}}\begin{bmatrix} a+id \\ \mathbf{b}+i\mathbf{c} \end{bmatrix}=\left(\underline{|\Lambda}\right)^{-}\Lambda$$

Let's go back to the equation $X'=\Lambda\mathbb{X}\Lambda^*$. By switching to a frame that moves at the speed represented by the $\left(\underline{|\Lambda}\right)^{-}$, we obtain:

$$\mathbb{X}''=\left(\underline{|\Lambda}\right)^{-}\Lambda\mathbb{X}\left[\left(\underline{|\Lambda}\right)^{-}\Lambda\right]^*=\frac{1}{\sqrt{a^2+c^2}}\begin{bmatrix} a \\ i\mathbf{c} \end{bmatrix}\mathbb{X}\frac{1}{\sqrt{a^2+c^2}}\begin{bmatrix} a \\ -i\mathbf{c} \end{bmatrix} \tag{12.47}$$

The conclusion is that **one can find such a way to move forward as to turn on the spot**, which raises our doubts about the Baylis' theory of "Lorentz rotation".

The question arises: What does the 'Lorentz rotation' represent?

If we substitute $\mathbf{u}=\mathbf{b}/a$ in the equation (12.15), we get $\mathbf{v}=2\mathbf{u}/(1+u^2)$. Hence, we get close to associating the obtained velocity with the ratio of momentum and kinetic energy. It is enough to multiply the numerator and denominator by the same mass. And here we come to the essence of the difference between the relativistic transformation proposed by us and the Lorentz transformation proposed by Professor Baylis:

According to Professor Baylis, the definition of velocity results from the relationship between kinetic energy and momentum. We traditionally define it as a road travelled in time. In the case of the description of the movement of a single object, it does not matter, but in the case of the movement described in relation to other objects, it will have serious consequences for relativistic mechanics. It seems that energy should always be the product of coupled paravectors, so it should be described with a real paravector. A paravector that describes the kinetic energy of an object in motion will always have a positive scalar component (energy) and a real momentum vector. Here it makes sense to rotate or nullify the vector component when composing the speed of the observer's, because the results of the multiplication operation of the mutually coupled paravectors will be different for each observed energy-momentum paravector. In the formula (12.47) we have the same rotation for every 4-vector $\mathbb{X}$, so it is the observer who makes the Euclidean rotation in space. It is particularly well visible in the case of the description of a purely spatial vector (improper paravector). According to the model created by Professor Baylis, such situations are possible, and with us - they are not.

By choosing the model of complex space, we enter an unknown territory, but the mathematics is largely simplified, which bodes well.

Chapter 13

Summary

In this paper is shown that it is possible to build a mathematical model including Electrodynamics and the Special Theory of Relativity, consistent with experiments, but different from the model considered by the science mainstream to be the only correct one. Experiments carried out in various variants show that light in a vacuum always moves at a constant speed of about 300,000 km/s. With this in mind, we assumed that the electromagnetic wave in any frame in a vacuum must be described by the same mathematical formula - the wave equation, and we found a simple linear transformation satisfying the invariance of this equation. The trouble is, it's a complex transformation, not a real one. Its matrix can be represented by a complex paravector. This transformation is close to the known Lorentz transformation, and in the direction of motion both are identical. To distinguish it from the classical Lorentz transformation, we called mine the relativistic transformation[1]. In the course of work, it turned out that our transformation matrices are already known as paravectors and that they are dealt with by physicists using Clifford's algebra. Paravectors are an excellent tool for describing objects in space-time because they naturally extend the algebra of vectors. Paravectors combine the features of numbers and vectors:

- Together with the addition and multiplication, paravectors form a ring, so they have properties similar to numbers.
- Paravectors meet geometrical relationships such as parallelism, perpendicularity or angles, which makes them spatially imaginable just like vectors.
- Maxwell's equations written in a paravector form naturally refer to the operator-vector notation introduced by Olivier Heaviside.

Although at the beginning we started from the assumption that physical space-time is real and we were looking for solutions in real quantities, mathematical aesthetics steadily directed the discussion towards a complex space. At this moment, we can say with full conviction that space-time has a complex structure. It is difficult to imagine an imaginary direction because our immediate experiences are about stationary phenomena in the sense of the speed of light, and those taking place at enormous speeds reach the observer through the transmitted energy, which by nature is always real. However, everyone is well aware of physical phenomena such as magnetic fields and gyroscopic phenomena that testify to complex space-time. Since imaginary coordinates never exist alone (they must always be accompanied by real coordinates) and are not independent of real coordinates, we treated them as auxiliary coordinates and not as independent dimensions in an algebraic sense. For this reason, we interpreted the imaginary spatial components as indicators of deformation of real quantities. The situation is much more problematic with the imaginary time component which we have failed to interpret. Real time is orderly - it has its direction and pace, and complex numbers are not ordered. Although for an immobile observer, that is one living in real time, the complex time can be ordered along the real component, as of now this is only a hypothesis and it requires a lot of research. Besides, it is not certain whether it makes sense at all to order the times of different frames that do not have direct

[1] The mathematical name is a paravector orthogonal transformation

physical relationships with each other, or can ordering be introduced only at the interface of the energy interaction of these frames? Certainly, the proper time of any frame is real and orderly. The interpretation of velocity as a paravector angle between four-vectors prompts us to give different 'directions' to the time axes of frames that are in mutual motion. The flow of time in different 'directions' can only take place in complex space-time when the frames are in motion with respect to each other. This does not mean, however, that these times move at different speeds. The real time is the local dimension. It is related to the object. The times of spatially distant objects can be synchronized only when they are at rest. In the case when objects move, each displacement causes the disynchronization of the previously established simultaneity. However, it is possible to coordinate objects that are in motion in relation to each other. This can be done when and where they meet. In the presented example of an explosion in the center of a spherical laboratory, the objectives for any observer were the beginning and the end of the experiment, that is, the place and time of the explosion and the meeting of particles. Meanwhile, the particles were not simultaneous and each observer saw them in their own way.

In Newtonian physics, time is a dimension independent of anything. Observers can objectively place all phenomena in time. All they need to do is synchronize their clocks with each other in advance. They can coordinate the place of the phenomenon by translation, rotation, or Gallileo transformation of their coordinate frames. The introduction of a variable time in the classical SR made it necessary to give up the concept of simultaneity, which was natural for a human being immersed in objective time. In order to make observations, there must be some objective references provided. The speed of light in a vacuum is such an objective quantity. The relative speed of two observers is also objective. If the speed of observer A relative to observer B is v, then the speed of observer B is the same with respect to A. SR reversed the principles - time and space are variable, but equally variable, so that their ratio (velocity) is constant. And at this point we encounter an interpretation difficulty, because in the current SR, space is variable only in the direction of relative motion (Lorentz contraction). The perpendicular directions do not change. The Lorentz transformation introduces anisotropy. Theoretically, a deformation of moving objects should occur. Complex space-time is more flexible, as opposed to the rigid real space-time of the SR. Another reason, for real space-time rigidity is the assumption that it is an affine space - observers in their coordinate systems on their timelines try to order the spatially distant events in the same way, and this is impossible.

Differences between the complex model and the current SR

Complex space-time is not a Cartesian space with a coordinate frame in which the observer can locate all the phenomena taking place in this space. An affine space is a subjective space of the observer. He marks the origin of the coordinate system and places the observed objects in this system. For the observer, only what energetically connects him with the observed objects makes physical sense. For space-time, this approach creates a huge limitation. The formulas we analysed describe physical interactions. However, the most important reason for departing from the affine space is the property of time which is its directed dynamics. Time does not stand still and does not turn back. Time can be divided into intervals (whiles), but an interval can never be built from moments (points). In other words, in the presented theory, time has a discrete structure, hence our space-time is a vector space and this is the basic difference between complex space-time and Minkowski's space-time.

The results of applying complex transformations in the theory of high velocity appear to be promising. As the most important, it should be noted that the complex model is consistent with the SR postulates, which does not mean, however, that both theories are equivalent. The postulates of the classical SR are so general that to deny them would be at least unwise. However, due to their general nature, they cannot be treated as axioms and a strict theory cannot be built on them.

1st. postulate: *The laws of physics take the same form in all inertial frames of reference* is valid for ANY physical theory. It is a necessary condition that should be met by any correctly formulated physical law and transformation equations of reference systems, regardless of what these laws refer to. This postulate should therefore be treated as a guideline for checking the universality of the mathematical model.

2nd. postulate: *The speed of light in a vacuum is independent of the kinematic state of any inertial observer* results directly from the first postulate, because if the wave equation expresses a law of nature, and the speed of the electromagnetic wave is its parameter, then by demanding the invariance of the wave equation, we also demand the invariance of the speed of light.

On the one hand, a complex model complies with the 2nd postulate, because we have shown invariance of the wave equation, but at the same time it doesn't comply, because the real speed of the object may exceed 300,000 km/s. This apparent paradox is caused by a different structure of space-time in which our theory is built. From the perspective of this work, it is clearly visible that the creators of the classic SR informally made a third assumption, which limited it a lot and resulted in numerous paradoxes. The assumption is: **Space-time is real**[2]. Our space-time is a complex structure, and therefore, although our real component of the speed of light may exceed (but not arbitrarily) the value of 300,000 km/s, measuring it in the light source frame the observer always yields the same result. In complex space-time the speed of light is constant. In the classical theory the assumption of real space-time results in the necessity to decompose the description of phenomena into components parallel and perpendicular to the direction of relative motion. This assumption, seemingly natural, is in fact contradictory to the postulate of universality of the laws of physics. The complex model eliminates this dissonance, and in addition, as far as the direction of motion and the description of energy go, it is in line with the current theory. In our considerations, this inconsistency points to the advantages of the complex model, the conformality (shape invariance) of which is immediately noticeable. While time dilation is observed, no space deformation has been noticed.

The complex model explains the concept of time dilation, which has been controversial since its introduction. As has been emphasized many times, and which we demonstrated clearly, it is prohibited to compare four-vectors from different reference systems, let alone their components (e.g. time). We can only compare the phases with each other, because they are invariant, and the phase interval is always equal to the interval of elapsed proper time. It cannot be ruled out that the proper times of all inertial reference frames elapse at the same rate. The fact that faster particles are seen as if they lived longer than identical, but slower ones, is an illusion similar to the fact that we see the mast of a yacht on the horizon as shorter than the mast of the same yacht in the harbour. This illusion only works when the observed object is in motion, just as the mast is short when the yacht is far away. This is because the local times of the objects moving in relation to each other flow in different 'directions'.

In complex model the energy is always a real quantity, and because the information from the surrounding world reaches the observer thanks to energy, it seems to him that the world is real while it is more compound. The real energy should follow the classical Lorentz transformation, but it does not. While the differences in the EM theory between the presented model and the classical relativistic electric field theory are cosmetic, the difference between the mechanics in both models is significant. These differences are shown in the table below

	Classic SR	Complex model
Momentum	$\mathbf{p} = m\gamma\mathbf{v}$	$\mathbf{p} = m\gamma^2\mathbf{v}$
Energy	$E = \gamma m$	$E = m\gamma^2(1+v^2)/2$

where $\gamma = (1-v^2)^{-1/2}$.

One more spectacular difference from the classic STR should be noted. With us, the formula for the equivalence of mass and energy is

$$E = \frac{mc^2}{2}$$

Mathematical aesthetics is also important. Obtaining the results presented above was possible only thanks to the paravector calculus. The algebra of the complex model naturally extends the concepts known from the most intuitive Euclidean geometry, which was emphasize in Chapter 2, and at the same time is closely related to the Clifford Cl_3 algebra.

The creators of SR, guided by the mathematical analysis of space-time dependencies, added time to spatial dimensions, which was a great and shocking discovery. Time is no longer stiff and independent of anything. However, the jigsaw puzzle that they put together has gaps due to the too literal transfer of space properties by mathematicians to time. After all, the physical properties of time and space are completely

[2] In the sense of the set R^4

different. Continuing the comparison to a puzzle, it can be said that flat puzzles turn out to be spatial blocks which, arranged in an additional imaginary dimension, fit together perfectly.

Differences between the complex model and the current Theory of Electricity and Magnetism

In electrodynamics, the complex model made it possible to abandon the vector potential, the Lorenz gauge condition and the current density in the Maxwell-Ampere equation, which was questioned even during Maxwell's lifetime. Instead, it was necessary to enter the scalar quantity $e = \mathbf{vE}$, called **scalar induction** by us, which does not occur in classical electrodynamics due to the adopted Lorenz gauge. This allowed for the symmetrization of the field equations (corresponding in the classical theory to Maxwell's equations) for fields from stationary charges and currents flowing in closed circuits, because in these cases scalar induction does not occur, and we deal with these cases in practice. From Maxwell's equations modified in this way, it is possible to derive all the laws of electrodynamics, including the Ampere's law. The lack of current density in the Maxwell-Ampere equation does not prevent the fundamental laws of the electromagnetic field from composing the wave equation, with the use of complex transformations. Neither vector potentials nor a Lorenz gauge condition are needed. The potential and charge density fields are invariant and real. On the basis of the theory described in this way, the basic premise for the search for a magnetic monopole loses its sense and it becomes clear why the magnetic charge has not been discovered, and if our reasoning is correct, then it cannot be discovered.

Epilogue

Although I am still far from declaring that the model proposed above is complete the obtained results seem promising enough that I decided to publish them. By creating the complex model I went back to the beginning of the evolution of the prevailing theory, that is: starting from the electric field equations, I developed alternative principles of relativistic transformation, trying not to lead to contradictions with known experimental results. Although initially I conducted my considerations assuming that physical space-time is real, mathematical aesthetics constantly directed me towards complex space. At the moment I can say with full conviction that space-time has a complex structure.

I was moving along my own path but I did not work in a vacuum. I drew much inspiration from the results of Professor William Baylis. I do not rule out that Baylis is right, but certain details, and above all intuition supported by a sense of mathematical aesthetics, indicate that high-velocity space-time is a complex structure after all. The main objection I make to W. Baylis's theory is that by combining the boosts he can turn in place. Based on the similarity of the geometry of complex space to Euclidean geometry, one can expect that by maneuvering his vehicle appropriately, the observer can arbitrarily change the direction of the motion of the object observed outside, that is, that this object can move relative to the observer in any direction at any speed. However, by combining the boosts, this never happens to all observed objects simultaneously and equally. The same change of direction of all observed objects takes place only in the case of the observer's rotation, which is why rotations and boosts are different groups of issues.

Since the scope of verified correctness is, so far, very limited, I admit that I approach my results with caution. Apart from checking the invariance of the laws and covariance of the most general equations of physics, I did not deal with more detailed issues. I leave this to specialists, whose interest I count on.

Chapter 14

Appendixes

Appendix 1. From discrete time to complex space-time

In the current SR, the time axis is a continuous set of real numbers. This assumption is impossible to test since the clocks measure the time in steps. As technology advances, these steps become shorter and shorter, but it seems that continuous measurement will never be achieved. We may assume that there is a hypothetical shortest time interval below which it is impossible to go - a quantum of time[1] , which we call a *while*. On the other hand, the stepwise advancement of time is close to our intuition. The history of the Earth is divided into epochs, the history of civilization is divided into periods, the year is divided into seasons, etc. The most precise clocks keep time in rhythm with the vibrations of the atom. To measure the time, there is a while determined by the beginning and ending moments: two consecutive moments. In the cinema, two frames of the film are needed to obtain the effect of motion. With just one photo-picture, nothing can be said about the movement. Motion in physics is described by differential equations that describe changes in the position or energy of an object in the infinitesimal time interval. The same goes for the transformational formulas. The Galilean transformation in Newtonian mechanics takes the form:

$$t' = t \qquad , \qquad \mathbf{x}' = \mathbf{x} - \mathbf{v}t \qquad \text{where} \quad \mathbf{x} = (x, y, z)$$

This formula is true only assuming that the systems are coordinated, that is, at the initial moment $t = 0$ they overlap, the object is at the point with $\mathbf{x}$ coordinates, and the clocks of both systems are synchronized. If we coordinate both systems at the moment $t'_0 \longleftrightarrow t_0$ and at the point $\mathbf{x}'_0 \longleftrightarrow \mathbf{x}_0$ the formula has the following form

$$t' - t'_0 = t - t_0 \qquad , \qquad \mathbf{x}' - \mathbf{x}'_0 = \mathbf{x} - \mathbf{x}_0 - \mathbf{v}(t - t_0)$$

The above formula is more general and it shows what cannot be seen in the previous formula, that the transformation applies to a specific time interval and not only to the coordinates of points, but also to vectors. The first formula imposes the coordination of the coordinates of both systems, while the second one is true even without the coordination of time and the coordinate system. Coordinate changes are objective no matter where the observers are and where they place the origins of their reference frames. While it is obvious in the affine Euclidean space, in the Lorentz transformation one should pay attention to this fact, because it is omitted in textbooks, and any attempts to synchronize the systems are backbreaking.

[1] Some physicists call it *Planck's time*

The Loretz transformation is commonly presented as a system of equations:

$$\begin{aligned} ct' &= \frac{ct - xv/c}{\sqrt{1-(v/c)^2}} \\ x' &= \frac{x - vt}{\sqrt{1-(v/c)^2}} \\ y' &= y \\ z' &= z, \end{aligned}$$

which are burdened with additional assumptions:

1. The relative speed of the systems is parallel to the OX and OX 'axes.
2. At the $t_0 = t'_0 = 0$ moment the origins of the coordinate systems coincide,

thus giving the impression that they are the transformational dependences of affine space. However, when we take more general formulas

$$\begin{aligned} ct' &= \gamma(ct - \boldsymbol{\beta}\mathbf{x}) \quad , \quad \text{gdzie} \quad \gamma = \frac{1}{\sqrt{1-v^2}}, \quad \boldsymbol{\beta} = \frac{\mathbf{v}}{c} \\ \mathbf{x}' &= \mathbf{x} + \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta}\mathbf{x})\boldsymbol{\beta} - \gamma\boldsymbol{\beta}t \end{aligned}$$

where the relative velocity of the systems has any direction, we will see that further explanations are necessary. The observer can place the object in any coordinate system he constructed - the coordinates (t, **x**) of the point in which the object is located are completely independent. However, if we want to describe the movement of this object between the coordinates Δt and $\Delta\mathbf{x}$, it is necessary to put the limiting relation $|\Delta\mathbf{x}/\Delta t| \leq c$. Besides, if we treat the spatial coordinates here as the coordinates of the position of the object in the frame, then we get a shift on the time axis depending on the dot product **vx** which makes it difficult to interpret, because what is the meaning of the dot product of the vector and the coordinates of the point? All points on the plane perpendicular to the velocity, regardless of their distance from the observer (the origin of the system), are simultaneous, and others are not. This desynchronisation is not due to the distance from the observer, but to the direction of movement.

If we do not coordinate the systems with each other and in the original system we choose an event starting at t_0 moment, at $\mathbf{x}_0$ point and ending at t, at $\mathbf{x}$ point, then an observer in the OX system will write that it took place in the $(t' - t'_0, \mathbf{x}' - \mathbf{x}'_0)$ space-time interval. The transformation formulas remain the same, but in place of the $(t, \mathbf{x}(x, y, z))$ variables there are $(\Delta t, \Delta\mathbf{x})$ vectors. So we transform the vector space, not the affine one. This subtle change on the one hand refines the assumptions for the considerations, and on the other hand gives us much more freedom, because vectors can be enriched with additional properties of multivectors, which cannot be done with points. In this way, the discrete advancement of time gives us the basis for extending the domain from a real space-time to a complex one. For this reason, from the perspective of mathematical properties, we treat space-time in the most general way - as a 4-dimensional complex continuous structure that has three spatial dimensions and a fourth scalar dimension - prototime. When describing the behavior of physical objects in such a mathematical environment, we cannot allow ourselves to be arbitrary, because any physical object has its own properties, which include a discrete value of energy, and many clues indicate that its proper time also has a discrete value. Such an assumption is objective due to the methodology of time measurement, in contrast to the assumption of continuity of proper time.

Philosopher Professor Herb Spencer [16] has written many on the discrete nature of time. This monograph is in complete agreement with his views on time.

Appendix 2. A reference to formalisms used by other authors

Until the end of the 19th century, physical quantities were divided into scalars and vectors. In the 20th century, theoretical physics was dominated by the unintuitive tensor formalism. There were also attempts to use quaternion calculus [1, 2, 3], [18]. For some time, scientists have been looking for a more intuitive language to describe phenomena in space-time. There are number of papers written in the language of multivector calculus in journals. The main precursors of this direction are Professors David Hestenes and William Baylis. To describe the properties of physical objects, they use concepts taken from the Clifford Algebra Cl_3, whose elements consist of scalars, vectors, bivectors and trivectors. David Hestenes, and many other authors after him, refer to it as Geometric Algebra (GA). William Baylis calls it the Algebra of Physical Space (APS). Although equations describing physical phenomena using multivectors are much clearer than tensor equations, multivector multiplication is an obstacle in performing calculations. Internal and external products are used, which can be scalar or vector products, depending on the configuration. D.Hestenes [11], W.Baylis [6] and the authors of [9] note that (p) element of GA as sum of a number (s), vector ($\mathbf{v}$), bivector ($\mathbf{b}$) and a trivector or pseudoscalar (t).

$$p = s + \mathbf{v} + \mathbf{b} + t \tag{14.1}$$

Operations on multivectors become clearer if we separate the scalar and vector part.

$$p = \begin{bmatrix} s+t \\ \mathbf{v}+\mathbf{b} \end{bmatrix} \tag{14.2}$$

In the notation traditionally used in GA, the multiplication is

$$p_1 p_2 = \begin{bmatrix} s_1+t_1 \\ \mathbf{v}_1+\mathbf{b}_1 \end{bmatrix} \begin{bmatrix} s_2+t_2 \\ \mathbf{v}_2+\mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} (p_1 p_2)_S \\ (p_1 p_2)_V \end{bmatrix}, \tag{14.3}$$

and

$$\begin{aligned} (p_1p_2)_S &= s_1 s_2 + s_1 t_2 + \mathbf{v}_1 \cdot \mathbf{v}_2 + \mathbf{v}_1 \wedge \mathbf{b}_2 + \mathbf{b}_1 \wedge \mathbf{v}_2 + \mathbf{b}_1 \cdot \mathbf{b}_2 + t_1 s_2 + t_1 \cdot t_2 \\ (p_1p_2)_V &= s_1 \mathbf{v}_2 + s_1 \mathbf{b}_2 + \mathbf{v}_1 s_2 + \mathbf{v}_1 \wedge \mathbf{v}_2 + \mathbf{v}_1 \cdot \mathbf{b}_2 + \mathbf{v}_1 \cdot t_2 + \mathbf{b}_1 s_2 + \\ & + \mathbf{b}_1 \cdot \mathbf{v}_2 + \mathbf{b}_1 \times \mathbf{b}_2 + \mathbf{b}_1 \cdot t_2 + t_1 \cdot \mathbf{v}_2 + t_1 \cdot \mathbf{b}_2 \end{aligned}$$

where, $\mathbf{v}_1 \cdot \mathbf{v}_2$ is an inner product, $\mathbf{v}_1 \wedge \mathbf{v}_2$ is an exterior product, and $\mathbf{v}_1 \times \mathbf{v}_2$ is a cross product.

The differences between the components are more readable if the sum (14.1) is written in complex form

$$p = s + \mathbf{v} + i\mathbf{b} + it \tag{14.4}$$

William Baylis called such complex multivectors paravectors because they have many properties of vectors. Instead of four components of a multivector, we can only got two: a complex scalar (α) and a complex vector ($\boldsymbol{\beta}$).

$$p = \begin{bmatrix} \alpha \\ \boldsymbol{\beta} \end{bmatrix} = \begin{bmatrix} s+it \\ \mathbf{v}+i\mathbf{b} \end{bmatrix} \tag{14.5}$$

In this notation, the multiplication of multivectors can be described

$$p_1 p_2 = \begin{bmatrix} (p_1 p_2)_S \\ (p_1 p_2)_V \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \boldsymbol{\beta}_1 \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \boldsymbol{\beta}_2 \end{bmatrix} = \begin{bmatrix} s_1+it_1 \\ \mathbf{v}_1+i\mathbf{b}_1 \end{bmatrix} \begin{bmatrix} s_2+it_2 \\ \mathbf{v}_2+i\mathbf{b}_2 \end{bmatrix} = \tag{14.6}$$

$$\begin{aligned} (p_1p_2)_S &= s_1 s_2 + i s_1 t_2 \mathbf{v}_1 \mathbf{v}_2 + i\mathbf{v}_1 \mathbf{b}_2 + i\mathbf{b}_1 \mathbf{v}_2 - \mathbf{b}_1 \mathbf{b}_2 + i t_1 s_2 - t_1 t_2 \\ (p_1p_2)_V &= s_1 \mathbf{v}_2 + i s_1 \mathbf{b}_2 + \mathbf{v}_1 s_2 + i\mathbf{v}_1 \times \mathbf{v}_2 - \mathbf{v}_1 \times \mathbf{b}_2 + i\mathbf{v}_1 t_2 + i\mathbf{b}_1 s_2 - \\ & - \mathbf{b}_1 \times \mathbf{v}_2 - i\mathbf{b}_1 \times \mathbf{b}_2 - \mathbf{b}_1 t_2 + i t_1 \mathbf{v}_2 - t_1 \mathbf{b}_2 \end{aligned}$$

$$= \begin{bmatrix} (s_1+it_1)(s_2+it_2) + (\mathbf{v}_1+i\mathbf{b}_1)(\mathbf{v}_2+i\mathbf{b}_2) \\ (s_1+it_1)(\mathbf{v}_2+i\mathbf{b}_2) + (\mathbf{v}_1+i\mathbf{b}_1)(s_2+it_2) + i(\mathbf{v}_1+i\mathbf{b}_1)\times(\mathbf{v}_2+i\mathbf{b}_2) \end{bmatrix} = \begin{bmatrix} \alpha_1\alpha_2 + \boldsymbol{\beta}_1\boldsymbol{\beta}_2 \\ \alpha_1\boldsymbol{\beta}_2 + \boldsymbol{\beta}_1\alpha_2 + i\boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2 \end{bmatrix}$$

The complex notation is definitely friendlier. By performing the above operation, we will see that it can be described in a simple form

$$\begin{bmatrix}\alpha_1\\ \boldsymbol{\beta}_1\end{bmatrix}\begin{bmatrix}\alpha_2\\ \boldsymbol{\beta}_2\end{bmatrix}=\begin{bmatrix}\alpha_1\alpha_2+\boldsymbol{\beta}_1\boldsymbol{\beta}_2\\ \alpha_1\boldsymbol{\beta}_2+\alpha_2\boldsymbol{\beta}_1+i\boldsymbol{\beta}_1\times\boldsymbol{\beta}_2\end{bmatrix}=\begin{bmatrix}\alpha_3\\ \boldsymbol{\beta}_3\end{bmatrix} \tag{14.7}$$

This, matrix notation of paravectors turned out to be very effective because

- paravectors have their matrix representation, so they have all matrix properties, which reduces the need to carry out many proofs,
- paravector notation naturally corresponds to the 4-vectors that are used in textbooks for SR,
- the clear separation of the scalar and vector part of the paravector highlights their completely different properties, which is obliterated in the tensor notation,
- the mutual relation of the scalar and vector parts is closely related to the determinant of the matrix and constitutes a paravector,
- calculations performed in this notation are transparent.

In the book [15] J.Scott called the above multiplication the algebraic multiplication of complex four-vectors and he showed the wide possibilities of using the algebra of complex four-vectors to describe physical issues.

The relationship between our notation and some others found in the scientific literature is explained below. For a reader familiar with the calculus of paravectors, it may be important to compare our notation with the formalism used by William Baylis, and we refer to his papers [8].

Table 14.1: William Baylis's equivalents

W.E.Baylis		This book	
paravector	$q=a+id+\mathbf{b}+i\mathbf{c}$	paravector	$\Gamma=\begin{bmatrix}a+id\\ \mathbf{b}+i\mathbf{c}\end{bmatrix}$
bar conjugation *)	$\overline{q}=a+id-\mathbf{b}-i\mathbf{c}$	reversion	$\Gamma^{-}=\begin{bmatrix}a+id\\ -\mathbf{b}-i\mathbf{c}\end{bmatrix}$
Hermitean conjugation	$q^{\dagger}=a-id+\mathbf{b}-i\mathbf{c}$	conjugation	$\Gamma^{*}=\begin{bmatrix}a-id\\ \mathbf{b}-i\mathbf{c}\end{bmatrix}$
gradient operator (or paragradient)	$\partial=\partial/\partial t-\nabla$	reversed differential operator (or 4-gradient)	$\partial^{-}=\begin{bmatrix}\partial/\partial t\\ -\nabla\end{bmatrix}$
spatially reversed gradient operator	$\partial=\partial/\partial t+\nabla$	differential operator (or 4-divergence)	$\partial=\begin{bmatrix}\partial/\partial t\\ \nabla\end{bmatrix}$
		determinant	$\det\Gamma=\Gamma\Gamma^{-}$
		module **)	$\lvert\Gamma\rvert=\sqrt{\det\Gamma}$
norm	$\lVert q\rVert=\sqrt{a^2+b^2+c^2+d^2}$		
		vigor	$\mathrm{vig}\Gamma=\Gamma\Gamma^{*}$

*) or Clifford conjugation **) it exists in the set of proper or singular paravectors only!

For readers familiar with multivector algebra (Geometric Algebra) the following explanations are valid. Scalars and trivectors are called by us complex scalars and we denote $\alpha=a+id$, where $i=a_{123}$ is a unity trivector. Vectors and bivectors are called by us real and imaginary vectors respectively, and their sum is a complex vector.

In the next table we compare our notation with the multivector notation used by D. Hestenes in the article [11].

Table 14.2: David Hestenes equivalents

Multivectors		Paravectors	
multivector	$q = [q]_0 + [q]_1 + [q]_2 + [q]_3$	paravector	$\Gamma = \begin{bmatrix} a + id \\ \mathbf{b} + i\mathbf{c} \end{bmatrix}$
scalar	$a = [q]_0$	real scalar	a
vector	$\underline{\mathbf{b}} = [q]_1$	real vector	$\mathbf{b}$
bivector	$\underline{\mathbf{C}} = [q]_2$	imaginary vector	$i\mathbf{c}$
trivector	$D = de_{123} = [q]_3$	imaginary scalar	id
conjugation	$\overline{q}$	reversion	Γ^-
reversion	q^*	conjugation	Γ^*
involution	$\widehat{q}$		$(\Gamma^-)^*$
bivector coordinates	$C^{23}\mathbf{e}_{23}, C^{31}\mathbf{e}_{31}, C^{12}\mathbf{e}_{12}$		$ic^1\mathbf{e}_1, ic^2\mathbf{e}_2, ic^3\mathbf{e}_3,$
trivector coordinates	de_{123}		id
inner products	$\underline{\mathbf{b}}_1 \cdot \underline{\mathbf{b}}_2$ $\underline{\mathbf{b}}_1 \cdot \underline{\mathbf{C}}_2 = -\underline{\mathbf{C}}_2 \cdot \underline{\mathbf{b}}_1$ $\underline{\mathbf{C}}_1 \cdot \underline{\mathbf{C}}_2 = \underline{\mathbf{C}}_2 \cdot \underline{\mathbf{C}}_1$ $D_1 \cdot \underline{\mathbf{b}}_2$ $D_1 \cdot \underline{\mathbf{C}}_2$		$\mathbf{b}_1\mathbf{b}_2$ $-\mathbf{b}_1 \times \mathbf{c}_2$ $-\mathbf{c}_1\mathbf{c}_2$ $id_1\underline{\mathbf{b}}_2$ $-d_1\mathbf{c}_2$
exterior products	$\underline{\mathbf{b}}_1 \wedge \underline{\mathbf{b}}_2$ $\underline{\mathbf{b}}_1 \wedge \underline{\mathbf{C}}_2 = \underline{\mathbf{C}}_2 \wedge \underline{\mathbf{b}}_1$		$i\mathbf{b}_1 \times \mathbf{b}_2$ $i\mathbf{b}_1\mathbf{c}_2$
vector product	$\underline{\mathbf{C}}_1 \times \underline{\mathbf{C}}_2 = -\underline{\mathbf{C}}_2 \times \underline{\mathbf{C}}_1$		$-i\mathbf{c}_1 \times \mathbf{c}_2$

Unit multivectors are represented by matrices:

Scalar $\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$ Vector $\begin{bmatrix} 0 & r_x & r_y & r_z \\ r_x & 0 & -ir_z & ir_y \\ r_y & ir_z & 0 & -ir_x \\ r_z & -ir_y & ir_x & 0 \end{bmatrix}$

Bivector $\begin{bmatrix} 0 & ir_x & ir_y & ir_z \\ ir_x & 0 & -r_z & r_y \\ ir_y & r_z & 0 & -r_x \\ ir_z & -r_y & r_x & 0 \end{bmatrix}$ Trivector $\begin{bmatrix} i & & & 0 \\ & i & & \\ & & i & \\ 0 & & & i \end{bmatrix}$

where $|\mathbf{r}| = 1$.

Appendix 3. Quasi-real space-time (real time and imaginary vectors)

It has been shown that it is not possible for the 4-vector coordinates to be special paravectors.

As we know from Chapter 2, there is a class of special paravectors which have real scalar and imaginary vector components. These paravectors with addition and multiplication operations form a division ring. In order for the structure to be a field, it only needs the commutation of multiplication. Such paravectors describe the Euclidean rotation. It is very tempting to check whether it is possible to build space-time on the basis of such paravectors. Since all special paravectors have a module, the space made up of such paravectors can be normalized. Since the vectors are imaginary, we say that space-time is quasi-real. Based on the current knowledge about paravectors, it is obvious that when writing the velocity with the $V^s = \frac{1}{\sqrt{1+v^2}}\begin{bmatrix} 1 \\ i\mathbf{v} \end{bmatrix}$ paravector[2] , the wave equation is invariant with respect to the transformation represented by this paravector. Below we will repeat for special paravectors the reasoning we used previously for proper paravectors.

The orthogonal transformation takes the form of

$$\begin{pmatrix} t' \\ i\mathbf{x}' \end{pmatrix} = \frac{1}{\sqrt{1+v^2}}\begin{bmatrix} 1 \\ i\mathbf{v} \end{bmatrix}\begin{pmatrix} t \\ i\mathbf{x} \end{pmatrix} \tag{14.8}$$

which is equivalent to a system of real equations

$$t' = \frac{1}{\sqrt{1+v^2}}(t - \mathbf{v}\mathbf{x}) \qquad \text{and} \qquad \mathbf{x}' = \frac{1}{\sqrt{1+v^2}}(\mathbf{x} + \mathbf{v}t - \mathbf{v}\times\mathbf{x}) \tag{14.9}$$

We can see that the transformation written in this way is internal to the set equinumerous with R^4 (quasi-real space-time) and everything would work if it were not for the dilation factor, which is different than in the applicable STR. As a result of composing paravectors corresponding to the velocity, we obtain

$$V_1^s V_2^s = \frac{1}{\sqrt{1+v_1^2}}\begin{bmatrix} 1 \\ i\mathbf{v}_1 \end{bmatrix}\frac{1}{\sqrt{1+v_2^2}}\begin{bmatrix} 1 \\ i\mathbf{v}_2 \end{bmatrix} = \frac{1}{\sqrt{1+\left(\frac{\mathbf{v}_1+\mathbf{v}_2-\mathbf{v}_1\times\mathbf{v}_2}{1-\mathbf{v}_1\mathbf{v}_2}\right)^2}}\begin{bmatrix} 1 \\ i\,\frac{\mathbf{v}_1+\mathbf{v}_2-\mathbf{v}_1\times\mathbf{v}_2}{1-\mathbf{v}_1\mathbf{v}_2} \end{bmatrix} \tag{14.10}$$

As it is not difficult to calculate in this case a result of the composing velocities is not limited.

$$v^2 = \left(\frac{\mathbf{v}_1+\mathbf{v}_2-\mathbf{v}_1\times\mathbf{v}_2}{1-\mathbf{v}_1\mathbf{v}_2}\right)^2 = \frac{\left(1+v_1^2\right)\left(1+v_2^2\right)}{(1-\mathbf{v}_1\mathbf{v}_2)^2} - 1 \tag{14.11}$$

For example, by combining the parallel velocities $v_1 = 0.8$ and $v_2 = 0.8$, we get the resultant velocity of $v \approx 4.9$, which is inconsistent with our scientific knowledge.

Below we will perform a time interval transformation.

$$\begin{pmatrix} \Delta t \\ 0 \end{pmatrix} = \frac{1}{\sqrt{1+v^2}}\begin{bmatrix} 1 \\ -i\mathbf{v} \end{bmatrix}\begin{pmatrix} \Delta t' \\ i\Delta\mathbf{x}' \end{pmatrix} = \frac{1}{\sqrt{1+v^2}}\begin{pmatrix} t' + \mathbf{v}\Delta\mathbf{x}' \\ i(\Delta\mathbf{x}' - \mathbf{v}\Delta t' + \mathbf{v}\times\Delta\mathbf{x}') \end{pmatrix} \tag{14.12}$$

From the vector part of the above formula, we obtain

$$\Delta\mathbf{x}' - \mathbf{v}\Delta t' + \mathbf{v}\times\Delta\mathbf{x}' = 0.$$

Since the result of the vector product is perpendicular to both $\mathbf{v}$ and $\Delta\mathbf{x}$, therefore it must be:

$$\Delta\mathbf{x}' - \mathbf{v}\Delta t' = 0 \qquad \text{and} \qquad \mathbf{v}\times\Delta\mathbf{x}' = 0. \tag{14.13}$$

In the motion equation of a point in the R^3 space the observer does not have a dilation factor, and the motion, just like in the complex space-time, is described by the Galilean formula, which looks good.

[2]The upper index *s* means that the paravector is special

When examining the wave equation, we face another problem.

In Chapter 4, we said that the solution of a homogeneous wave equation $\partial\partial^- A(X) = 0$ is satisfied by any function:

$$A(\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix}) \qquad \text{lub} \qquad A(\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix})$$

such that A is a paravector and paravector $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is singular. If $\alpha = 1$, and $\beta = -\mathbf{c}$ and $c^2 = 1$, we interpreted the vector $\mathbf{c}$ (Chpt. 3) as the speed of the wave (the speed of light). The wave front met the phase compliance condition

$$\begin{bmatrix} 1 \\ -\mathbf{c} \end{bmatrix}\begin{pmatrix} t - t_0 \\ \mathbf{x} - \mathbf{x}_0 \end{pmatrix} = 0.$$

Here we face a serious problem. In the case of special paravectors, the wave front should satisfy the equation:

$$C^{s-}\mathbb{X}^s = \begin{bmatrix} 1 \\ -i\mathbf{c} \end{bmatrix}\begin{pmatrix} t - t_0 \\ i(\mathbf{x} - \mathbf{x}_0) \end{pmatrix} = 0$$

Note that because paravectors C^{s-} and $\mathbb{X}^s$ are not singular, it is only satisfied for $X^s = X_0^s$.

However, if we write down the wave front as

$$\begin{bmatrix} 1 \\ i\mathbf{c} \end{bmatrix}\begin{pmatrix} t - t_0 \\ i(\mathbf{x} - \mathbf{x}_0) \end{pmatrix} = 0, \tag{14.14}$$

then we get a system of equations

$$\begin{cases} \Delta t - \mathbf{c}\Delta\mathbf{x} = 0 \\ \Delta\mathbf{x} + \mathbf{c}\Delta t - \mathbf{c} \times \Delta\mathbf{x} = 0 \end{cases} \tag{14.15}$$

The first equation shows that the vectors $\mathbf{c}$ and $\Delta\mathbf{x}$ should have the same direction, and the second equation shows that directions are opposite.

We can obtain the wave equation $\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi = 0$ in a different way, consistently for special paravectors. If we denote the differentiation operator as:

$$\partial^s = \begin{bmatrix} \frac{\partial}{\partial t} \\ i\nabla \end{bmatrix}$$

hence, the homogeneous wave equation is:

$$\partial^s\partial^s\varphi = \begin{bmatrix} \frac{\partial^2}{\partial t^2} - \nabla^2 \\ 2i\frac{\partial}{\partial t}\nabla \end{bmatrix}\varphi = 0 \qquad \text{or} \qquad \partial^{s-}\partial^{s-}\varphi = \begin{bmatrix} \frac{\partial^2}{\partial t^2} - \nabla^2 \\ -2i\frac{\partial}{\partial t}\nabla \end{bmatrix}\varphi = 0, \tag{14.16}$$

where we get an additional condition in the spatial part $\frac{\partial}{\partial t}\nabla\varphi = 0$.

In case of the transformation

$$X^{s\prime} = V^s X^s \qquad \text{or} \qquad \begin{pmatrix} t' \\ i\mathbf{x}' \end{pmatrix} = \frac{1}{\sqrt{1+v^2}}\begin{bmatrix} 1 \\ i\mathbf{v} \end{bmatrix}\begin{pmatrix} t \\ i\mathbf{x} \end{pmatrix} \tag{14.17}$$

the following identities can be proved in the same way as we did in Chapter 4

$$\partial^s A^s(X^s) = V^{s-}\partial^{s\prime}\varphi(V^{s-}X^{s\prime}) \tag{14.18}$$

$$\partial^{s-} A^s(X^s) = \partial^{s\prime-}V^s A^s(V^{s-}X^{s\prime}) \tag{14.19}$$

Hence we conclude that the wave equation (14.16) is not invariant with respect to the transformation (14.17).

The description of the Doppler effect differs from the current theory in terms of frequency only in the dilation factor.

$$\Theta = \begin{bmatrix} \omega \\ -i\mathbf{k} \end{bmatrix}\begin{pmatrix} t \\ i\mathbf{x} \end{pmatrix} = \frac{1}{\sqrt{1+v^2}}\begin{bmatrix} \omega \\ -i\mathbf{k} \end{bmatrix}\begin{bmatrix} 1 \\ -i\mathbf{v} \end{bmatrix}\begin{pmatrix} t' \\ i\mathbf{x}' \end{pmatrix} = \begin{bmatrix} \omega' \\ -i\mathbf{k}' \end{bmatrix}\begin{pmatrix} t' \\ i\mathbf{x}' \end{pmatrix}$$

that is, the new frequency is

$$\omega' = \frac{\omega - \mathbf{vk}}{\sqrt{1+v^2}}.$$

We should also check whether the Maxwell equations can be represented by a special paravector. The electrostatic field equation would be

$$\partial^{s-} E^s = \rho \qquad \text{or} \qquad \begin{array}{rcl} \nabla \mathbf{E} & = & \rho \\ \frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{E} & = & 0 \end{array} \tag{14.20}$$

According to the identity (14.19), the electric field $\mathbf{E}$ seen by an observer moving at velocity -$\mathbf{v}$ has the value of

$$\mathbf{E}' = \frac{1}{\sqrt{1+v^2}}(\mathbf{E} + \mathbf{v} \times \mathbf{E}) \tag{14.21}$$

It can be seen that when trying to write Maxwell's equations, we also failed to get any meaningful result, because instead of the magnetic field, we got an additional component of the electric field. Although the relativistic transformation is an internal operation in a real space-time, we once again come to the critical point when it turns out that the equations corresponding to the Maxwell's equations do not fit the EM theory.

The use of special paravectors for notation of coordinates and velocities seemed very tempting, primarily because special paravectors with summation and multiplication form a division ring. When doing multiplication and summation, we are always in space-time, which we can treat as real. Unfortunately, this hypothesis turned out to be a dead end. Attempts to use special paravectors to describe wave phenomena ended in a failure:

- it was impossible to write the wave front equation,
- composing velocities allowed superluminal speeds,
- trying to write Maxwell's equations yielded absurd results.

Therefore, we do not see any sense in continuing the search in this direction.

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