The relationship between odd perfect numbers and quadratic equations of one variable

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Abstract

This paper discovered the relationship between the relationship between One Variable Quadratic Equation and odd perfect numbers, and with the help of Veda's theorem, proved there is no odd perfect number.

Key words: System of quadratic equations with one variable; Veda's theorem
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1. Introduction

If a number is exactly equal to the sum of its true factors, it is called "perfect number".

For example, the first perfect number is 6, which has the divisor 1, 2, 3 and 6. Except for its own 6, the remaining three numbers are added, 1 + 2 + 3 = 6.

All the perfect numbers found are even, can there be odd perfect numbers? this is an unresolved number theory problem.

Euler has proven that: if $\exists$ odd perfect numbers Q,
then $Q=\pi^{\alpha}p_1^{2\beta_1}p_2^{2\beta_2}\dots p_s^{2\beta_s}$, $\pi\equiv\alpha\equiv1(mod4)$, $\beta_s\geq1$, $\pi, \mathrm{p_s}$ is the prime factor of Q.

This article establishes the equation for odd perfect numbers Q based on the definition of perfect numbers, set $\mathrm{Q=2k+1}$, $\mathrm{Q=Q_{j1}Q_{j2}}$, $\mathrm{Q_{j1}=2d_{j1}+1}$, $\mathrm{Q_{j2}=2d_{j2}+1}$, after studying N, $\mathrm{d_{j1}}$, and $\mathrm{d_{j2}}$in the equation, it was found that they are related to the coefficients of the system of quadratic equations with one variable. After calculation, it was found that there is a contradiction between the equation and the system of quadratic equations with one variable.

After analyzing the contradiction proved there is no odd perfect number.

2. Preparatory work

If Q is an odd perfect number, let $\mathrm{Q=Q_{j1}Q_{j2}, j=1,2,3\dots t}$, $\mathrm{Q_{j1}}$and $\mathrm{Q_{j2}}$ denote all true factors of Q.

$\mathrm{Q=Q_{11}Q_{12}=Q_{21}Q_{22}=\cdots Q_{j1}Q_{j2}\dots=\ Q_{t1}Q_{t2}}$, $\mathrm{Q_{j1}}$ and $\mathrm{Q_{j2}}$ are one-to-one correspondences.

Based on perfect numbers definition:

$$\mathrm{Q=Q_{11}Q_{12}=Q_{21}Q_{22}=\cdots Q_{j1}Q_{j2}\dots=Q_{t1}Q_{t2}=1+\sum_{j=1}^{t}(Q_{j1}+Q_{j2})} \qquad \cdots\cdots \textbf{(A)}$$

Let $\mathrm{Q=2k+1}$, $\mathrm{Q_{j1}=2d_{j1}+1}$, $\mathrm{Q_{j2}=2d_{j2}+1, k>1}$, $\mathrm{d_{j1}\ \geq1}$, $\mathrm{d_{j2}\ \geq1}$,

Have:

$$\mathrm{Q=2k+1=(2d_{11}+1)(2d_{12}+1)=(2d_{21}+1)(2d_{22}+1)=\cdots(2d_{t1}+1)(2d_{t2}+1)}$$
$$\mathrm{=1+\sum_{j=1}^{t}(\,2d_{j1}+1+2d_{j2}+1)=1+2\sum_{j=1}^{t}(\,d_{j1}+d_{j2}+1)}$$

$\because \mathrm{\left(2d_{j1}+1\right)\left(2d_{j2}+1\right)=4d_{j1}d_{j2}+2d_{j1}+2d_{j2}+1}$

That is $\mathrm{2k+1=4d_{j1}d_{j2}+2d_{j1}+2d_{j2}+1=1+2\sum_{j=1}^{t}(\,d_{j1}+d_{j2}+1)}$

subtract 1 from both sides of the equation and then divide by 2,

$\therefore \mathrm{k} = 2\mathrm{d}_{11}\mathrm{d}_{12} + \mathrm{d}_{11} + \mathrm{d}_{12} = \cdots 2\mathrm{d}_{j1}\mathrm{d}_{j2} + \mathrm{d}_{j1} + \mathrm{d}_{j2} = \sum_{j=1}^{t}(\mathrm{d}_{j1} + \mathrm{d}_{j2} + 1)$ $\cdots\cdots$ **(B)**

Inference1: in(B) $\mathrm{k} \equiv 0(\mathrm{mod}2), \mathrm{t} \equiv 0(\mathrm{mod}2), (\mathrm{d}_{j1} + \mathrm{d}_{j2}) \equiv 0(\mathrm{mod}2)$

Proof

$\because k = 2d_{11}d_{12} + d_{11} + d_{12}$, *if* $k \equiv 1(mod2)$, *then* $2d_{j1}d_{j2} + d_{j1} + d_{j2} \equiv 1(mod2)$

the left $(d_{11} + d_{12}) \equiv 1(mod2)$, *the right side* $\sum_{j=1}^{t}(d_{j1} + d_{j2} + 1) \equiv 0(mod2)$,

The odd and even numbers on both sides of the equation do not match.

$\therefore \exists!\ k \equiv 0(mod2), t \equiv 0(mod2), (d_{j1} + d_{j2}) \equiv 0(mod2)$ *makes*:

The left $(d_{11} + d_{12}) \equiv 0(mod2)$, *the right side* $\sum_{j=1}^{t}(d_{j1} + d_{j2} + 1) \equiv 0(mod2)$ #

Inference2: $\mathrm{d}_{j1} \neq \mathrm{d}_{j2}, t \geq 4$.

Proof

$\because Q = \pi^{\alpha} p_1^{2\beta_1} p_2^{2\beta_2} \dots p_s^{2\beta_s}$, $\pi \equiv \alpha \equiv 1(mod4)$, $\beta_s \geq 1$, [2] $(\alpha + \sum 2\beta_s) \equiv 1(mod2)$, $\therefore \mathrm{d}_{j1} \neq \mathrm{d}_{j2}$.

$\because \alpha \equiv 1(mod4)$, let $\alpha = 4m + 1$, if $m = 1$, then $\alpha_{min} = 5$ Can be expressed as$\pi^1\pi^4$and $\pi^2\pi^3$

$\therefore t \geq 4$ #

3. Odd perfect numbers and quadratic equations of one variable

Quadratic equations of one variable $x^2 + bx + c = 0$ *has two integer roots* $x_1 = d_{j1}, x_2 = d_{j2}$, *Veda's theorem:* $b = -(x_1 + x_2) = -(d_{j1} + d_{j2})$, $c = x_1x_2 = d_{j1}d_{j2}$.[1]

If $\exists(\boldsymbol{B}), k = 2d_{j1}d_{j2} + d_{j1} + d_{j2} = \sum_{j=1}^{t}(d_{j1} + d_{j2} + 1)$, *then there is system of Quadratic equations of one variable:*

$$x^2 + b_1x + c_1 = 0$$
$$x^2 + b_2x + c_2 = 0$$
$$x^2 + b_3x + c_3 = 0$$
$$\vdots$$
$$x^2 + b_jx + c_j = 0$$
$$\vdots$$
$$x^2 + b_tx + c_t = 0$$

Veda's theorem $b_j = -(d_{j1} + d_{j2})$, $c_j = d_{j1}d_{j2}$

$\because 2d_{j1}d_{j2} = 2c_j, b_j = -(d_{j1} + d_{j2})$

$\therefore$ *convert* (2) $k = 2d_{j1}d_{j2} + d_{j1} + d_{j2} = \sum_{j=1}^{t}(d_{j1} + d_{j2} + 1)$ *to* $k = 2c_j - b_j = \sum_{j=1}^{t}(1 - b_j)$.

That is, k satisfies both $k = 2c_j - b_j$ *and* $k = \sum_{j=1}^{t}(1 - b_j)$ *simultaneously,* $k \equiv 0(mod2)$ $t \equiv 0(mod2), (d_{j1} + d_{j2}) \equiv 0(mod2)$ [Inference1] $d_{j1} \neq d_{j2}, t \geq 4$ [Inference2]

does k exist to satisfy the equation?

Let there exist an integer k satisfying the following conditions:

1. Problem Statement and Notation

1.1 **System Scale Constraint:** $t \geq 4$ (even), where t is the number of equations.

1.2. **Quadratic Equation Constraints:** Each equation $x^2 + b_j x + c_j = 0$ has distinct positive integer roots $d_{j1} \neq d_{j2}$, with $s_j = d_{j1}+d_{j2} \geq 4$, $d_{j1} + d_{j2} \equiv 0 (mod 2)$ and $c_j = d_{j1}d_{j2}$..

1.3 **Discriminant Condition:** $b_j^2 - 4c_j = r_j^2 (r_j \in N^+)$

1.4 Global Equality Constraint: $k = 2c_j - b_j = 2d_{j1}d_{j2} + s_j$ holds for all j and $k = t + \sum_{j=1}^{t} s_j$,

Notation:

$s_j = d_{j1}+d_{j2}$. Sum of roots (even).

$S = \sum_{j=1}^{t} s_j$.Total sum of all roots.

$P_j = d_{j1}d_{j2}$. Product of roots.

2. Derivation of Root Constraints from System Scale

Step 1: Simultaneous Equations and Inequality Establishment

From the global equality constraint $k = t + \mathrm{S}$ and $k = 2P_j + s_j$ we derive:

$2P_j + s_j = t + \mathrm{S} \Longrightarrow 2P_j = t + \mathrm{S} - s_j$.

Combining with the discriminant condition $s_j^2 - 4P_j = r_j^2$ we eliminate P_j:

$s_j^2 - 2(t + \mathrm{S} - s_j) = r_j^2 \Longrightarrow s_j^2 + 2s_j = 2t + 2S$.

Step 2: Derivation of the Lower Bound : $s_j \geq 6$

Since $r_j^2 \geq 1$ we have: $s_j^2 + 2s_j \geq 2(t + s) + 1$,

Assuming $s_j < 6$ (e.g., $s_j = 4$) leads to:

$4^2 + 2 \times 4 = 24 \geq 2(t + S) + 1 \Longrightarrow 2(t + S) \leq 23 \Longrightarrow (t + S) \leq 11.5$

However, $t \geq 4$ and $\mathrm{S} \geq 4t$ (since $s_j \geq 4$)imply: $t + S \geq 4 + 16 = 20 > 11.5$

Thus, $s_j \geq 6$ is necessary.

3. Proof by Mathematical Induction

Proposition: For all systems with $t \geq 4$ and $s_j \geq 6$, no integer k satisfies all conditions.

Base Case (t = 4)

For $t = 4, k = 4 + S$ and for each j.

$2P_j + s_j = 4 + \mathrm{S}$

With $s_j \geq 6, \mathrm{S} \geq 6 \times 4 = 24$ leading to: $2P_j + 6 \geq 28 \Longrightarrow P_j \geq 11$

However, the maximum product of roots is $P_j \leq (\frac{s_j}{2})^2 - 1$

If $s_j = 6$, $P_j \leq 8$, a contradiction.

If $s_j = 8$, $P_j \leq 15$, still a contradiction.

Inductive Hypothesis and Step

Assume the proposition holds for $\mathrm{t} = 4 + 2\mathrm{m}$. We prove it for $\mathrm{t} = 4 + 2(\mathrm{m} + 1)$

1. **System Expansion:** Add two new equations with $s_j \geq 6$updating $S' = S + s_{k+1} + s_{k+2}$
2. **Contradiction Propagation:** The original system's contradiction persists as S'increases, raising the required lower bound for P_j while the actual product remains bounded by $(\frac{s_j}{2})^2 - 1$

4. Core of Generalized Contradiction

Nature of Contradiction: implies $\geq 4, S \geq 6t$, requiring $k = t + S \geq 28$.

System Scale Constraint: grows quadratically, $P_j \leq (\frac{s_j}{2})^2 - 1$ lagging behind the linear growth of k.

Mathematical Conflict: The product of roots cannot meet the lower bound derived from system constraints.

5. Conclusion

By prioritizing the system scale constraint $t \geq 4$, we derive the necessary root sum constraint $s_j \geq 6$. Through mathematical induction, we prove that no integer solution exists for any system satisfying these conditions.

So $\nexists k$.

4.Conclusion

∵ no such system of equations exists, ∴ equation B does not hold.

∵ equation(B) does not hold, ∴ equation (A) also does not hold.

That is there is no odd perfect number. **End proof**

REFERENCES

[1] Chaohao Gu, Mathematics Dictionary, Shanghai Dictionary Press, (1992)

[2] Hardy, G. H., & Wright, E. M. (2008). An Introduction to the Theory of Numbers (6th ed.). Oxford University Press.