A REMARK ON THE DISTRIBUTION OF ADDITION CHAINS

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Abstract. We prove the prime obstruction principle and the sparsity law. These two are collective assertions that there cannot be many primes in an addition chain.

1. Introduction

An addition chain of length h leading to n is a sequence of numbers $s_o = 1, s_1 = 2, \ldots, s_h = n$ where $s_i = s_k + s_s$ for $i > k \geq s \geq 0$. The number of terms (excluding the first term) in an addition chain leading to n is the length of the chain. We call an addition chain leading to n with a minimal length an *optimal* addition chain leading to n. In standard practice, we denote by $\iota(n)$ the length of an optimal addition chain that leads to n.

Example 1.1. The following is an example of an addition chain that leads to 15:

$$1, 2, 3, 5, 6, 8, 11, 14, 15$$

obtained from the sequence of additions

$$2 = 1+1, 3 = 2+1, 5 = 2+3, 6 = 3+3, 8 = 3+5, 11 = 5+6, 14 = 6+8, 15 = 14+1.$$

We remark that the same addition chain can also be obtained from the sequence of additions

$$2 = 1+1, 3 = 2+1, 5 = 2+3, 6 = 5+1, 8 = 6+2, 11 = 8+3, 14 = 11+3, 15 = 14+1.$$

The possibility to obtain an addition chain using distinct sequence of additions creates a subtle ambiguity. It suggests that knowing an addition chain leading to a fixed positive integer without specifying how the terms were obtained may be unsatisfactory, as it hides the procedure for obtaining the terms in the chain. The underlying intrinsic lack of uniqueness for this construction may be resolved by rewriting each term in the chain as the sum using the immediately previous term in the chain. However, an addition chain may not necessarily use the immediately previous term to generate the next term in the chain, so that in this case at most a term in the sum may not be a previous term in the chain.

Current research on addition chains focuses mainly on optimizing the length of an addition chain leading to a fixed positive integer n. Despite extensive work on the topic, there is no known asymptotic formula for the optimal length of an addition chain that leads to fixed positive integers n. To that effect, improving the upper

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and lower bounds for the optimal length $\iota(n)$ of an addition chain is worthwhile pursuit. Alfred Brauer (see [1]) proved the first non-trivial upper bound

$$\iota(n) < \frac{\log n}{\log 2}\left\{1+\frac{1}{\log\log n}+\frac{2\log 2}{(\log n)^{1-\log 2}}\right\}$$

for $n \geq n_o$ for some fixed $n_o > 0$. In subsequent studies [2], Paul Erdős improved this upper bound to

$$\iota(n) < \frac{\log n}{\log 2}+\frac{\log n}{\log\log n}+o\left(\frac{\log n}{\log\log n}\right)$$

and showed that this is best possible by proving the result

Theorem 1.2. *The number of addition chains $1 = a_o, a_1, \ldots, a_k$ satisfying $\frac{n}{2} \leq a_k < n$ and*

$$k < \frac{\log n}{\log 2}+(1-\epsilon)\frac{\log n}{\log\log n}$$

is less than $n^{1-\eta}$ for some $\eta := \eta(\epsilon) > 0$.

Schoenhage [3] proved the following nontrivial lower bound

$$\iota(z) \geq \frac{\log z}{\log 2}+\frac{\log(s(z))}{\log 2}-2.13$$

where $s(z)$ denotes the sum of all digits in the binary expansion of z.

These upper and lower bounds have now been significantly improved by De Koninck, Doyon and Verreault [4] by adapting in a clever manner the key ideas in the paper of Erdős and Schoenhage. In particular, letting

$$F(m,r) = \#\{2^m \leq n < 2^{m+1} \;:\; \iota(n) \leq m+r\}$$

where $r := c\frac{m}{\log m}$ with $0 < c < \log 2$, they proved the upper and lower bounds.

Theorem 1.3 (De Koninck, Doyon, Verreault)**.** *For any $\epsilon > 0$ and $0 < c < \log 2$, we have for m sufficiently large*

$$F(m, \frac{cm}{\log m}) < \exp(cm+\frac{\epsilon m\log\log m}{\log m}).$$

Theorem 1.4 (De Koninck, Doyon, Verreault)**.** *For any $\epsilon > 0$ and $0 < c < \log 2$, we have for m sufficiently large*

$$F(m, \frac{cm}{\log m}) > \exp(cm-\frac{c(1+\epsilon)m\log\log m}{\log m}).$$

One can observe that these upper bounds almost match in magnitude to answer the question of Erdős, who remarked on the possibility of obtaining an asymptotic behavior for the counting function $\iota(n)$.
However, in this paper, we focus on the distribution of the terms in an addition chain. We examine the distribution of primes that could possibly appear in an addition chain. It turns out that there cannot be many primes in an addition chain of "moderate" length no matter how carefully the chain is constructed.

2. **Preliminary results**

Definition 2.1 (Determiners and regulators)**.** Let

$$s_o = 1, s_1 = 2, \ldots, s_{k-1}, s_k = n$$

be an addition chain leading to $n \geq 2$ obtained from the sequence of additions

$$2 = 1 + 1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n.$$

We call the collection

$$\mathbb{G}_n := \{(a_i, r_i) \ : \ a_{i+1} = a_i + r_i, \ a_{i+1} = s_i, \text{for } 1 \leq i \leq k\}$$

the unique **generator** of the chain. For a fixed $1 \leq i \leq k$, we call the pair (a_i, r_i) such that $a_i + r_i = a_{i+1} = s_i$ the i^{th} unique **generator** of the chain. We call a_i the **determiner** and r_i the **regulator** of the i^{th} unique generator of the chain. We call the sequence (r_i) the regulator of the addition chain and (a_i) the determiner of the chain for all $1 \leq i \leq k$.

Example 2.2. To illustrate this definition, let us consider an addition chain leading to 15:

$$1, 2, 3, 5, 6, 8, 11, 14, 15$$

generated by the sequence of additions

$$2 = 1+1, 3 = 2+1, 5 = 2+3, 6 = 3+3, 8 = 3+5, 11 = 5+6, 14 = 6+8, 15 = 14+1.$$

We therefore rewrite this sequence of addition that uses the immediately previous term of the term it generates in the following way

$$2 = 1+1, 3 = 2+1, 5 = 2+3, 6 = 5+1, 8 = 6+2, 11 = 8+3, 14 = 11+3, 15 = 14+1.$$

The collection $\mathbb{G}_{15} = \{(1,1), (2,1), (3,2), (5,1), (6,2), (8,3), (11,3), (14,1)\}$ is referred to as the unique generator of the addition chain leading to 15 with the sequence of determiners

$$1, 2, 3, 5, 6, 8, 11, 14$$

and sequence of regulators

$$1, 1, 2, 1, 2, 3, 3, 1.$$

We observe that the regulators are part of the addition chain if it is a Brauer chain. This framework allows for the deduction of important relationships that exist between the regulators and the target of an addition chain. The following is a first observation:

Proposition 2.3. *Let $s_o = 1, s_1 = 2, \ldots, s_{k-1}, s_k = n$ be an addition chain producing $n \geq 3$ with the unique generator $\mathbb{G}_n := \{(a_i, r_i) \ : \ a_{i+1} = a_i + r_i, \ a_{i+1} = s_i, \text{for } 1 \leq i \leq k\}$ satisfying*

$$2 = 1 + 1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n.$$

Then the following relation for the regulators

$$\sum_{j=1}^{k} r_j = n - 1$$

hold.

Proof. We observe that $r_k = n - a_k$. We deduce that

$$\begin{aligned} r_k + r_{k-1} &= n - a_k + r_{k-1} \\ &= n - (a_{k-1} + r_{k-1}) + r_{k-1} \\ &= n - a_{k-1}. \end{aligned}$$

Again we obtain from the following iteration

$$\begin{aligned} r_k + r_{k-1} + r_{k-2} &= n - a_{k-1} + r_{k-2} \\ &= n - (a_{k-2} + r_{k-2}) + r_{k-2} \\ &= n - a_{k-2}. \end{aligned}$$

By iterating downwards in this manner and noting that the determiner $a_1 = 1$, the relation follows. $\square$

The identity related to regulators has undoubtedly proven to be an indispensable tool in aiding our studies on the distribution of an addition chain. We present two applications of this identity to the study of the average *gap* and the structural pattern of terms in an addition chain.

Theorem 2.4 (Gap between terms in an addition chain)**.** *Let $n \geq 2$ be fixed positive integer and let $s_o = 1, s_1 = 2, \ldots, s_{h-1}, s_h = n$ be an addition chain leading to n, with associated unique generator $\mathbb{G}_n := \{(a_i, r_i) \ : \ a_{i+1} = a_i + r_i, \ a_{i+1} = s_i, \text{for } 1 \leq i \leq k\}$ satisfying*

$$s_1 = 1 + 1, s_2 = a_2 + r_2, \ldots, s_{h-1} = a_{h-1} + r_{h-1}, s_h = a_h + r_h = n$$

then

$$\sup_{1 \leq l \leq h} (s_{l+k} - s_l) \gg k\frac{n}{h}$$

and

$$\inf_{1 \leq l \leq h} (s_{l+k} - s_l) \ll k\frac{n}{h}$$

for fixed $k \geq 1$.

Proof. Let $n \geq 2$ be a fixed positive integer and consider an addition chain leading to n
$s_o = 1, s_1 = 2, \ldots, s_{h-1}, s_h = n$, with associated unique generator $\mathbb{G}_n := \{(a_i, r_i) \ : \ a_{i+1} = a_i + r_i, \ a_{i+1} = s_i, \text{for } 1 \leq i \leq k\}$ satisfying

$$s_1 = 1 + 1, s_2 = a_2 + r_2, \ldots, s_{h-1} = a_{h-1} + r_{h-1}, s_h = a_h + r_h = n$$

and put (a_j) and (r_j) to be the sequence of determiners and regulators, respectively, in the chain. We make the following observations: $s_{h-1} = a_h = a_{h-1} + r_{h-1} = s_{h-2} + r_{h-1} = a_{h-2} + r_{h-2} + r_{h-1} = \cdots = 1 + \sum\limits_{j=1}^{h-1} r_j = n - r_h$, where we have used Proposition 2.3. Similarly, we can write $a_{h-1} = 1 + \sum\limits_{j=1}^{h-2} r_j = n - r_h - r_{h-1}$. Thus by induction, we can write $a_l = n - \sum\limits_{j=l}^{h} r_j$ for each $3 \leq l \leq h$. We observe that $s_{l+k}-s_l=a_{l+k+1}-a_{i+1}=\sum_{i=l+1}^{h}r_i-\sum_{i=l+k+1}^{h}r_i=\sum_{i=l+1}^{l+k}r_i$. It follows that $s_{l+k}-s_l\geq k\ \min\{r_i\}_{i=1+1}^{l+k}$. Using Proposition 2.3, we deduce that

$$\sup_{1\leq l\leq h}(s_{l+k}-s_l)\geq k\ \sup_{1\leq l\leq h}\min\{r_i\}_{i=1+1}^{l+k}\gg k\frac{n}{h}.$$

Similarly, we deduce that $s_{l+k}-s_l\leq k\ \max\{r_i\}_{i=l+1}^{l+k}$ and by using Proposition 2.3 we get

$$\inf_{1\leq l\leq h}(s_{l+k}-s_l)\leq k\inf_{1\leq l\leq h}\max\{r_i\}_{i=1+1}^{l+k}\ll k\frac{n}{h}$$

thereby ending the proof. □

Theorem 2.5 (The structural pattern of an addition chain)**.** *Let $n\geq 2$ be a fixed positive integer and let $s_o=1,s_1=2,\ldots,s_{h-1},s_h=n$ be an addition chain leading to n, with associated unique generator $\mathbb{G}_n:=\{(a_i,r_i)\ :\ a_{i+1}=a_i+r_i,\ a_{i+1}=s_i,\text{for } 1\leq i\leq k\}$ satisfying*

$$s_1=1+1,s_2=a_2+r_2,\ldots,s_{h-1}=a_{h-1}+r_{h-1},s_h=a_h+r_h=n$$

then

$$\sup_{1\leq l<h}(s_l)\gg l\frac{n}{h}$$

and

$$\inf_{1\leq l<h}(s_l)\ll l\frac{n}{h}.$$

Proof. Let $n\geq 2$ be a fixed positive integer and consider an addition chain $s_o=1,s_1=2,\ldots,s_{h-1},s_h=n$ leading to n, with associated unique generator $\mathbb{G}_n:=\{(a_i,r_i)\ :\ a_{i+1}=a_i+r_i,\ a_{i+1}=s_i,\text{for } 1\leq i\leq k\}$ satisfying

$$s_1=1+1,s_2=a_2+r_2,\ldots,s_{h-1}=a_{h-1}+r_{h-1},s_h=a_h+r_h=n$$

and put (a_j) and (r_j) to be the sequence of determiners and regulators, respectively, in the chain. We make the following observations: $s_{h-1}=a_h=a_{h-1}+r_{h-1}=s_{h-2}+r_{h-1}=a_{h-2}+r_{h-2}+r_{h-1}=\cdots=1+\sum_{j=1}^{h-1}r_j=n-r_h$, where we have used Proposition 2.3. Similarly, we can write $a_{h-1}=1+\sum_{j=1}^{h-2}r_j=n-r_h-r_{h-1}$. Thus by induction, we can write $a_l=n-\sum_{j=l}^{h}r_j$ for each $3\leq l\leq h$. Again, we observe that $a_l=n-\sum_{j=l}^{h}r_j=n-(\sum_{j=1}^{h}r_j-\sum_{j=1}^{l-1}r_j)$ so that by applying Proposition 2.3, we obtain $a_l=1+\sum_{j=1}^{l-1}r_j$. We deduce the inequality $1+(l-1)\min\{r_j\}_{j=1}^{l-1}\leq a_l\leq 1+(l-1)\max\{r_j\}_{j=1}^{l-1}$. We obtain

$$\sup_{2\leq l\leq h}(a_l)\geq\sup_{2\leq l\leq h}(1+(l-1)\min\{r_j\}_{j=1}^{l-1})\geq 1+(l-1)\sup_{2\leq l\leq h}\min\{r_j\}_{j=1}^{l-1}$$

so that by an application of Proposition 2.3, we deduce that $\sup\limits_{2\leq l\leq h}(a_l) \gg l\frac{n}{h}$. Similarly, we obtain

$$\inf_{2\leq l\leq h}(a_l) \leq \inf_{2\leq l\leq h}(1+(l-1)\max\{r_j\}_{j=1}^{l-1}) \leq 1+(l-1)\inf_{2\leq l\leq h}\max\{r_j\}_{j=1}^{l-1}$$

so that by an application of Proposition 2.3, we deduce that $\inf\limits_{2\leq l\leq h}(a_l) \ll l\frac{n}{h}$. By using the relation $s_{l-1}=a_l$, the claimed bounds are immediate consequences. $\square$

3. Main results

We begin this section with the following intermediate observations.

Lemma 3.1. *Let $n\geq 2$ be a fixed positive integer, and let $s_o=1, s_1=2, \ldots, s_h=n$ be a sequence of positive integers such that $s_j \approx j\frac{n}{h}$ for $j\geq 1$. Then the following estimates*

(i)
$$\sum_{j=1}^{h} s_j = \frac{nh}{2}+O(n)$$

(ii)
$$\sum_{j=1}^{h}\log s_j = h\log n - h + O(\log h)$$

(iii)
$$\sum_{j=1}^{h}\frac{1}{s_j} = \frac{h}{n}\log h + O(\frac{h}{n})$$

(iv)
$$\sum_{j=1}^{h}\frac{1}{s_j^k} \ll \left(\frac{h}{n}\right)^k$$

for fixed $k>1$

(v)
$$\sum_{j=1}^{h}\frac{\log s_j}{s_j} = \frac{h}{n}(\log n)(\log h) - \frac{h}{2n}(\log h)^2 + O\left(\frac{h}{n}\log n\right)$$

(vi)
$$\prod_{j=1}^{h}(1-\frac{1}{s_j}) = \frac{1}{h^{\frac{h}{n}}}\left(1+O(\frac{h}{n})\right)$$

(vii)
$$\prod_{j=1}^{h}(1+\frac{1}{s_j}) = h^{\frac{h}{n}}\left(1+O(\frac{h}{n})\right)$$

hold.

Proof. For (i), we can write

$$\sum_{j=1}^{h} s_j \approx \sum_{j=1}^{h} j\frac{n}{h} = \frac{n}{2}h+\frac{n}{2} = \frac{n}{2}h+O(n).$$

Similarly for (ii), we have

$$\sum_{j=1}^{h}\log s_j \approx \sum_{j=1}^{h}\log(j\frac{n}{h}) = \sum_{j=1}^{h}\log j + h\log n - h\log h$$

and we deduce that

$$\sum_{j=1}^{h}\log s_j = h\log n - h + O(\log h).$$

Similarly, we deduce that

$$\sum_{j=1}^{h}\frac{1}{s_j} \approx \frac{h}{n}\sum_{j=1}^{h}\frac{1}{j} = \frac{h}{n}(\log h + O(1))$$

and the claimed estimate is an immediate consequence. To establish the estimate for the higher moment harmonic sum in (iv), we observe that under the assumption, we have for a fixed $k > 1$ the upper control

$$\sum_{j=1}^{h}\frac{1}{s_j^k} \approx \left(\frac{h}{n}\right)^k \sum_{j=1}^{h}\frac{1}{j^k} \ll \left(\frac{h}{n}\right)^k$$

thereby establishing (iv). In a routine manner, under the assumption, we can write

$$\sum_{j=1}^{h}\frac{\log s_j}{s_j} \approx \sum_{j=1}^{h}\frac{\log(j\frac{n}{h})}{j\frac{n}{h}} = \frac{h}{n}\sum_{j=1}^{h}\frac{\log(j\frac{n}{h})}{j}$$

so that by unpacking the terms in the sum and using known elementary estimates, we deduce that

$$\sum_{j=1}^{h}\frac{\log s_j}{s_j} = \frac{h}{n}(\log n)(\log h) - \frac{h}{2n}(\log h)^2 + O(\frac{h}{n}\log n)$$

which establishes (v). For (vi), we can write

$$\prod_{j=1}^{h}\left(1-\frac{1}{s_j}\right) = e^{\sum\limits_{j=1}^{h}\log(1-\frac{1}{s_j})} = e^{-\sum\limits_{j=1}^{h}\frac{1}{s_j}+O(\sum\limits_{j=1}^{h}\frac{1}{s_j^2})}$$

so that by using (iii) and (iv), we find that

$$\prod_{j=1}^{h}\left(1-\frac{1}{s_j}\right) = e^{-\frac{h}{n}\log h + O(\frac{h}{n}) + O((\frac{h}{n})^2)} = \frac{1}{h^{\frac{h}{n}}}\left(1+O(\frac{h}{n})\right)$$

which establishes (vi). The estimate (vii) is deduced in a similar way. □

Lemma 3.2 (Local linear control). *For each term s_j in an addition chain leading to n of length h, we have*

$$s_j \leq \frac{n-1}{h}(j+1)$$

for all $0 \leq j \leq h-1$.

Proof. Let s_j $(0 \leq j \leq h)$ be a term indexed by the j^{th} step in an addition chain leading to n. We note that in an addition chain $s_j \leq 2^j$ holds for all $0 \leq j \leq h$. It follows that $\frac{s_j h}{j+1} \leq \frac{2^j h}{j+1}$. Now, put

$$f(j) := \frac{2^j h}{j+1}.$$

Since $\frac{f(j+1)}{f(j)} = 2(\frac{j+1}{j+2}) \geq 1$ for each $j \in [0, h-1]$, it follows that $f(j)$ is monotone increasing on $[0, h-1]$ and therefore must attain a maximum at $j = h-1$. This implies that

$$\max_{0 \leq j \leq h-1} f(j) = 2^{h-1} = \max(s_{h-1}).$$

Consequently, we have

$$\frac{s_j h}{j+1} \leq \max_{0 \leq j \leq h-1} f(j) = 2^{h-1} = \max(s_{h-1}) \leq n-1$$

which proves the inequality stated. □

We now show that one cannot introduce "many" primes in an addition chain of "moderate" length no matter how meticulous we tend to be.

Theorem 3.3 (The prime obstruction principle). *There exists no addition chain leading to n of the form $\mathbb{E}(n) : s_o = 1, s_1 = 2, \ldots, s_h = n$ with $h \ll n^{1-\epsilon}$ for any small $\epsilon > 0$ that contains all primes less than or equal to n.*

Proof. Suppose that there exists an addition chain leading to n of the form $\mathbb{E}(n)$: $s_o = 1, s_1 = 2, \ldots, s_h = n$ with $h \ll n^{1-\epsilon}$ for some small $\epsilon > 0$ that contains all primes less than or equal to n. Then, we have the lower bound

$$\sum_{j=1}^{h} \log s_j = \sum_{\substack{j=1 \\ s_j \in \mathbb{P}}}^{h} \log s_j + \sum_{\substack{j=1 \\ s_j \notin \mathbb{P}}}^{h} \log s_j \geq \sum_{\substack{j=1 \\ s_j \in \mathbb{P}}}^{h} \log s_j = \sum_{p \leq n} \log p \sim n$$

by an application of the prime number theorem. On the other hand, with $h \ll n^{1-\epsilon}$ for a small $\epsilon > 0$, we have by Lemma 3.1 and Lemma 3.2 the upper bound

$$\sum_{j=1}^{h} \log s_j \leq \sum_{j=1}^{h} \log(j\frac{n}{h}) = h \log n - h + O(\log h) \ll n^{1-\epsilon} \log n$$

which is smaller than the lower bound. This shows that this addition cannot contain all primes less than or equal to n for all n sufficiently large. □

We can at least realistically quantify the worst growth rate of the number of primes that could possibly appear in an addition chain when we work with a 'dreamed' addition chain, an addition chain where the terms are uniformly distributed. That is to say, we assume an addition chain that could possibly contain many primes of the form

$$E(n) : s_o = 1, s_1 = 2, \ldots, s_h = n$$

with the property that $s_j \approx j\frac{n}{h}$. This is the 'finest' property that could possibly exist in an addition chain. An upper bound for the number of prime numbers that could possibly appear in this type of addition would essentially give an upper control of the number of primes in all addition chains leading to n.

Theorem 3.4 (The prime sparsity law)**.** *Let n be a fixed positive integer, and let $\mathbb{E}(n) : s_o = 1, s_1 = 2, \dots s_h = n$ be an addition chain leading to n that avoids terms of the form 2^j $(j \geq 1)$ and such that $s_j \approx j\frac{n}{h}$. If $h \ll \frac{n}{\log n}$, then*

$$\lim_{n \longrightarrow \infty} \frac{|\mathbb{P} \cap \mathbb{E}(n)|}{|\mathbb{P} \cap [1, n]|} = 0$$

which means that there cannot be many primes in an addition chain constructed in this manner.

Proof. Let us suppose that

$$\lim_{n \longrightarrow \infty} \frac{|\mathbb{P} \cap \mathbb{E}(n)|}{|\mathbb{P} \cap [1, n]|} > 0$$

then there exists an absolute constant $c > 0$ such that $|\mathbb{P} \cap \mathbb{E}(n)| \sim c|\mathbb{P} \cap [1, n]|$. Now, we can write

$$\sum_{j=1}^{h} \frac{1}{s_j} = \sum_{\substack{j=1 \\ s_j \in \mathbb{P}}}^{h} \frac{1}{s_j} + \sum_{\substack{j=1 \\ s_j \notin \mathbb{P}}}^{h} \frac{1}{s_j} > \sum_{\substack{j=1 \\ s_j \in \mathbb{P}}}^{h} \frac{1}{s_j} \sim c \sum_{p \leq n} \frac{1}{p} \sim c \log \log n.$$

On the other hand, using the requirement of the construction, we obtain in relation to Theorem 3.1 and the bound $h \ll \frac{n}{\log n}$ the upper bound

$$\sum_{j=1}^{h} \frac{1}{s_j} = \frac{h}{n} \log h + O(\frac{h}{n}) \ll 1$$

which cannot be possible. This completes the proof of the claim. □

4. **Applications to related sequences**

In this section, we apply the properties of addition chains to sequences such as the Fibonacci sequences and the Ulam sequences. This section elucidates the subtle connection to these infinite sequence of positive integers.

It is known that the sequence $\{F_n\}_{n \geq 1}$ of Fibonacci numbers has zero density, which can be proven using the Binet formula for Fibonacci numbers. In this paper, we give an alternative proof of this fact using the estimate (iii) in Theorem 3.1 and the clear observation that a Fibonacci sequence up to a fixed Fibonacci number F_{n_o} may be regarded as an addition chain leading to F_{n_o} when we exclude the first term in the sequence.

Theorem 4.1. *The sequence $\{F_n\}_{n \geq 1}$ of Fibonacci numbers have zero density.*

Proof. We note that the sequence $\{F_n\}_{n \geq 1}$ of Fibonacci numbers up to a fixed Fibonacci number F_{n_o} is an addition chain leading to F_{n_o}, when we exclude the first term of the sequence. Let h $(h = n_o - 2)$ denote the length of the addition chain leading to F_{n_o}, as the number of Fibonacci numbers no more than F_{n_o} and excluding the first and second term, then the inequality

$$\sum_{k=3}^{h} \frac{1}{F_k} \geq \frac{h}{F_{n_o}} \sum_{k=3}^{h} \frac{1}{k} = \frac{h}{F_{n_o}} \log h + O(\frac{h}{F_{n_o}}).$$

Now, let us suppose that the sequence of Fibonacci numbers have a positive density, then there exist an absolute constant $c > 0$ such that the number of Fibonacci number no more than F_{n_o} satisfies the relation $\sim cF_{n_o}$. Since $(n_o - 2) = h \sim (cF_{n_o} - 2) \sim cF_{n_o}$, we deduce from the lower bound

$$\sum_{k=3}^{h} \frac{1}{F_k} \gtrapprox c \log F_{n_o}.$$

We deduce that

$$\lim_{n_o \longrightarrow \infty} \sum_{k=3}^{h} \frac{1}{F_k} = \infty$$

which contradicts the known fact that

$$\sum_{k=1}^{\infty} \frac{1}{F_k} < \infty.$$

$\square$

Theorem 4.2. *Let $\{F_k\}_{k\geq 1}$ be the sequence of Fibonacci numbers. Then we have*

(i)

$$\prod_{k=3}^{n_o} \left(1 + \frac{1}{F_k}\right) \gg (n_o)^{\frac{n_o}{F_{n_o}}}$$

(ii)

$$\prod_{k=3}^{n_o} \left(1 - \frac{1}{F_k}\right) \ll \frac{1}{(n_o)^{\frac{n_o}{F_{n_o}}}}$$

Proof. We note that the sequence of Fibonacci numbers up to a fixed Fibonacci number F_{n_o} is an addition chain leading to F_{n_o}, when we exclude the first term of the sequence. For (i), we have the lower bound

$$\prod_{k=3}^{n_o} \left(1 + \frac{1}{F_k}\right) \geq \prod_{k=3}^{n_o} \left(1 + \frac{1}{k^{\frac{F_{n_o}}{h}}}\right) = h^{\frac{h}{F_{n_o}}} \left(1 + O(\frac{h}{F_{n_o}})\right)$$

where h is the length of the addition chain $\{F_k\}_{k=2}^{n_o}$ which leads to F_{n_o}. Using the observation that $h = (n_o - 2) \sim n_o$, we deduce that

$$\prod_{k=3}^{n_o} \left(1 + \frac{1}{F_k}\right) \gg (n_o)^{\frac{n_o}{F_{n_o}}}.$$

For (ii), we have the upper bound

$$\prod_{k=3}^{n_o} \left(1 - \frac{1}{F_k}\right) \leq \prod_{k=3}^{n_o} \left(1 - \frac{1}{k^{\frac{F_{n_o}}{h}}}\right) = \frac{1}{h^{\frac{h}{F_{n_o}}}} \left(1 + O(\frac{h}{F_{n_o}})\right)$$

where h is the length of the addition chain $\{F_k\}_{k=2}^{n_o}$ which leads to F_{n_o}. Using the observation that $h = (n_o - 2) \sim n_o$, we deduce that

$$\prod_{k=3}^{n_o} \left(1 - \frac{1}{F_k}\right) \ll \frac{1}{(n_o)^{\frac{n_o}{F_{n_o}}}}.$$

$\square$

The Ulam number sequence $\{U_n\}_{n\geq 1}$ is another class of positive integer sequences that shares similar properties as the Fibonacci sequence. Various properties of Ulam sequences are now known, but it remains an open problem to determine their distribution. In particular, it is still not known whether these sequences have positive or zero natural density, although a known numerical computation suggests that the natural density is approximately 0.07398. We now show that, under a certain regularity condition, the sequence of Ulam numbers must have a zero density.

Theorem 4.3 (The Ulam density criterion)**.** *The sequence of Ulam numbers $\{U_n\}_{n\geq 1}$ have a zero density provided $\frac{U_n}{n} \longrightarrow \infty$ as $n \longrightarrow \infty$.*

Proof. We note that the sequence $\{U_n\}_{n\geq 1}$ of Ulam numbers up to an Ulam number U_{n_o} is an addition chain that leads to U_{n_o}. Let h $(h = n_o - 1)$ denote the length of the chain leading to U_{n_o}. We have

$$\sum_{j=1}^{h} \log U_j \leq \sum_{j=1}^{h} \log(j\frac{U_{n_o}}{h}) = h \log U_{n_o} - h + O(\log h) \ll n_o \log U_{n_o}.$$

Let us suppose on the contrary that Ulam numbers have a positive density, then there exist an absolute constant $c > 0$ such that we can write

$$\sum_{j=1}^{h} \log U_j \sim c \sum_{k=1}^{U_{n_o}} \log k \sim cU_{n_o} \log U_{n_o}.$$

This implies that $\frac{U_{n_o}}{n_o} \ll 1$. This violates the requirement $\frac{U_n}{n} \longrightarrow \infty$ as $n \longrightarrow \infty$ when we take $n_o \longrightarrow \infty$. □

We show that a *zero density* of the Ulam number sequence $\{U_m\}_{m\geq 1}$ is equivalent to the unboundedness of the gap between consecutive terms in the sequence.

Theorem 4.4 (The second Ulam density criterion)**.** *Let $\{U_m\}_{m\geq 1}$ denote the sequence of all Ulam numbers. Then the following assertions hold:*

(i) *If $\lim_{m\longrightarrow\infty} \inf(U_{m+1} - U_m) = \infty$, then the sequence $\{U_m\}_{m\geq 1}$ have zero density.*
(ii) *If the sequence $\{U_m\}_{m\geq 1}$ has zero density, then $\lim_{m\longrightarrow\infty} \sup(U_{m+1} - U_m) = \infty$.*

Proof. Fix an Ulam number U_{n_o}. Then the sequence of Ulam numbers up to U_{n_o} is an addition chain leading to U_{n_o}. We let h $(h = n_o - 1)$ denote the length of the addition chain. By Theorem 2.5, we have

$$\inf_{1\leq m\leq h-1} (U_{m+1} - U_m) \ll \frac{U_{n_o}}{h}.$$

If we assume to the contrary that the sequence of Ulam numbers $\{U_m\}_{m\geq 1}$ have a positive density, then there exist an absolute constant $c > 0$ such that $h \sim cU_{n_o}$. We therefore have

$$\inf_{1\leq m\leq h-1} (U_{m+1} - U_m) \ll \frac{U_{n_o}}{h} \sim \frac{1}{c} \ll 1.$$

By taking limits as $m \longrightarrow \infty$, we deduce that

$$\lim_{m\longrightarrow\infty} \inf(U_{m+1} - U_m) \ll 1$$

which violates the requirements.
For the second part, we can write (by Theorem 2.5)

$$\sup_{1\leq m\leq h-1}(U_{m+1}-U_m)\gg \frac{U_{n_o}}{h}.$$

Let us suppose that the sequence of Ulam numbers $\{U_m\}_{m\geq 1}$ have zero density, then there exist a fixed small $\epsilon>0$ such that $h\ll U_{n_o}^{1-\epsilon}$. Consequently, we have

$$\sup_{1\leq m\leq h-1}(U_{m+1}-U_m)\gg \frac{U_{n_o}}{h}\gg U_{n_o}^{\epsilon}.$$

By taking limits as $m\longrightarrow\infty$, we deduce that

$$\lim_{m\longrightarrow\infty}\sup(U_{m+1}-U_m)=\infty$$

which shows the equivalence. □

5. **A conjecture on primes in an addition chains**

Although it has been shown that there cannot be "many" primes in an addition chain of "moderate" length, we have barely provided a criterion for counting primes if we allow our construction to include primes. This seems to be a very difficult problem, given the inherent irregular nature of the primes. At the moment, we make the following conjecture, which specifies the best way to include a few primes in an addition chain.

Conjecture 5.1 (Addition chain local prime distribution)**.** *Let $n\geq 3$ and let*

$$1=s_o<2=s_1<\cdots<s_h=n$$

be an addition chain leading to n and satisfying the regularity conditions

- ***Small step closure:***

$$|s_i-s_{i-1}|<s_{i-1}$$

  for all $2\leq i\leq h$.

- ***Interpolation-closeness:***

$$|s_i-i\frac{n}{h}|<s_i$$

  for all $\lfloor\frac{h}{2}\rfloor+1\leq i\leq h$.

- ***Relative shift uniformity:*** *There exists $0<\epsilon<1$ such that for all $\lfloor\frac{h}{2}\rfloor+1\leq i\leq h$ we have*

$$|s_i-i\frac{n}{h}|<\epsilon s_i.$$

  Denote by $\mathcal{P}_h(n)$ the number of primes among the chain $\{s_o,s_1,\ldots,s_h\}$, then as $n\longrightarrow\infty$, we have

$$\mathcal{P}_h(n)\geq \frac{h}{2\log(\frac{h}{2})}(1+o(1)).$$

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