ON A SPECIFIC FAMILY OF ORTHOGONAL POLYNOMIALS OF BERNSTEIN-SZEGÖ TYPE

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Abstract. We study certain weight functions on $[-1,1]$ which are particular cases of the general weights considered by Bernstein and Szegö. These weights are numbered by two positive integers and when these integers tend to infinity, these weights approximate weight functions on $\mathbb{R}$ considered by Ismail and Valent. We also consider modifications of these weight functions by a continuous variable $a>0$. These ideas are then used to find finite analogs of some improper integrals first studied by Glaisher and Ramanujan.

1. Introduction

Entry 4.123.6 in Gradshteyn and Ryzhik's *Table of Integrals, Series, and Products* [9] reads

$$\int\limits_0^\infty \frac{\sin(ax)\sinh(bx)}{\cos(2ax)+\cosh(2bx)}\, x^{p-1}dx=\frac{\Gamma(p)}{(a^2+b^2)^{p/2}}\,\sin\left(p\tan^{-1}\frac{a}{b}\right)\sum_{k=0}^\infty\frac{(-1)^k}{(2k+1)^p},\quad p>0. \tag{1}$$

Limiting case $p\to+0$ of (1) is

$$\int\limits_0^\infty \frac{\sin(x)\sinh(x/a)}{\cos(2x)+\cosh(2x/a)}\,\frac{dx}{x}=\frac{\tan^{-1}a}{2}. \tag{2}$$

Also, when $a=b$ and $p/4\in\mathbb{N}$ one finds from (1)

$$\int\limits_0^\infty \frac{\sin(x)\sinh(x)}{\cos(2x)+\cosh(2x)}\, x^{4k-1}dx=0,\quad k\in\mathbb{N}.$$

Integrals of this type were first studied by Glaisher [7]. Recently, they were studied in connection with integrals of Dedekind eta-function in [8],[14],[6].

In [12], we have generalized (2) as

$$\int\limits_0^1 \frac{\sin\big(n\sin^{-1}t\big)\sinh\big(n\sinh^{-1}(t/a)\big)}{\cos\big(2n\sin^{-1}t\big)+\cosh\big(2n\sinh^{-1}(t/a)\big)}\,\frac{dt}{t\sqrt{(1-t^2)(1+t^2/a^2)}}=\frac{\tan^{-1}a}{2}, \tag{3}$$

where n is a positive odd integer , and we have also shown that

$$\int\limits_0^1 \frac{\cos(n\sin^{-1}t)\cosh(n\sinh^{-1}t)}{\cos(2n\sin^{-1}t)+\cosh(2n\sinh^{-1}t)}\,\frac{dt}{\sqrt{1-t^4}}=0, \tag{4}$$

where n is a positive even integer. Our proof was based on explicit calculations using the fact, that the roots of $\cos\big(2n\sin^{-1}t\big)+\cosh\big(2n\sinh^{-1}(t/a)\big)$ (which is a polynomial in t) can be determined in closed form (see also Section 9 of the present paper for a similar calculation). Two alternative proofs of (3) can be found in [20] and [16]. Motivation for considering such integrals came from the mapping $\alpha_z=2n\sinh^{-1}\sin\frac{\pi z}{2n}$ encountered in the theory of Dirichlet problem on finite nets [15], as discussed in Section 5 of [12]. Note that $\alpha_z\sim\pi z$, when $n\to\infty$.

In this paper, we will be concerned with integration formulas similar to the following:

Theorem 1. *Let n and m be positive odd integers. Then*

$$\int\limits_{-1}^1 \frac{\sin(n\sin^{-1}\sqrt{t})\sinh(m\sinh^{-1}\sqrt{t})}{\cos(2n\sin^{-1}\sqrt{t})+\cosh(2m\sinh^{-1}\sqrt{t})}\,\frac{t^j\,dt}{\sqrt{1-t^2}}=\begin{cases}\pi/2, & j=-1,\\ 0, & j=0,1,\dots,\frac{m+n-2}{2},\end{cases} \tag{5}$$

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$$\int_{-1}^{1} \frac{\cos(n\sin^{-1}\sqrt{t})\cosh(m\sinh^{-1}\sqrt{t})}{\cos(2n\sin^{-1}\sqrt{t})+\cosh(2m\sinh^{-1}\sqrt{t})}\, t^j\, dt = 0, \quad j=0,1,\ldots,\tfrac{m+n-4}{2}. \tag{6}$$

Our proof of Theorem 1 is based on the theory of orthogonal polynomials in its elementary form. In particular, we will use the concept of so-called Bernstein-Szegö polynomials, brief account of which is given in Section 2. Modifications of Theorem 1 by a continuous parameter $a>0$ will be studied in Section 5, Theorem 4. The case of even parameters n and m is treated in Section 6. More complicated weight functions are studied in Section 7. Resulting Gauss quadrature formulas are given in Section 8, where we also discuss a finite analog of the generating function formula of Kuznetsov from [11].

In a series of papers [1],[10], Berg, Valent, and Ismail have considered orthogonal polynomials on $\mathbb{R}$ related to elliptic functions. Weight function for these orthogonal polynomails is equivalent to

$$\frac{1}{\cos\left(2\sqrt{x}\right)+\cosh\left(2\sqrt{x/a}\right)} \tag{7}$$

after rescaling of the variable. Recently, there has been a flurry of activity in studying different aspects of the integrals with weight function (7), e.g. [2],[11],[21],[3],[13],[18],[22]. The weight functions

$$\frac{1}{\cos(2n\sin^{-1}\sqrt{t})+\cosh(2m\sinh^{-1}\sqrt{t})}\,\frac{1}{\sqrt{1-t^2}}, \tag{8}$$

numbered by two positive integers n and m, approximate the weight functions (7) in the limit $n,m\to\infty$. For example, (5) is a finite analog of

$$\int_{\mathbb{R}} \frac{\sin\left(\sqrt{x}\right)\sinh\left(\sqrt{x/a}\right)}{\cos\left(2\sqrt{x}\right)+\cosh\left(2\sqrt{x/a}\right)}\, x^j\, dx = \begin{cases}\pi/2, & j=-1,\\ 0, & j=0,1,2,\ldots\end{cases}$$

As can be seen from (5), $\sin(n\sin^{-1}\sqrt{t})\sinh(m\sinh^{-1}\sqrt{t})$ is an $(m+n)/2$-th degree orthogonal polynomial corresponding to the weight function (8). Thus, our paper provides an elementary setting for the orthogonal polynomials studied in [1],[10].

There is an integral looking similar to (2) but somewhat different

$$\int_{0}^{\infty} \frac{\sin(kx)}{\cos(x)+\cosh(x)}\,\frac{dx}{x} = \frac{\pi}{4}, \tag{9}$$

where k is a positive odd integer. It was submitted by Ramanujan to the Journal of the Indian Mathematical Society as problem number 353 [17]. More information on the history of (9), and also on the Ismail and Valent integral, including further references can be found in [2]. [2] also contains direct proof of (9) using contour integration. In Section 9, we will prove a finite analog of (9) that contains an additional integer parameter. When this integer parameter goes to infinity, one recovers (9) in the limit.

2. General Bernstein-Szego polynomials

In this section, we closely follow the book [19]. The trigonometric polynomial in θ of degree k is

$$g(\theta) = a_0 + \sum_{j=1}^{k}\left\{a_j\cos(j\theta)+b_j\sin(j\theta)\right\}.$$

Theorem 2 ([19], Theorem 1.2.2)**.** *Let $g(\theta)$ be a trigonometric polynomial with real coefficients which is nonnegative for all real values of θ and $g(\theta)\not\equiv 0$. Then a representation $g(\theta)=|h(e^{i\theta})|^2$ exists such that $h(z)$ is a polynomial of the same degree as $g(\theta)$, with $h(z)\neq 0$ in $|z|<1$, and $h(0)>0$. This polynomial is uniquely determined. If $g(\theta)$ is a cosine polynomial, $h(z)$ is a polynomial with real coefficients.*

Let $\rho(t)$ be a polynomial of precise degree l and positive in $[-1,1]$. Then, orthonormal polynomials $p_k(t)$, which are associated with weight functions

$$w(t) = \frac{1}{\rho(t)\sqrt{1-t^2}}$$

can be calculated explicitly provided $l < 2k$. Namely, let $\rho(\cos\theta) = |h(e^{i\theta})|^2$ be the normalized representation of $\rho(\cos\theta)$ in the sense of Theorem 2. Then, writing $h(e^{i\theta}) = c(\theta) + is(\theta)$, $c(\theta)$ and $s(\theta)$ real, we have

$$p_k(t) = \sqrt{\frac{2}{\pi}}\,\mathrm{Re}\left\{e^{ik\theta}\overline{h(e^{i\theta})}\right\} = \sqrt{\frac{2}{\pi}}\left\{c(\theta)\cos k\theta + s(\theta)\sin k\theta\right\}. \tag{10}$$

These formulas must be modified for $l = 2k$ by multiplying the right-hand member of (10) by a certain constant factor. However, we will only consider $l < 2k$.

3. Some properties of the function $\cos(2n\sin^{-1}\sqrt{t}) + \cosh(2m\sinh^{-1}\sqrt{t})$

The integral $I(n,m,j)$ on the left-hand side of (5) satisfies

$$I(n,m,j) = (-1)^{j-1}\,I(m,n,j).$$

This symmetry means that it is enough to consider $m \geq n$. The expression

$$\rho(t) = \cos(2n\sin^{-1}\sqrt{t}) + \cosh(2m\sinh^{-1}\sqrt{t}) \tag{11}$$

is an even polynomial in t of degree

$$\deg\rho = \begin{cases} m, & m > n,\\ 2\lfloor \frac{m}{2}\rfloor, & m = n.\end{cases}$$

One can write for $t \in [-1,1]$

$$\rho(t) = \left|\sqrt{2}\,\cos\bigl(n\sin^{-1}\sqrt{t} - im\sinh^{-1}\sqrt{t}\bigr)\right|^2.$$

Using logarithmic form of the functions $\sin^{-1}$, $\sinh^{-1}$, we obtain the representation

$$\sqrt{2}\,\cos\bigl(n\sin^{-1}\sqrt{\cos\theta} - im\sinh^{-1}\sqrt{\cos\theta}\bigr) = i^n e^{-i(n+m)\theta/2}\,h\bigl(e^{i\theta}\bigr),$$

$$h(z) = \frac{1}{\sqrt{2}}\Biggl\{\left(\sqrt{z^2+1}+1\right)^{\frac{m+n}{2}}\left(\sqrt{z^2+1}+z\right)^{\frac{m-n}{2}} + (-1)^n\left(\sqrt{z^2+1}-1\right)^{\frac{m+n}{2}}\left(\sqrt{z^2+1}-z\right)^{\frac{m-n}{2}}\Biggr\}. \tag{12}$$

Obviously, $h(z)$ is a polynomial in z.

Lemma 3. (i) $\deg h = \deg\rho$.
(ii) $h(0) > 0$.
(iii) $h(z) \neq 0$ *in* $|z| < 1$.

Proof. Parts (i) and (ii) are obvious. Part (iii) can be proved by Rouché's theorem as follows. Let

$$f(z) = (-1)^n\left(1+\sqrt{z^2+1}\right)^{m+n}\left(z+\sqrt{z^2+1}\right)^{m-n}/z^{m+n}.$$

Unlike $h(z)$, $f(z)$ is a multivalued function. We choose brunch cuts on $[i,+i\infty)$ and $[-i,-i\infty)$. The roots of $h(z)$ coincide with the roots of equation $f(z)-1=0$. Consider the contour C composed of 4 arcs: two arcs of radius 1 centered at the origin, and two arcs of small radius ε centered at $\pm i$. We will show that $|f(z)| > 1$ on C. Since $f(z)$ does not have zeroes inside the unit circle, according to Rouché's theorem it will follow that $f(z)-1$ does not vanish inside the unit circle.

One can easily show that $|f(z)| > 1$ when $|z| = 1$, with the exception of two points $\pm i$. The arc around $+i$ can be parametrized as

$$z = i + \varepsilon e^{-2i\varphi}, \quad \varphi \in (0,\pi/2).$$

Since

$$\left|f(i+\varepsilon e^{-2i\varphi})\right| = 1 + 2\sqrt{\varepsilon}\,(m\cos\varphi + n\sin\varphi) + O(\varepsilon), \quad \varphi\in(0,\pi/2).$$

and $m\cos\varphi + n\sin\varphi$ is strictly positive when $\varphi\in(0,\pi/2)$ and m, n positive, we deduce that $|f(z)| > 1$ on the arc around $+i$. The arc around $-i$ is dealt with in the same manner. □

Hence, $\rho(\cos\theta) = |h(e^{i\theta})|^2$ is the normalized representation of $\rho(\cos\theta)$ in the sense of Theorem 2. The resulting formulas are simpler when m and n have the same parity: when they are both odd (the next Section), or both even, Section 6.

4. Proof of Theorem 1

Defining the functions

$$\xi = \cos(n\sin^{-1}\sqrt{t})\cosh(m\sinh^{-1}\sqrt{t}), \tag{13a}$$

$$\eta = \sin(n\sin^{-1}\sqrt{t})\sinh(m\sinh^{-1}\sqrt{t}), \tag{13b}$$

we find from (12) taking into account that m and n are odd

$$\sqrt{2}(i\xi+\eta) = e^{i(n+m)\theta/2}\overline{h(e^{i\theta})}. \tag{14}$$

Using (10) we find two orthogonal polynomials

$$p_{\frac{m+n}{2}}(t) = 2\pi^{-1/2}\eta = 2\pi^{-1/2}\,\sin(n\sin^{-1}\sqrt{t})\sinh(m\sinh^{-1}\sqrt{t}), \tag{15a}$$

$$p_{\frac{m+n}{2}+1}(t) = 2\pi^{-1/2}\big(\eta\, t - \xi\sqrt{1-t^2}\big). \tag{15b}$$

This settles $j \geq 0$ in both equations (5), (6). To deal with $j=-1$ in (5) we will need the kernel polynomials ([4], Chapter I, eq. 4.11) defined as

$$K_k(t,u) = \sum_{j=0}^{k} p_j(t)p_j(u) = \frac{\varkappa_k}{\varkappa_{k+1}}\,\frac{p_{k+1}(t)p_k(u)-p_k(t)p_{k+1}(u)}{t-u}, \tag{16}$$

where $\varkappa_j$ is the leading coefficient of $p_j(t)$. Since $p_{\frac{m+n}{2}}(0)=0$, this simplifies to

$$K_{\frac{m+n}{2}}(t,0) = -\varkappa_{\frac{m+n}{2}}\varkappa^{-1}_{\frac{m+n}{2}+1}\,p_{\frac{m+n}{2}+1}(0)\,p_{\frac{m+n}{2}}(t)/t$$

From (13) and (15), one can work out the values of the constants in the last formula

$$\varkappa_{\frac{m+n}{2}} = \pi^{-1/2}(-1)^{(n-1)/2}\,2^{m+n-1}, \qquad \varkappa_{\frac{m+n}{2}+1} = 2\varkappa_{\frac{m+n}{2}}, \qquad p_{\frac{m+n}{2}+1}(0) = -2\pi^{-1/2}. \tag{17}$$

To finish the proof, we use the reproducing property of the kernel polynomials with $k=(m+n)/2$

$$\int_{-1}^{1} K_k(t,0)\,\frac{dt}{\rho(t)\sqrt{1-t^2}} = 1.$$

5. Generalization that includes an additional continuous parameter $a>0$

Theorem 1 can be generalized. Let $t\in[-a,1]$. It is known that the substitution

$$t = \frac{1}{2}\left\{1-a-(1+a)\cos\theta\right\}, \quad \theta\in[0,\pi]$$

uniformizes the square root expression

$$\sqrt{(1-t)(a+t)} = \frac{1}{2}(a+1)\sin\theta.$$

After some tedious but quite straightforward algebra we obtain the representation

$$\sqrt{2}\,\cos\big(n\sin^{-1}\sqrt{t} - im\sinh^{-1}\sqrt{t/a}\big) = i^n\,e^{-i(n+m)\theta/2}\left(\frac{a+1}{2\sqrt{a}}\right)^m h(e^{i\theta}),$$

$$h(z) = \frac{1}{\sqrt{2}}\Bigg\{\left(\sqrt{z^2+2bz+1}+1+bz\right)^{\frac{m+n}{2}}\left(\sqrt{z^2+2bz+1}+b+z\right)^{\frac{m-n}{2}} + (-1)^n\left(\sqrt{z^2+2bz+1}-1-bz\right)^{\frac{m+n}{2}}\left(\sqrt{z^2+2bz+1}-b-z\right)^{\frac{m-n}{2}}\Bigg\},$$

where we have denoted $b=\frac{1-a}{1+a}$. This is a generalization of the representation (12). One can show that this representation also satisfies all three conditions of Lemma 3.

Theorem 4 (1*). *Let n and m be positive odd integers and $a>0$. Then*

$$\int_{-a}^{1} \frac{\sin\bigl(n\sin^{-1}\sqrt{t}\bigr)\sinh\bigl(m\sinh^{-1}\sqrt{t/a}\bigr)}{\cos\bigl(2n\sin^{-1}\sqrt{t}\bigr)+\cosh\bigl(2m\sinh^{-1}\sqrt{t/a}\bigr)}\frac{t^j\,dt}{\sqrt{(1-t)(1+t/a)}} = \begin{cases}\pi/2, & j=-1,\\ 0, & j=0,1,\dots,\frac{m+n-2}{2};\end{cases} \tag{18}$$

$$\int_{-a}^{1} \frac{\cos(n\sin^{-1}\sqrt{t})\cosh(m\sinh^{-1}\sqrt{t/a})}{\cos(2n\sin^{-1}\sqrt{t})+\cosh(2m\sinh^{-1}\sqrt{t/a})}\,t^j\,dt = 0, \quad j=0,1,\dots,\tfrac{m+n-4}{2}. \tag{19}$$

The symmetric case $m=n$ of (18) also follows from (3) and the identity $\tan^{-1}(x)+\tan^{-1}(1/x)=\pi/2$.

Equation (18) has an additional (integer) parameter m compared to (3). However, this extra parameter comes at a cost: The integration range now covers the entire interval $[-a,1]$. There does not seem to be a closed-form evaluation of the integral in (18) when the integration range is $[0,1]$ (in other words, there do not seem to be any extensions of (3) that include an additional parameter).

6. The case of even integers m and n

Again, one can restrict consideration to $m\ge n$. Then, the degree of the polynomial $\rho(t)$ (11) is $\deg\rho=m$. Taking into account that n and m are even we get with (13a),(13b)

$$-\sqrt{2}(\xi-i\eta)=e^{i(n+m)\theta/2}\overline{h\bigl(e^{i\theta}\bigr)}.$$

where $h(z)$ is given by (12). The difference from (14) is the phase factor of -1 instead of i. Orthogonal polynomials of interest are

$$p_{\frac{m+n}{2}}(t)=2\pi^{-1/2}\,\xi=2\pi^{-1/2}\cos(n\sin^{-1}\sqrt{t})\cosh(m\sinh^{-1}\sqrt{t}),$$

$$p_{\frac{m+n}{2}+1}(t)=2\pi^{-1/2}\bigl(\xi t+\eta\sqrt{1-t^2}\bigr).$$

Since $p_{\frac{m+n}{2}+1}(0)=0$, equation (16) simplifies to

$$K_{\frac{m+n}{2}}(t,0)=\varkappa_{\frac{m+n}{2}}\varkappa^{-1}_{\frac{m+n}{2}+1}\,p_{\frac{m+n}{2}}(0)\,p_{\frac{m+n}{2}+1}(t)/t,$$

where

$$\varkappa_{\frac{m+n}{2}}=\pi^{-1/2}(-1)^{n/2}2^{m+n-1},\qquad \varkappa_{\frac{m+n}{2}+1}=2\varkappa_{\frac{m+n}{2}},\qquad p_{\frac{m+n}{2}}(0)=2\pi^{-1/2}.$$

The resulting theorem is stated in general form, modified by a continuous parameter:

Theorem 5. *Let n and m be positive even integers and $a>0$. Then*

$$\int_{-a}^{1} \frac{\sin(n\sin^{-1}\sqrt{t})\sinh(m\sinh^{-1}\sqrt{t/a})}{\cos(2n\sin^{-1}\sqrt{t})+\cosh(2m\sinh^{-1}\sqrt{t/a})}\,t^j\,dt = \begin{cases}\pi/2, & j=-1\\ 0, & j=0,1,\dots,\frac{m+n-4}{2},\end{cases} \tag{20}$$

$$\int_{-a}^{1} \frac{\cos(n\sin^{-1}\sqrt{t})\cosh(m\sinh^{-1}\sqrt{t/a})}{\cos(2n\sin^{-1}\sqrt{t})+\cosh(2m\sinh^{-1}\sqrt{t/a})}\frac{t^j\,dt}{\sqrt{(1-t)(1+t/a)}}=0, \quad j=0,1,\dots,\tfrac{m+n-2}{2}. \tag{21}$$

(21) is a two-parameter generalization of (4).

7. Some other theorems

One can take as $\rho(t)$ a product of several expressions like (11) with different integer parameters. Here we restrict our attention to the simplest of such functions

$$\rho(t)=\bigl\{\cos(2n\sin^{-1}\sqrt{t})+\cosh(2m\sinh^{-1}\sqrt{t})\bigr\}^2,$$

where m,n are positive integers. From (14) we immediately obtain the representation

$$2(\xi-i\eta)^2=e^{i(n+m)\theta}\,\overline{h^2\bigl(e^{i\theta}\bigr)}$$

with $h(z)$ defined by (12). This leads to

$$p_{m+n}(t)=(8/\pi)^{1/2}(\xi^2-\eta^2),$$

$$p_{m+n+1}(t)=tp_{m+n}(t)+4\pi^{-1/2}\xi\eta\sqrt{1-t^2}.$$

Since $p_{m+n+1}(0)=0$, (16) simplifies to

$$K_{m+n}(t,0)=\varkappa_{m+n}\varkappa_{m+n+1}^{-1}\,p_{m+n}(0)\,p_{m+n+1}(t)/t,$$

where

$$\varkappa_{m+n}=(2\pi)^{-1/2}(-1)^n\,2^{2m+2n-1},\qquad \varkappa_{m+n+1}=2\varkappa_{m+n},\qquad p_{m+n}(0)=4\pi^{-1/2}.$$

Thus, we obtain after redefining the integers n,m (again, the theorem is stated in general form, modified by a continuous parameter):

Theorem 6. *Let n and m be positive even integers. Then*

$$\int_{-a}^{1}\frac{\sin(n\sin^{-1}\sqrt{t})\sinh(m\sinh^{-1}\sqrt{t/a})}{\{\cos(n\sin^{-1}\sqrt{t})+\cosh(m\sinh^{-1}\sqrt{t/a})\}^2}\,t^j\,dt=\begin{cases}\pi/2, & j=-1,\\ 0, & j=0,1,\dots,\frac{n+m}{2}-2.\end{cases}\tag{22}$$

(22) is related to the integral

$$\int_{0}^{\infty}\frac{\sin(x\sin\alpha)\sinh(x\cos\alpha)}{\{\cosh(x\cos\alpha)+\cos(x\sin\alpha)\}^2}\frac{dx}{x}=\frac{\alpha}{2},\tag{23}$$

mentioned in section 7 of [12]. Integrals similar to (23) were also studied in [5].

8. Application to certain Gauss quadratures

Theorem 7. *Let n and m be positive odd integers, and define*

$$\alpha_z=2n\sinh^{-1}\sin\frac{\pi z}{2n},\quad \beta_z=2m\sinh^{-1}\sin\frac{\pi z}{2m}.\tag{24}$$

Then for any polynomial $p(t)$ of degree at most $m+n-1$

$$\int_{-1}^{1}\frac{p(t)}{\cos(2n\sin^{-1}\sqrt{t})+\cosh(2m\sinh^{-1}\sqrt{t})}\frac{dt}{\sqrt{1-t^2}}=\frac{\pi}{2mn}p(0)$$
$$+\frac{2\pi}{n}\sum_{i=1}^{\lfloor\frac{n}{2}\rfloor}\frac{\tanh\frac{\alpha_{2i}}{2n}}{\sinh\left(\frac{m}{n}\alpha_{2i}\right)}p\Big(\sin^2\frac{\pi i}{n}\Big)+\frac{2\pi}{m}\sum_{j=1}^{\lfloor\frac{m}{2}\rfloor}\frac{\tanh\frac{\beta_{2j}}{2m}}{\sinh\left(\frac{n}{m}\beta_{2j}\right)}p\Big(-\sin^2\frac{\pi j}{m}\Big).$$

Proof. According to Theorem 1, the $k=(m+n)/2$-th degree orthonormal polynomial corresponding to the weight function $1/\{\rho(t)\sqrt{1-t^2}\}$, where $\rho(t)=\cos(2n\sin^{-1}\sqrt{t})+\cosh(2m\sinh^{-1}\sqrt{t})$, is

$$p_k(t)=2\pi^{-1/2}\,\sin(n\sin^{-1}\sqrt{t})\sinh(m\sinh^{-1}\sqrt{t}).$$

Its k roots x_s are

$$0;\quad \sin^2\frac{\pi i}{n},\quad i=1,2,\dots,\tfrac{n-1}{2};\quad -\sin^2\frac{\pi j}{m},\quad j=1,2,\dots,\tfrac{m-1}{2}.$$

Gauss quadrature formula [19] now takes the form

$$\int_{-1}^{1}p(t)\frac{dt}{\rho(t)\sqrt{1-t^2}}=\sum_{s=1}^{k}w_s p(x_s),\quad w_s=\frac{\varkappa_{k+1}}{\varkappa_k p_{k+1}(x_s)p_k'(x_s)}.$$

The factors entering the formula for the weights w_s can be calculated using formulas (13),(15),(17) from Section 4. □

This theorem can be extended to a pair of positive even integers using the results of Section 6, and also to include an additional parameter $a>0$. Applying these theorems to certain polynomials, one can obtain finite analogs of generating functions in Kuznetsov's paper [11], e.g.,

Theorem 8. *Let n, m and u be integers such that $|u|<n$, and α_z be defined as in (24). Then*

$$\int_{-1}^{1}\frac{\cos\left(2u\sin^{-1}\sqrt{t}\right)}{\cos\left(2n\sin^{-1}\sqrt{t}\right)+\cosh\left(2m\sinh^{-1}\sqrt{t}\right)}\frac{dt}{\sqrt{1-t^2}}=\frac{\pi}{2n}\sum_{j=1}^{2n}\frac{(-1)^{j-1}}{\coth\frac{\alpha_j}{2n}}\left\{\tanh\frac{m\alpha_j}{2n}\right\}^{(-1)^j}\cdot\cos\frac{\pi j u}{n}.$$

As well as some other integrals with $\cos\left(2u\sin^{-1}\sqrt{t}\right)$ replaced by $\sin\left(2u\sin^{-1}\sqrt{t}\right)/\sqrt{t(1-t)}$. Theorem 8 is a non-symmetric $m\neq n$ extention of Theorem 4 from [12]. The case $u=0$ is a finite analog of Ismail and Valent's integral [10]. When $n=1$, we obtain the following curious formula:

Corollary 9. *For any positive even integer* m

$$\int_{-1}^{1}\frac{1}{1+2t+\cos\left(m\sin^{-1}\sqrt{t}\right)}\frac{dt}{\sqrt{1-t^2}}=\frac{\pi}{\sqrt{8}}\frac{\left(\sqrt{2}+1\right)^m+1}{\left(\sqrt{2}+1\right)^m-1}.$$

9. Finite analog of the integral in Ramanujan's question 353

Theorem 10. *Let n be a positive even integer and k a positive odd integer. Then*

$$\int_{0}^{1}\frac{\sin(kn\sin^{-1}t)}{\cos(n\sin^{-1}t)+\cosh(n\sinh^{-1}t)}\frac{dt}{t}=\frac{\pi}{4}. \tag{25}$$

Proof. Let $n=2\nu$, $k=2\mu+1$, where ν is a positive integer, and μ is a nonnegative integer. Similar to that of section 3 of [12], or by other means, one can derive the partial fractions expansion

$$\frac{1}{\cos(2\nu\sin^{-1}t)+\cosh(2\nu\sinh^{-1}t)}\frac{\sin(2\nu\sin^{-1}t)}{t\sqrt{1-t^2}}=\frac{1}{\nu}\sum_{j=1}^{\nu}\frac{i-\cos\frac{\pi(2j-1)}{2\nu}}{2t^2\cos\frac{\pi(2j-1)}{2\nu}+i\sin^2\frac{\pi(2j-1)}{2\nu}}.$$

Further calculations assume that ν is even. For ν odd, calculations are similar, except that one has to take special care of the term with $j=(\nu+1)/2$. Thus, define

$$q_j=\frac{1-\sin\frac{\pi(2j-1)}{2\nu}}{\cos\frac{\pi(2j-1)}{2\nu}}e^{-i\frac{\pi(2j-1)}{2\nu}},\qquad j=1,2,\dots,\nu.$$

Obviously,

$$|q_j|<1,\qquad j=1,2,\dots,\nu.$$

We are going to make change of variables

$$t=\sin(\varphi/2),\quad \varphi\in(0,\pi),$$

in the integral (9). Thus $2t^2=1-\cos\varphi$, and $4\sqrt{1-t^2}\,dt=(1+\cos\varphi)\,d\varphi$. By simple algebra

$$\frac{1+\cos\varphi}{(1-\cos\varphi)\cos\frac{\pi(2j-1)}{2\nu}+i\sin^2\frac{\pi(2j-1)}{2\nu}}=\frac{-2}{(1-q_j)\cos\frac{\pi(2j-1)}{2\nu}}\left(1+(1+q_j)\frac{1-q\cos\varphi}{1-2q_j\cos\varphi+q_j^2}\right).$$

According to well known formulas

$$\frac{1-q\cos\varphi}{1-2q_j\cos\varphi+q_j^2}=\sum_{r=0}^{\infty}q_j^r\cos(r\varphi),$$

$$\frac{\sin(k\nu\varphi)}{\sin(\nu\varphi)}=1+2\sum_{l=1}^{\mu}\cos(2\nu l\varphi).$$

Thus, the integral (9) becomes

$$I=\frac{1}{2\nu}\sum_{j=1}^{\nu}\frac{\cos\frac{\pi(2j-1)}{2\nu}-i}{(1-q_j)\cos\frac{\pi(2j-1)}{2\nu}}\int_{0}^{\pi}\left(1+(1+q_j)\sum_{r=0}^{\infty}q_j^r\cos(r\varphi)\right)\left(1+2\sum_{l=1}^{\mu}\cos(2\nu l\varphi)\right)d\varphi.$$

The integrals are easily calculated using orthogonality of cosines on $(0,\pi)$:

$$I=\frac{\pi}{2\nu}\sum_{j=1}^{\nu}f(j),\quad f(j)=\frac{\cos\frac{\pi(2j-1)}{2\nu}-i}{(1-q_j)\cos\frac{\pi(2j-1)}{2\nu}}\left(2+q_j+(1+q_j)\sum_{l=1}^{\mu}q_j^{2\nu l}\right).$$

Trivial algebra (under the transformation $j\to n+1-j$ the expressions $\sin\frac{\pi(2j-1)}{2\nu}$ and $q_j^{2\nu}$ do not change, and $\cos\frac{\pi(2j-1)}{2\nu}$ changes sign) shows that

$$f(j)+f(n+1-j)=1,\qquad j=1,2,\dots,\nu.$$

Thus $\sum_{j=1}^{n}f(j)=\nu/2$, and $I=\pi/4$. □

One can obtain a finite analog of Theorem 4.2 from [2] multiplying the integrand in (25) by t^{4b}, $b \in \mathbb{N}$.

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