Fibonacci Sequence
Genesis of the Golden Ratio
Genesis of the Lucas Sequence
Genesis of Binet's Formula

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I - Warning

In this article, we will discuss several properties, both known and novel. Specifically, the genesis of the golden ratio from the Fibonacci sequence. But more importantly, to demonstrate that the Fibonacci sequence itself also originates from the very framework of the golden ratio, if I may use that expression. We will reveal new insights into the connection that unites the Fibonacci sequence with the Lucas sequence and Binet's formula.

II - Brief Presentation of the Results

Note: For now, let's set aside the derivation of [Phi] from the Fibonacci sequence (it's a bit more extensive, and we will address it in the third section. But before delving further, here is a succinct illustration of the approach that will allow us to tackle these questions in a novel way, it seems to us. Let Phi, the golden ratio, be such that:

$$\phi = \frac{1}{2}\left(1 + \sqrt{5}\right)$$

Consider the following function :

$$f[n] = \phi^n$$

We will have:

$$\phi^1 = \frac{1}{2}\left(1 + 1\sqrt{5}\right) \implies 1 * 1 = \mathbf{1}$$
$$\phi^2 = \frac{1}{2}\left(3 + 1\sqrt{5}\right) \implies 3 * 1 = \mathbf{3}$$
$$\phi^3 = \frac{1}{2}\left(4 + 2\sqrt{5}\right) \implies 4 * 2 = \mathbf{8}$$
$$\phi^4 = \frac{1}{2}\left(7 + 3\sqrt{5}\right) \implies 7 * 3 = \mathbf{21}$$
$$\phi^5 = \frac{1}{2}\left(11 + 5\sqrt{5}\right) \implies 11 * 5 = \mathbf{55}$$

Etc.

In red : Fibonacci sequence

In blue: the Lucas sequence

Currently, we only have the even-order (appearance) numbers of the Fibonacci sequence

1 ; **1** ; 2 ; **3** ; 5 ; **8** ; 13 ; **21** ; etc.

But we will see that it is easy to generate from a Fibonacci number both the preceding number and the following one

This is particularly due to solving an equation related to a very important property of the Fibonacci sequence, whose solution will allow us to retrieve both the golden ratio and Binet's formula.

III - Demonstrations

1 - General Framework

Like all natural numbers, Fibonacci numbers can be expressed in the form:

$$\mathbf{n = \frac{a^2-b^2}{d}}$$

Examples :

$$5^2 = \frac{3^2-2^2}{1}\ ;\ 8^2 = \frac{5^2-3^2}{2}\ ;\ 13^2 = \frac{8^2-5^2}{3}\ ;\ \text{etc.}$$

In general, we will have:

$$n = \frac{a^2-b^2}{d}\ ;\ \text{avec}\quad a = \frac{n+d}{2}\quad \text{et}\quad b = \frac{n-d}{2}$$

2 - An exclusive property

This will provide us, among other things, this exclusive property in the Fibonacci sequence:

- For numbers whose appearance order is odd, we will write:

  Example 1 : **5 * 1 = 3 * 2 - 1**

  Example 2 : **13 * 3 = 8 * 5 - 1**

- Pour les nombres dont l'ordre est pair, on écrira :

  Example 1 : **8 * 2 = 5 * 3 + 1**

  Example 2 : **21 * 5 = 13 * 8 + 1**

3 - The central equation

This will yield:

$$n*d + 1 = a*b$$

Recall that:

$$n = (a + b)$$
$$d = (a - b)$$

The solutions to these two equations provide us with:

$$b = -2\,a + \sqrt{-4 + 5\,a^2}\,)$$
$$a = -2\,b + \sqrt{4 + 5\,b^2}\,)$$

4 - Genesis of the Lucas Sequence

Every number in the Lucas sequence can be expressed in these two ways:

- *1st formula:*

$$L = \sqrt{4 + 5 * F^{2}}$$

Or

$$L = \sqrt{-4 + 5 * F^{2}}$$

Examples

$$7 = \sqrt{4 + 5 * 3^{2}}$$
$$11 = \sqrt{-4 + 5 * 5^{2}}$$

- *2nd formula:*

$$\mathbf{L = F_n + 2*F_{n-1}}$$

L is a number in the Lucas sequence
F is a number in the Fibonacci sequence

5 - Genesis of Binet’s Formula

It is thus necessary that $(4 + 5\,a^2)$ and $(-4 + 5\,a^2)$ be perfect squares of natural numbers.

Let us examine the following case (k is a natural number):

$$k^2 = 4 + 5\,a^2$$

Recall that Pythagorean triples (x , y , z) satisfy this property (**m** and **n** being natural numbers):

$$x = n^2 - m^2$$
$$y = 2*m*n$$
$$z = n^2 + m^2$$

- This property can therefore be extended to real numbers. And thus it can be expressed concerning the triplet of real numbers $(k\ ,\ 2\ ,\ \sqrt{5\ a^2})$

$$\sqrt{5\ a^2} = n^2 - m^2$$
$$2 = 2*m*n$$
$$k = n^2 + m^2$$

We will have:

$$\left(m^2 + n^2\right)^2 = (2\ mn)^2 + \left(m^2 - n^2\right)^2$$

Which gives us:

$$2 = 2*m*n \implies m = \frac{1}{n}$$

Here is the genesis of the famous Binet's formula:

$$m^2 + n^2 = \mathbf{k} = \mathbf{n^2 + \left(\frac{1}{n}\right)^2}$$

First consequence:

As seen above ϕ^3 allows us to recover the Fibonacci number 8

$$\phi^3 = \frac{1}{2}\left(4 + 2\sqrt{5}\right) \implies 4 * 2 = \mathbf{8}$$

Thus: $\phi^3 \implies \mathbf{8}$

We will have this:

$$\frac{1}{\left(\phi^3\right)^2} + \left(\phi^3\right)^2 = 18$$

And so:

$$18 + 8 = 26 \implies \frac{26}{2} = \mathbf{13}$$
$$18 - 8 = 10 \implies \frac{10}{2} = \mathbf{5}$$

Which gives us:

$$\mathbf{5}\ ,\ \mathbf{8}\ ,\ \mathbf{13}$$

6 - Genesis of the Golden Ratio ϕ

Now, let's consider the specific case where:

$$\mathbf{b = 1}$$

We will have this equation :

$$1 == -2\ a + \sqrt{4 + 5\ a^2}$$

And knowing that:

$$k = \sqrt{4 + 5 a^2} = n^2 + \left(\frac{1}{n}\right)^2$$

One of the solutions is the renowned Golden Ratio:

$$a = \phi = \frac{1}{2}\left(1 + \sqrt{5}\right)$$

2 - Other Properties to Explore

1st Property

his exclusive property can also be studied for the Fibonacci sequence.
However, the same implications as previously will be found :

$$\frac{n_2^2 - 1}{n_1} = n_3 \implies \text{exemple :} \ \frac{13^2 - 1}{8} = 21$$

Or

$$\frac{n_2^2 + 1}{n_1} = n_5 \implies \text{exemple :} \ \frac{8^2 + 1}{5} = 13$$

2nd Property

But it is particularly this other solution of a variant of the equation we have just studied:

$$\left(\sqrt{5} + 2\right)$$

This number has properties very similar to the golden ratio. It could be considered its own twin.

a) - For Fibonacci numbers of odd order (1st, 3rd, 5th, etc.), the product of this number with a Fibonacci number results in exactly (in its integer part) another Fibonacci number.

b) - For numbers of even order (2nd, 4th, 6th, etc.), you only need to add 1 to the integer part to exactly find another Fibonacci number.

3rd Property

The sum of the squares of two successive Fibonacci numbers is always a Fibonacci number.
It exhibits certain different properties that have implications in other fields.
I will discuss this in another article.
Examples :

$$2^2 + 3^2 = 13$$
$$3^2 + 5^2 = 34$$

4th Property ::

We will explore a novel formula that connects the number **Pi** to **Phi**, but for now only from an approximation standpoint.

$$\text{Pi} \simeq \frac{22}{7}$$

We saw at the beginning of this presentation the following:

$$\phi^4 = \frac{1}{2}\left(7 + 3\sqrt{5}\right) \Longrightarrow 7 * 3 = \mathbf{21}$$

We will therefore have: $\mathbf{3 = \frac{-7+2\,\phi^4}{\sqrt{5}}}$ et $7\ \mathbf{= \frac{-3+2\,\phi^4}{\sqrt{5}}}$

And also : $\mathbf{Pi \simeq \frac{22}{7} = 3 + \frac{1}{7}}$

The solution to this equation will give us:

$$\pi \simeq \frac{-7+2\,\phi^4}{\sqrt{5}} + \frac{1}{-3\sqrt{5}+2\,\phi^4}$$

It is possible to further refine the relationship between π and ϕ
Stay tuned.

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