A Three Field Theory of Gravity

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Abstract

We propose a tensor-vector-scalar field theory of gravity defining a corresponding Lagrange function. The theory is completely relativistic in the sense, that for a flat metric all additional fields vanish. After considerations of stability and convergence we develop the systems of differential equations for two cases.

In the static case of a central mass we start with the Schwarzschild metric as a first approximation and get a behavior of gravity generated by the scalar field corresponding to the observation of galaxy rotation for sufficiently large distances from the center without the need of Dark Matter. Using this result a second approximation gives a qualitatively stronger gravity for very large distances, which could describe the cohesion of galaxy clusters. When the distance tends to infinity, i.e., on cosmological scales, gravity returns to a Newton-like decay. This result also coincides with observations.

In cosmology we find a homogeneous, isotropic, flat universe, that in the long term does not expand, but shows a time contraction towards the past generating the observed redshift of light from far distant sources that corresponds to an accelerated expansion of the universe in the Robertson-Walker metric, but without Dark Energy.

1 Introduction

There are different attempts to improve general relativity and describe gravity without the need of dark matter by relativistic theories based on the Einstein tensor field and extended by vector and scalar fields, for example the Brans-Dicke theory [1], TeVeS by Bekenstein [2] or newer versions [3] of Modified Newtonian dynamics and Modified Gravity by Moffat [4]. In most cases scalar fields represent a generalization of the gravitational constant. In our theory, the scalar field should fill the gap between the observed gravity and that calculated from visible matter by the Newton/Einstein theory. This scalar field is coupled to the vector field by the scalar potential while the vector field is coupled to the tensor field.

In the next section we define the Lagrange function consisting of three parts representing tensor, vector and scalar fields, while the fourth part corresponds to the stress-energy tensor. The following section starts with a decomposition of the coupling between tensor and vector field. It turns out that the term quadratic in the vector field components represents a mass term, where the mass is given by $\frac{3}{2}$ of the coupling constant. Next, we discuss the relation of the resulted system of equations to general relativity for small coupling constants. In the case of a flat metric we investigate the hyperbolic character of the equations for the vector field and determine the number of degrees of freedom for the remaining system using the Hamilton-Dirac analysis.

The following section is devoted to the spherically symmetric case of a central mass M. First, it turns out that we have to use Cartesian instead of spherical coordinates because of their intrinsic warping. We derive the time dependent equations, but reduce the consideration to the static case. Starting with the Schwarzschild metric we describe the behavior of the vector field and the scalar field in powers and logarithms of the distance r to the central mass. In a first approximation the scalar field generates a gravity proportional to $M^{\frac{1}{2}}r^{-1}$ multiplied by a logarithmic factor with a negative exponent for sufficiently large r. This corresponds to observations of galaxy rotation, which in the outer regions can be described by the square root of Newtonian gravity multiplied by a constant. Using this result for a second approximation of the metric, we get a higher rate of gravity proportional to $M^{\frac{1}{4}}r^{-\frac{1}{2}}$ again multiplied by a logarithmic term for larger distances. This result corresponds to solutions of the p-Laplacian for $p = 5$, which is used in [5] to describe galaxy clusters, where the gravity given by the first approximation is to weak. Finally, for distances tending to infinity, the mass term dominates the equation for the vector field. Then, the scalar field generates a gravity proportional to r^{-2}, which corresponds to Newtonian gravity.

In the section about cosmology we apply the theory to a homogeneous, isotropic universe, dominated by mass neglecting radiation. In contrast to the Robertson-Walker metric we need a variable factor for the time component. Using again the Ansatz of a power series depending on t, as a first approach we get no expansion but a time contraction with a factor $t^{-2-\epsilon}$ for an arbitrary small positive ϵ. Since a possible solution vanishes in the case $\epsilon = 0$, we extent the Ansatz by logarithmic factors looking for a metric with warping as small as possible. Again, we see no expansion, while in the time contraction the factor t^{-2} is multiplied by a logarithmic term with an arbitrary small negative exponent. For observed light from the early universe this gives a redshift that suggests an accelerated expansion of the universe if interpreted in the Robertson-Walker metric as derived from observations.

2 Lagrange formulation

In a Minkowski space with coordinates $x^\alpha|_{\alpha=0}^3$, where for simplicity we identify x^0 with time t even if it should be $\frac{1}{c}x^0$, we will describe gravity by the metric tensor $g = g_{\alpha,\beta}|_{\alpha,\beta=0}^3$ and two additional fields, a vector field $\mathcal{U} = u_\alpha|_{\alpha=0}^3$ and a scalar field v. For the metric we will use the signature $(-,+,+,+)$, i.e. in absence of matter the metric should be $g = \mathrm{diag}(-1,+1,+1,+1)$.

The Christoffel symbols $\Gamma^\gamma_{\alpha,\beta}$, the Ricci tensor $\mathcal{R}_{\alpha\beta}$ and the Ricci scalar R will be considered as differential operators

$$\Gamma^\gamma_{\alpha\beta}(\partial) = \tfrac{1}{2}\, g^{\gamma\sigma}\,(\partial_\alpha g_{\sigma\beta} + \partial_\beta g_{\alpha\sigma} - \partial_\sigma g_{\alpha\beta}) \quad (1)$$

$$\mathcal{R}_{\alpha\beta}(\partial) = \partial_\gamma \Gamma^\gamma_{\alpha\beta}(\partial) - \partial_\beta \Gamma^\gamma_{\alpha\gamma}(\partial) + \Gamma^\gamma_{\alpha\beta}(\partial)\,\Gamma^\sigma_{\gamma\sigma}(\partial) - \Gamma^\gamma_{\alpha\sigma}(\partial)\,\Gamma^\sigma_{\gamma\beta}(\partial) \quad (2)$$

with $\partial = \partial_\alpha|_{\alpha=0}^3$. One can show, that $\mathcal{R}_{\alpha\beta}$ is symmetric in α,β, what is only nontrivial for the second term of (2). To preserve this symmetry for more general differential operators $D = D_\alpha|_{\alpha=0}^3$ we define

$$\mathcal{R}_{\alpha\beta}(D) := \tfrac{1}{2}\left[\mathcal{R}_{\alpha\beta}(\partial)|_{\partial\to D} + \mathcal{R}_{\beta\alpha}(\partial)|_{\partial\to D}\right] \quad (3)$$

$$R(D) := g^{\alpha\beta}\,\mathcal{R}_{\alpha\beta}(D) \quad (4)$$

The tensor part $\mathcal{L}_T$ of the Lagrangian can now be defined by

$$\mathcal{L}_T := \tfrac{1}{2}\, R(\partial + \mu\,\mathcal{U}) \quad (5)$$

where $\partial + \mu\mathcal{U} = \partial_\alpha + \mu\, u_\alpha|_{\alpha=0}^3$ and μ is the coupling constant between the tensor and the vector field. The vector part $\mathcal{L}_V$ and the scalar part $\mathcal{L}_S$ of the Lagrangian are

$$\mathcal{L}_V := -\tfrac{1}{4}\,(\nabla_\alpha u_\beta - \nabla_\beta u_\alpha)\left(\nabla^\alpha u^\beta - \nabla^\beta u^\alpha\right) - \tfrac{1}{2}\,\nabla_\alpha u^\alpha\,\nabla_\beta u^\beta \quad (6)$$

$$\mathcal{L}_S := -\tfrac{1}{2}\,\nabla_\alpha v\,\nabla^\alpha v \,-\, V(\mathcal{U})\,v^2, \quad V(\mathcal{U}) = \tfrac{1}{2}\left(m^2 + \nu^2\,\nabla_\alpha u^\alpha\,\nabla_\beta u^\beta\right) \quad (7)$$

where ν describes the strength of the coupling between the scalar and the vector field. The covariant derivative of a vector or a scalar field is defined by

$$\nabla_\alpha u_\beta = \partial_\alpha u_\beta - \Gamma^\gamma_{\alpha\beta}(\partial)\,u_\gamma, \quad \nabla_\alpha u^\beta = \partial_\alpha u^\beta + \Gamma^\beta_{\alpha\gamma}(\partial)\,u^\gamma, \quad \nabla_\alpha v = \partial_\alpha v \quad (8)$$

Note that $\nabla_\alpha u^\alpha \, \nabla_\beta u^\beta = \left(\sum_\gamma \nabla_\gamma u^\gamma\right)^2 \geq 0$.

Together with the matter part $\mathcal{L}_M$, which is defined later in the usual way, depending only on the metric g we can write the complete Lagrangian

$$\mathcal{L} = \mathcal{L}_T + \mathcal{L}_V + \mathcal{L}_S + \mathcal{L}_M \tag{9}$$

and the action

$$\mathcal{S} = \int \mathcal{L} \, \epsilon \, d^4x, \quad \epsilon = \sqrt{-\det g} \tag{10}$$

3 Field Equations

First, we shall decompose the expression $R(\partial + \mu\mathcal{U})$ in the tensor part of the Lagrangian. For

$$\Gamma^\gamma_{\alpha\beta}(\partial + \mu\mathcal{U}) = \Gamma^\gamma_{\alpha\beta}(\partial) + \mu \, \Upsilon^\gamma_{\alpha\beta} \tag{11}$$

with

$$\begin{aligned} \Upsilon^\gamma_{\alpha\beta} &= \tfrac{1}{2} \, g^{\gamma\sigma} \left(u_\alpha g_{\sigma\beta} + u_\beta g_{\alpha\sigma} - u_\sigma g_{\alpha\beta}\right) \\ &= \tfrac{1}{2} \left(\delta^\gamma_\beta u_\alpha + \delta^\gamma_\alpha u_\beta - g_{\alpha\beta} u^\gamma\right) \end{aligned} \tag{12}$$

we have

$$\begin{aligned} &\mathcal{R}_{\alpha\beta}(\partial{+}\mu\mathcal{U}) \\ &= \partial_\gamma \left[\Gamma^\gamma_{\alpha\beta}(\partial) + \mu \, \Upsilon^\gamma_{\alpha\beta}\right] - \tfrac{1}{2} \, \partial_\beta \left[\Gamma^\gamma_{\alpha\gamma}(\partial) + \mu \, \Upsilon^\gamma_{\alpha\gamma}\right] \\ &\quad - \tfrac{1}{2} \, \partial_\alpha \left[\Gamma^\gamma_{\beta\gamma}(\partial) + \mu \, \Upsilon^\gamma_{\beta\gamma}\right] + \mu \, u_\gamma \left[\Gamma^\gamma_{\alpha\beta}(\partial) + \mu \, \Upsilon^\gamma_{\alpha\beta}\right] \\ &\quad - \tfrac{1}{2} \, \mu \, u_\beta \left[\Gamma^\gamma_{\alpha\gamma}(\partial) + \mu \, \Upsilon^\gamma_{\alpha\gamma}\right] - \tfrac{1}{2} \, \mu \, u_\alpha \left[\Gamma^\gamma_{\beta\gamma}(\partial) + \mu \, \Upsilon^\gamma_{\beta\gamma}\right] \\ &\quad + \left[\Gamma^\gamma_{\alpha\beta}(\partial) + \mu \, \Upsilon^\gamma_{\alpha\beta}\right] \left[\Gamma^\sigma_{\gamma\sigma}(\partial) + \mu \, \Upsilon^\sigma_{\gamma\sigma}\right] \\ &\quad - \left[\Gamma^\gamma_{\alpha\sigma}(\partial) + \mu \, \Upsilon^\gamma_{\alpha\sigma}\right] \left[\Gamma^\sigma_{\gamma\beta}(\partial) + \mu \, \Upsilon^\sigma_{\gamma\beta}\right] \\ \\ &= \mathcal{R}_{\alpha\beta}(\partial) + \mu \, \Big[\partial_\gamma \Upsilon^\gamma_{\alpha\beta} - \tfrac{1}{2} \, \partial_\beta \Upsilon^\gamma_{\alpha\gamma} - \tfrac{1}{2} \, \partial_\alpha \Upsilon^\gamma_{\beta\gamma} \\ &\quad + \Gamma^\gamma_{\alpha\beta}(\partial) \, \Upsilon^\sigma_{\gamma\sigma} + \Gamma^\sigma_{\gamma\sigma}(\partial) \, \Upsilon^\gamma_{\alpha\beta} - \Gamma^\gamma_{\alpha\sigma}(\partial) \, \Upsilon^\sigma_{\gamma\beta} - \Gamma^\sigma_{\gamma\beta}(\partial) \, \Upsilon^\gamma_{\alpha\sigma}\Big] \\ &\quad + \mu \, \left[u_\gamma \, \Gamma^\gamma_{\alpha\beta}(\partial) - \tfrac{1}{2} \, u_\beta \, \Gamma^\gamma_{\alpha\gamma}(\partial) - \tfrac{1}{2} \, u_\alpha \, \Gamma^\gamma_{\beta\gamma}(\partial)\right] \\ &\quad + \mu^2 \left[u_\gamma \Upsilon^\gamma_{\alpha\beta} - \tfrac{1}{2} \, u_\beta \Upsilon^\gamma_{\alpha\gamma} - \tfrac{1}{2} \, u_\alpha \Upsilon^\gamma_{\beta\gamma} + \Upsilon^\gamma_{\alpha\beta} \Upsilon^\sigma_{\gamma\sigma} - \Upsilon^\gamma_{\alpha\sigma} \Upsilon^\sigma_{\gamma\beta}\right] \end{aligned} \tag{13}$$

For the last term an easy calculation using (12) shows

$$\begin{aligned} \mathcal{R}_{\alpha\beta}(\mathcal{U}) &= u_\gamma \, \Upsilon^\gamma_{\alpha\beta} - \tfrac{1}{2} \, u_\beta \, \Upsilon^\gamma_{\alpha\gamma} - \tfrac{1}{2} \, u_\alpha \, \Upsilon^\gamma_{\beta\gamma} + \Upsilon^\gamma_{\alpha\beta} \, \Upsilon^\sigma_{\gamma\sigma} - \Upsilon^\gamma_{\alpha\sigma} \, \Upsilon^\sigma_{\gamma\beta} \\ &= -\tfrac{1}{2} \, u_\alpha \, u_\beta - g_{\alpha\beta} \, u_\gamma \, u^\gamma \end{aligned} \tag{14}$$

With the relations

$$
\begin{aligned}
\nabla_\gamma \Upsilon^\gamma_{\alpha\beta} - \nabla_\beta \Upsilon^\gamma_{\alpha\gamma} &= \partial_\gamma \Upsilon^\gamma_{\alpha\beta} - \partial_\beta \Upsilon^\gamma_{\alpha\gamma} \\
&\quad + \Gamma^\gamma_{\alpha\beta}(\partial)\, \Upsilon^\sigma_{\gamma\sigma} + \Gamma^\sigma_{\gamma\sigma}(\partial)\, \Upsilon^\gamma_{\alpha\beta} - \Gamma^\gamma_{\alpha\sigma}(\partial)\, \Upsilon^\sigma_{\gamma\beta} - \Gamma^\sigma_{\gamma\beta}(\partial)\, \Upsilon^\gamma_{\alpha\sigma} \\
\nabla_\gamma \Upsilon^\gamma_{\alpha\beta} - \nabla_\alpha \Upsilon^\gamma_{\beta\gamma} &= \partial_\gamma \Upsilon^\gamma_{\alpha\beta} - \partial_\alpha \Upsilon^\gamma_{\beta\gamma} \\
&\quad + \Gamma^\gamma_{\alpha\beta}(\partial)\, \Upsilon^\sigma_{\gamma\sigma} + \Gamma^\sigma_{\gamma\sigma}(\partial)\, \Upsilon^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\sigma}(\partial)\, \Upsilon^\sigma_{\gamma\alpha} - \Gamma^\sigma_{\gamma\alpha}(\partial)\, \Upsilon^\gamma_{\beta\sigma}
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\mathcal{L}_T &= \tfrac{1}{2}\, g^{\alpha\beta}\, \mathcal{R}_{\alpha\beta}(\partial + \mu\,\mathcal{U}) \\
&= \tfrac{1}{2}\, R(\partial) + \tfrac{1}{2}\,\mu\, g^{\alpha\beta} \left[\nabla_\gamma \Upsilon^\gamma_{\alpha\beta} - \tfrac{1}{2}\,\nabla_\beta \Upsilon^\gamma_{\alpha\gamma} - \tfrac{1}{2}\,\nabla_\alpha \Upsilon^\gamma_{\beta\gamma}\right] \\
&\quad + \tfrac{1}{2}\,\mu\, g^{\alpha\beta} \left[u_\gamma\, \Gamma^\gamma_{\alpha\beta}(\partial) - \tfrac{1}{2}\, u_\beta\, \Gamma^\gamma_{\alpha\gamma}(\partial) - \tfrac{1}{2}\, u_\alpha\, \Gamma^\gamma_{\beta\gamma}(\partial)\right] \\
&\quad - \tfrac{9}{4}\,\mu^2\, u_\gamma\, u^\gamma
\end{aligned} \tag{15}
$$

By renaming some indices we can now write

$$
\mathcal{L}_T = \tfrac{1}{2}\,\mathcal{L}_T^{(0)} + \tfrac{1}{2}\,\mu\,\mathcal{L}_T^{(1)} + \tfrac{1}{4}\,\mu^2\,\mathcal{L}_T^{(2)} + \tfrac{1}{2}\,\mu\,\nabla_\gamma \mathcal{K}^\gamma_T \tag{16}
$$

with

$$
\mathcal{L}_T^{(0)} = R(\partial) \tag{17}
$$

$$
\mathcal{L}_T^{(1)} = g^{\alpha\beta}\left[u_\gamma\, \Gamma^\gamma_{\alpha\beta}(\partial) - \tfrac{1}{2}\, u_\beta\, \Gamma^\gamma_{\alpha\gamma}(\partial) - \tfrac{1}{2}\, u_\alpha\, \Gamma^\gamma_{\beta\gamma}(\partial)\right] \tag{18}
$$

$$
= u_\gamma \left[g^{\alpha\beta}\, \Gamma^\gamma_{\alpha\beta}(\partial) - \tfrac{1}{2}\, g^{\alpha\gamma}\, \Gamma^\beta_{\alpha\beta}(\partial) - \tfrac{1}{2}\, g^{\beta\gamma}\, \Gamma^\alpha_{\alpha\beta}(\partial)\right] \tag{19}
$$

$$
\mathcal{L}_T^{(2)} = 9\, u_\gamma\, u^\gamma \tag{20}
$$

$$
\mathcal{K}^\gamma_T = g^{\alpha\beta}\, \Upsilon^\gamma_{\alpha\beta} - \tfrac{1}{2}\, g^{\alpha\gamma}\, \Upsilon^\beta_{\alpha\beta} - \tfrac{1}{2}\, g^{\beta\gamma}\, \Upsilon^\alpha_{\alpha\beta} = -3\, u^\gamma \tag{21}
$$

where $\mathcal{K}^\gamma_T$ was simply calculated using (12).

Having defined the vector field as massless, we now see a mass term

$$
\tfrac{1}{4}\,\mu^2\,\mathcal{L}_T^{(2)} = \left(\tfrac{3}{2}\,\mu\right)^2 u_\gamma\, u^\gamma
$$

generated by the coupling to the tensor field with a mass proportional to the coupling constant.

For the field equations we have to describe the variation of the action $\mathcal{S}$ by the variation of $\mathcal{L}\,\epsilon$ in the form

$$
\delta[\mathcal{L}\,\epsilon] = \left[\mathcal{M}^{(g)}_{\alpha\beta}\delta g^{\alpha\beta} + \mathcal{M}^{(u)}_\gamma \delta u^\gamma + \mathcal{M}^{(v)}\delta v + \nabla_\gamma \mathcal{K}^\gamma\right]\epsilon \tag{22}
$$

Then, the field equations are

$$
\mathcal{M}^{(g)}_{\alpha\beta} = 0 \;\text{ for }\; \alpha,\beta = 0,..,3, \quad \mathcal{M}^{(u)}_\gamma = 0 \;\text{ for }\; \gamma = 0,..,3, \quad \mathcal{M}^{(v)} = 0 \tag{23}
$$

First, we have

$$\delta[\mathcal{L}\,\epsilon] = \left[\delta\mathcal{L} - \tfrac{1}{2}\,\mathcal{L}\,g_{\alpha\beta}\,\delta g^{\alpha\beta}\right]\epsilon \tag{24}$$

since

$$\delta\epsilon = -\tfrac{1}{2}\,\epsilon^{-1}\delta(\det g) \quad \text{and} \quad \delta(\det g) = \det g\, g^{\alpha\beta}\delta g_{\alpha\beta} = \epsilon^2\, g_{\alpha\beta}\,\delta g^{\alpha\beta}$$

One can show [6] (appendix A) that

$$\mathcal{A}^\gamma = g^{\alpha\beta}\,\delta\Gamma^\gamma_{\alpha\beta}(\partial) - g^{\alpha\gamma}\,\delta\Gamma^\beta_{\alpha\beta}(\partial) = g_{\alpha\beta}\,\nabla^\gamma\delta g^{\alpha\beta} - \nabla_\beta\,\delta g^{\beta\gamma} \tag{25}$$

Using this relation for the zero part of $\mathcal{L}_T$ we get

$$\delta\mathcal{L}^{(0)}_T = \mathcal{R}_{\alpha\beta}(\partial)\,\delta g^{\alpha\beta} + \nabla_\gamma\left[g_{\alpha\beta}\,\nabla^\gamma\delta g^{\alpha\beta} - \nabla_\beta\,\delta g^{\beta\gamma}\right] \tag{26}$$

The variation of the next part gives

$$\begin{aligned}
\delta\mathcal{L}^{(1)}_T &= \delta u_\gamma\left[g^{\alpha\beta}\,\Gamma^\gamma_{\alpha\beta}(\partial) - \tfrac{1}{2}\,g^{\alpha\gamma}\,\Gamma^\beta_{\alpha\beta}(\partial) - \tfrac{1}{2}\,g^{\beta\gamma}\,\Gamma^\alpha_{\alpha\beta}(\partial)\right] \\
&\quad + u_\gamma\left[\delta g^{\alpha\beta}\,\Gamma^\gamma_{\alpha\beta}(\partial) - \tfrac{1}{2}\,\delta g^{\alpha\gamma}\,\Gamma^\beta_{\alpha\beta}(\partial) - \tfrac{1}{2}\,\delta g^{\beta\gamma}\,\Gamma^\alpha_{\alpha\beta}(\partial)\right] \\
&\quad + u_\gamma\left[g^{\alpha\beta}\,\delta\Gamma^\gamma_{\alpha\beta}(\partial) - \tfrac{1}{2}\,g^{\alpha\gamma}\,\delta\Gamma^\beta_{\alpha\beta}(\partial) - \tfrac{1}{2}\,g^{\beta\gamma}\,\delta\Gamma^\alpha_{\alpha\beta}(\partial)\right]
\end{aligned} \tag{27}$$

We can write

$$\begin{aligned}
\delta u_\gamma &= \delta(g_{\gamma\rho}u^\rho) = g_{\gamma\rho}\,\delta u^\rho - u^\rho\, g_{\gamma\sigma}\, g_{\tau\rho}\,\delta g^{\sigma\tau} = g_{\gamma\rho}\,\delta u^\rho - u_\tau\, g_{\gamma\sigma}\,\delta g^{\sigma\tau} \\
&= g_{\gamma\rho}\,\delta u^\rho - \tfrac{1}{2}\,u_\tau\, g_{\gamma\sigma}\,\delta g^{\sigma\tau} - \tfrac{1}{2}\,u_\sigma\, g_{\gamma\tau}\,\delta g^{\sigma\tau}
\end{aligned}$$

to preserve symmetry of indices in the δg terms. With $g_{\gamma\rho}g^{\alpha\gamma} = \delta^\alpha_\rho$ and applying (25) again we obtain

$$\begin{aligned}
\delta\mathcal{L}^{(1)}_T &= \left[g_{\gamma\rho}\,g^{\alpha\beta}\,\Gamma^\gamma_{\alpha\beta}(\partial) - \tfrac{1}{2}\,\Gamma^\beta_{\rho\beta}(\partial) - \tfrac{1}{2}\,\Gamma^\alpha_{\alpha\rho}(\partial)\right]\delta u^\rho \\
&\quad -\frac{1}{2}\left[g_{\gamma\sigma}\,g^{\alpha\beta}\,\Gamma^\gamma_{\alpha\beta}(\partial) - \tfrac{1}{2}\,\Gamma^\beta_{\sigma\beta}(\partial) - \tfrac{1}{2}\,\Gamma^\alpha_{\alpha\sigma}(\partial)\right]u_\tau\,\delta g^{\sigma\tau} \\
&\quad -\frac{1}{2}\left[g_{\gamma\tau}\,g^{\alpha\beta}\,\Gamma^\gamma_{\alpha\beta}(\partial) - \tfrac{1}{2}\,\Gamma^\beta_{\tau\beta}(\partial) - \tfrac{1}{2}\,\Gamma^\alpha_{\alpha\tau}(\partial)\right]u_\sigma\,\delta g^{\sigma\tau} \\
&\quad + \left[u_\gamma\,\Gamma^\gamma_{\alpha\beta}(\partial) - \tfrac{1}{2}\,u_\beta\,\Gamma^\gamma_{\alpha\gamma}(\partial) - \tfrac{1}{2}\,u_\alpha\,\Gamma^\gamma_{\gamma\beta}(\partial)\right]\delta g^{\alpha\beta} \\
&\quad + \left[g_{\alpha\beta}\,\nabla^\gamma\delta g^{\alpha\beta} - \tfrac{1}{2}\,\nabla_\beta\,\delta g^{\beta\gamma} - \tfrac{1}{2}\,\nabla_\alpha\,\delta g^{\alpha\gamma}\right]u_\gamma
\end{aligned} \tag{28}$$

The last line can be transformed by

$$\begin{aligned}
u_\gamma g_{\alpha\beta}\,\nabla^\gamma\delta g^{\alpha\beta} &= \nabla_\rho\left[u^\rho\, g_{\alpha\beta}\,\delta g^{\alpha\beta}\right] - \nabla_\rho u^\rho\, g_{\alpha\beta}\,\delta g^{\alpha\beta} \\
u_\gamma\left[\tfrac{1}{2}\,\nabla_\beta\,\delta g^{\beta\gamma} + \tfrac{1}{2}\,\nabla_\alpha\,\delta g^{\alpha\gamma}\right] &= \tfrac{1}{2}\,\nabla_\beta\left[u_\gamma\,\delta g^{\beta\gamma}\right] + \tfrac{1}{2}\,\nabla_\alpha\left[u_\gamma\,\delta g^{\alpha\gamma}\right] \\
&\quad -\tfrac{1}{2}\,\nabla_\beta u_\gamma\,\delta g^{\beta\gamma} - \tfrac{1}{2}\,\nabla_\alpha u_\gamma\,\delta g^{\alpha\gamma}
\end{aligned} \tag{29}$$

having used $\nabla^\gamma = g^{\rho\gamma}\,\nabla_\rho$ and $g^{\rho\gamma}\,u_\gamma = u^\rho$. After changing indices we get

$$\begin{aligned} \delta\mathcal{L}_T^{(1)} &= \left[u_\gamma\,\Gamma^\gamma_{\alpha\beta}(\partial) - \tfrac{1}{2}\,(u_\beta\,g_{\alpha\gamma} + u_\alpha\,g_{\beta\gamma})\,g^{\sigma\tau}\,\Gamma^\gamma_{\sigma\tau}(\partial) \right. \\ &\quad \left. + \tfrac{1}{2}\,(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) - \nabla_\gamma u^\gamma\,g_{\alpha\beta}\right]\delta g^{\alpha\beta} \\ &\quad + \left[g_{\gamma\rho}\,g^{\sigma\tau}\,\Gamma^\rho_{\sigma\tau}(\partial) - \Gamma^\rho_{\gamma\rho}(\partial)\right]\delta u^\gamma \\ &\quad + \nabla_\gamma\left[u^\gamma\,g_{\sigma\tau}\,\delta g^{\sigma\tau} - u_\rho\,\delta g^{\gamma\rho}\right] \end{aligned} \tag{30}$$

For the remaining parts we obtain

$$\begin{aligned} \delta\mathcal{L}_T^{(2)} &= 9\,\delta\left[g_{\gamma\rho}\,u^\rho\,u^\gamma\right] \\ &= 9\,g_{\gamma\rho}\left[\delta u^\rho\,u^\gamma + u^\rho\,\delta u^\gamma\right] - 9\,u^\gamma\,u^\rho\,g_{\gamma\sigma}\,g_{\rho\tau}\,\delta g^{\sigma\tau} \\ &= 9\left[2\,u_\gamma\,\delta u^\gamma - u_\alpha\,u_\beta\,\delta g^{\alpha\beta}\right] \end{aligned} \tag{31}$$

and

$$\delta\mathcal{K}_T^\gamma = -3\,\delta u^\gamma \tag{32}$$

From (26), (30), (31) and (32) we get

$$\begin{aligned} \delta\mathcal{L}_T &= \tfrac{1}{2}\left\{\mathcal{R}_{\alpha\beta}(\partial) + \mu\left[u_\gamma\,\Gamma^\gamma_{\alpha\beta}(\partial) - \tfrac{1}{2}\,(u_\beta\,g_{\alpha\gamma} + u_\alpha\,g_{\beta\gamma})\,g^{\sigma\tau}\,\Gamma^\gamma_{\sigma\tau}(\partial)\right.\right. \\ &\quad \left.\left. + \tfrac{1}{2}\,(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) - \nabla_\gamma u^\gamma\,g_{\alpha\beta}\right] - \tfrac{9}{2}\,\mu^2\,u_\alpha\,u_\beta\right\}\delta g^{\alpha\beta} \\ &\quad + \tfrac{1}{2}\,\mu\,\left[g_{\gamma\rho}\,g^{\sigma\tau}\,\Gamma^\rho_{\sigma\tau}(\partial) - \Gamma^\rho_{\gamma\rho}(\partial) + 9\,\mu\,u_\gamma\right]\delta u^\gamma \\ &\quad + \tfrac{1}{2}\,\nabla_\gamma\left[g_{\alpha\beta}\,\nabla^\gamma\delta g^{\alpha\beta} - \nabla_\beta\,\delta g^{\beta\gamma}\right. \\ &\qquad \left. +\ \mu\,\left(u^\gamma\,g_{\sigma\tau}\,\delta g^{\sigma\tau} - u_\rho\,\delta g^{\gamma\rho} - 3\,\delta u^\gamma\right)\right] \end{aligned} \tag{33}$$

We write the vector part of the Lagrangian (6) in the form

$$\mathcal{L}_V = -\tfrac{1}{2}\,\mathcal{L}_V^{(0)} - \mathcal{L}_V^{(1)} \tag{34}$$

with

$$\mathcal{L}_V^{(0)} = \tfrac{1}{2}\,(\nabla_\alpha u_\beta - \nabla_\beta u_\alpha)\,\left(\nabla^\alpha u^\beta - \nabla^\beta u^\alpha\right), \quad \mathcal{L}_V^{(1)} = \tfrac{1}{2}\,\nabla_\alpha u^\alpha\,\nabla_\beta u^\beta \tag{35}$$

The first part can be simplified by changing indices in two terms

$$\begin{aligned} \mathcal{L}_V^{(0)} &= \tfrac{1}{2}\,\left[\nabla_\alpha u_\beta\,\nabla^\alpha u^\beta + \nabla_\beta u_\alpha\,\nabla^\beta u^\alpha - \nabla_\alpha u_\beta\,\nabla^\beta u^\alpha - \nabla_\beta u_\alpha\,\nabla^\alpha u^\beta\right] \\ &= \nabla_\alpha u_\beta\,\left(\nabla^\alpha u^\beta - \nabla^\beta u^\alpha\right) = \nabla^\alpha u^\beta\,\left(\nabla_\alpha u_\beta - \nabla_\beta u_\alpha\right) \end{aligned} \tag{36}$$

With $u_\beta = g_{\beta\gamma}\, u^\gamma$ the variation gives

$$\begin{aligned}
\delta\mathcal{L}_V^{(0)} &= \delta g_{\beta\gamma}\nabla_\alpha u^\gamma \left(\nabla^\alpha u^\beta - \nabla^\beta u^\alpha\right) + g_{\beta\gamma}\nabla_\alpha \delta u^\gamma \left(\nabla^\alpha u^\beta - \nabla^\beta u^\alpha\right) \\
&\quad + \nabla_\alpha u_\beta \left(\nabla^\alpha \delta u^\beta - \nabla^\beta \delta u^\alpha\right) \\
&= -\,\delta g^{\rho\sigma}\, g_{\beta\rho}\, g_{\sigma\gamma}\nabla_\alpha u^\gamma \left(\nabla^\alpha u^\beta - \nabla^\beta u^\alpha\right) \\
&\quad + \nabla_\alpha \left[\delta u^\gamma \left(\nabla^\alpha u_\gamma - \nabla_\gamma u^\alpha\right)\right] - \,\delta u^\gamma\, \nabla_\alpha \left(\nabla^\alpha u_\gamma - \nabla_\gamma u^\alpha\right) \\
&\quad + \nabla_\alpha u_\beta \nabla_\gamma \left(g^{\gamma\alpha}\delta u^\beta - g^{\gamma\beta}\delta u^\alpha\right) \\
&= -\,\nabla_\alpha u_\sigma \left(\nabla^\alpha u_\rho - \nabla_\rho u^\alpha\right) \delta g^{\rho\sigma} \\
&\quad + \nabla_\alpha \left[\left(\nabla^\alpha u_\gamma - \nabla_\gamma u^\alpha\right) \delta u^\gamma\right] - \,\nabla_\alpha \left(\nabla^\alpha u_\gamma - \nabla_\gamma u^\alpha\right) \delta u^\gamma \\
&\quad + \nabla_\gamma \left[\nabla^\gamma u_\beta\, \delta u^\beta - \nabla_\alpha u^\gamma\, \delta u^\alpha\right] \\
&\quad - \left[\nabla_\gamma \nabla^\gamma u_\beta\, \delta u^\beta - \nabla_\gamma \nabla_\alpha u^\gamma\, \delta u^\alpha\right]
\end{aligned}$$

and after changing indices

$$\begin{aligned}
\delta\mathcal{L}_V^{(0)} &= -\,\nabla^\gamma u_\beta \left(\nabla_\gamma u_\alpha - \nabla_\alpha u_\gamma\right) \delta g^{\alpha\beta} - 2\,\nabla^\rho \left(\nabla_\rho u_\gamma - \nabla_\gamma u_\rho\right) \delta u^\gamma \\
&\quad + 2\,\nabla_\gamma \left[\left(\nabla^\gamma u_\rho - \nabla_\rho u^\gamma\right) \delta u^\rho\right] \qquad (37)
\end{aligned}$$

For the second part we get

$$\begin{aligned}
\delta\mathcal{L}_V^{(1)} &= \nabla_\rho u^\rho\, \nabla_\gamma \delta u^\gamma \\
&= \nabla_\gamma \left[\nabla^\rho u_\rho\, \delta u^\gamma\right] - \,\nabla_\gamma \left[\nabla^\rho u_\rho\right] \delta u^\gamma \qquad (38)
\end{aligned}$$

and together

$$\begin{aligned}
\delta\mathcal{L}_V &= -\tfrac{1}{2}\,\delta\mathcal{L}_V^{(0)} - \delta\mathcal{L}_V^{(1)} \\
&= \tfrac{1}{4}\,\left[\nabla^\gamma u_\beta \left(\nabla_\gamma u_\alpha - \nabla_\alpha u_\gamma\right) + \nabla^\gamma u_\alpha \left(\nabla_\gamma u_\beta - \nabla_\beta u_\gamma\right)\right] \delta g^{\alpha\beta} \\
&\quad + \nabla^\rho\, \nabla_\rho u_\gamma\, \delta u^\gamma - \,\nabla_\gamma \left[\left(\nabla^\gamma u_\rho - \nabla_\rho u^\gamma\right) \delta u^\rho + \nabla^\rho u_\rho\, \delta u^\gamma\right] \qquad (39)
\end{aligned}$$

Here we have used $\delta g^{\alpha\beta} = \delta g^{\beta\alpha}$ and changed the indices α, β in 1/2 of its coefficient to symmetrize the expression.

Splitting the scalar part of the Lagrangian (7) in the form

$$\mathcal{L}_S = -\tfrac{1}{2}\,\mathcal{L}_S^{(0)} - \mathcal{L}_S^{(1)} \qquad (40)$$

with

$$\mathcal{L}_S^{(0)} = \nabla_\alpha v\, \nabla^\alpha v, \quad \mathcal{L}_S^{(1)} = V(\mathcal{U})\, v^2, \qquad (41)$$

the variation gives

$$\begin{aligned}
\delta\mathcal{L}_S^{(0)} &= \delta\left[g^{\alpha\beta}\, \nabla_\alpha v\, \nabla_\beta v\right] \\
&= \delta g^{\alpha\beta}\, \nabla_\alpha v\, \nabla_\beta v + g^{\alpha\beta}\, \nabla_\alpha \delta v\, \nabla_\beta v + g^{\alpha\beta}\, \nabla_\alpha v\, \nabla_\beta \delta v \\
&= \nabla_\alpha v\, \nabla_\beta v\, \delta g^{\alpha\beta} + 2\, \nabla^\gamma v\, \nabla_\gamma \delta v \\
&= \nabla_\alpha v\, \nabla_\beta v\, \delta g^{\alpha\beta} + 2\, \nabla_\gamma \left[\nabla^\gamma v\, \delta v\right] - 2\, \nabla_\gamma \nabla^\gamma v\, \delta v \qquad (42)
\end{aligned}$$

and similar to $\delta\mathcal{L}_V^{(1)}$

$$\begin{aligned}
\delta\mathcal{L}_S^{(1)} &= \tfrac{1}{2}\,\delta\left[m^2\,v^2 + \nu^2\,\nabla_\gamma u^\gamma\,\nabla_\rho u^\rho\,v^2\right] \\
&= m^2\,v\,\delta v + \nu^2\,\nabla^\gamma u_\gamma\,\nabla^\rho u_\rho\,v\,\delta v + \nu^2\,\nabla_\gamma\left[\nabla^\rho u_\rho\,v^2\,\delta u^\gamma\right] \\
&\quad -\nu^2\,\nabla_\gamma\left[\nabla^\rho u_\rho\,v^2\right]\delta u^\gamma
\end{aligned} \tag{43}$$

Together we have

$$\begin{aligned}
\delta\mathcal{L}_S &= -\tfrac{1}{2}\,\delta\mathcal{L}_S^{(0)} - \delta\mathcal{L}_S^{(1)} \\
&= \left[\nabla_\gamma\nabla^\gamma v - m^2\,v - \nu^2\,\nabla^\gamma u_\gamma\,\nabla^\rho u_\rho\,v\right]\delta v \\
&\quad +\nu^2\,\nabla_\gamma\left[\nabla^\rho u_\rho\,v^2\right]\delta u^\gamma - \tfrac{1}{2}\,\nabla_\alpha v\,\nabla_\beta v\,\delta g^{\alpha\beta} \\
&\quad -\nabla_\gamma\left[\nabla^\gamma v\,\delta v + \nu^2\,\nabla^\rho u_\rho\,v^2\,\delta u^\gamma\right]
\end{aligned} \tag{44}$$

Finally, the variation of the matter part should give the stress energy tensor $\mathcal{T}$

$$\delta\left[\mathcal{L}_M\,\epsilon\right] = -\tfrac{1}{2}\,\kappa\,\mathcal{T}_{\alpha\beta}\,\epsilon\,\delta g^{\alpha\beta} \tag{45}$$

where $\kappa = 8\pi\,\frac{G}{c^4}$ with the gravitational constant G and the speed of light in vacuum c.

So we obtain the functions in (22) by (24), (33), (39), (44) and (45)

$$\begin{aligned}
\mathcal{M}_{\alpha\beta}^{(g)} &= \tfrac{1}{2}\,\mathcal{R}_{\alpha\beta}(\partial) - \tfrac{1}{2}\,\mathcal{L}\,g_{\alpha\beta} - \tfrac{1}{2}\,\kappa\,\mathcal{T}_{\alpha\beta} \\
&\quad + \tfrac{1}{2}\,\mu\left[u_\gamma\,\Gamma^\gamma_{\alpha\beta}(\partial) - \tfrac{1}{2}\,(u_\beta\,g_{\alpha\gamma} + u_\alpha\,g_{\beta\gamma})\,g^{\sigma\tau}\,\Gamma^\gamma_{\sigma\tau}(\partial)\right] \\
&\quad + \tfrac{1}{4}\,\mu\,(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) - \tfrac{1}{2}\,\mu\,\nabla_\gamma u^\gamma\,g_{\alpha\beta} - \tfrac{9}{4}\,\mu^2\,u_\alpha u_\beta \\
&\quad + \tfrac{1}{4}\,\left[\nabla^\gamma u_\beta\,(\nabla_\gamma u_\alpha - \nabla_\alpha u_\gamma) + \nabla^\gamma u_\alpha\,(\nabla_\gamma u_\beta - \nabla_\beta u_\gamma)\right] \\
&\quad - \tfrac{1}{2}\,\nabla_\alpha v\,\nabla_\beta v
\end{aligned} \tag{46}$$

$$\begin{aligned}
\mathcal{M}_\gamma^{(u)} &= \nabla_\rho\nabla^\rho u_\gamma + \tfrac{9}{2}\,\mu^2\,u_\gamma + \nu^2\,\nabla_\gamma\left[\nabla^\rho u_\rho\,v^2\right] \\
&\quad + \tfrac{1}{2}\,\mu\,\left[g_{\gamma\rho}\,g^{\sigma\tau}\,\Gamma^\rho_{\sigma\tau}(\partial) - \Gamma^\rho_{\gamma\rho}(\partial)\right]
\end{aligned} \tag{47}$$

$$\mathcal{M}^{(v)} = \nabla_\gamma\nabla^\gamma v - m^2\,v - \nu^2\,\nabla^\gamma u_\gamma\,\nabla^\rho u_\rho\,v \tag{48}$$

$$\begin{aligned}
\mathcal{K}^\gamma &= \tfrac{1}{2}\,\left[g_{\alpha\beta}\,\nabla^\gamma\delta g^{\alpha\beta} - \nabla_\beta\,\delta g^{\beta\gamma} + \mu\,\left(u^\gamma\,g_{\sigma\tau}\,\delta g^{\sigma\tau} - u_\rho\,\delta g^{\gamma\rho}\right)\right] \\
&\quad - (\nabla^\gamma u_\rho - \nabla_\rho u^\gamma)\,\delta u^\rho - \left(\nu^2\,v^2 + 1\right)\nabla^\rho u_\rho\,\delta u^\gamma - \tfrac{3}{2}\,\mu\,\delta u^\gamma \\
&\quad -\nabla^\gamma v\,\delta v
\end{aligned} \tag{49}$$

Inserting the parts of $\mathcal{L}$ from (7), (16) - (21), (36) we get the field equations defined by (23)

$$
\begin{aligned}
&\mathcal{R}_{\alpha\beta}(\partial) - \tfrac{1}{2}\, R(\partial)\, g_{\alpha\beta} - \tfrac{1}{2}\, \mu\, g^{\sigma\tau} \left[u_\gamma\, \Gamma^\gamma_{\sigma\tau}(\partial) - u_\sigma\, \Gamma^\gamma_{\tau\gamma}(\partial)\right]\, g_{\alpha\beta} \\
&+ \mu \left[u_\gamma\, \Gamma^\gamma_{\alpha\beta}(\partial) - \tfrac{1}{2}\, (u_\beta\, g_{\alpha\gamma} + u_\alpha\, g_{\beta\gamma})\, g^{\sigma\tau}\, \Gamma^\gamma_{\sigma\tau}(\partial)\right] \\
&+ \mu \left[\tfrac{1}{2}\, (\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) - \nabla_\gamma u^\gamma\, g_{\alpha\beta}\right] - \tfrac{9}{4}\, \mu^2\, (2\, u_\alpha u_\beta + u_\gamma u^\gamma\, g_{\alpha\beta}) \\
&+ \tfrac{1}{2}\, [\, \nabla^\gamma u_\beta\, (\nabla_\gamma u_\alpha - \nabla_\alpha u_\gamma) + \nabla^\gamma u_\alpha\, (\nabla_\gamma u_\beta - \nabla_\beta u_\gamma)] \\
&+ \tfrac{1}{2}\, \nabla^\gamma u^\rho\, (\nabla_\gamma u_\rho - \nabla_\rho u_\gamma)\, g_{\alpha\beta} - \nabla_\alpha v\, \nabla_\beta v + \tfrac{1}{2} \nabla_\gamma v\, \nabla^\gamma v\, g_{\alpha\beta} \\
&+ \tfrac{1}{2}\, m^2 v^2\, g_{\alpha\beta} + \tfrac{1}{2}\, \left(\nu^2 v^2 + 1\right)\, \nabla^\gamma u_\gamma \nabla^\rho u_\rho\, g_{\alpha\beta} \\
&\quad = \kappa\, \mathcal{T}_{\alpha\beta} \quad \text{for} \quad \alpha, \beta = 0, .., 3, \qquad (50)
\end{aligned}
$$

$$
\begin{aligned}
&\nabla_\rho \nabla^\rho u_\gamma + \tfrac{9}{2}\, \mu^2\, u_\gamma + \nu^2\, \nabla_\gamma \left[\nabla^\rho u_\rho\, v^2\right] \\
&\quad = \tfrac{1}{2}\, \mu \left[\Gamma^\rho_{\gamma\rho}(\partial) - g_{\gamma\rho}\, g^{\sigma\tau}\, \Gamma^\rho_{\sigma\tau}(\partial)\right] \quad \text{for} \ \ \gamma = 0, .., 3, \qquad (51)
\end{aligned}
$$

$$
\nabla_\gamma \nabla^\gamma v - m^2\, v - \nu^2\, \nabla^\gamma u_\gamma\, \nabla^\rho u_\rho\, v = 0 \qquad (52)
$$

4 Relation to General Relativity, the case of flat metric

We assume now, that μ and m are very small. For a fixed metric g we introduce an additional parameter λ with $0 \leq \lambda \leq 1$ and denote by $u_\alpha(x, \lambda)$, $v(x, \lambda)$ the solutions of equations (51), (52) for $\mu = \lambda\, \mu_0$, $m = \lambda\, m_0$ with some fixed μ_0, m_0. Omitting the part proportional to μ^2 in (51) the linearity of the equation yields the proportionality to μ for u_α. Putting this property into equation (52) we can also describe the dependence on λ of v:

$$
u_\alpha(x, \lambda) \sim \lambda\, u_\alpha(x, 1), \ v(x, \lambda) \sim v(y, 1), \ \nabla_\beta v(x, \lambda) \sim \lambda\, \nabla_\beta v(y, 1) \qquad (53)
$$

for $y = \lambda\, x$. With these results we can write (50) in the form

$$
\mathcal{R}_{\alpha\beta}(\partial) - \frac{1}{2}\, R(\partial)\, g_{\alpha\beta} - \kappa\, \mathcal{T}_{\alpha\beta} \sim \lambda^2\, \tilde{\mathcal{F}}^{(g)}_{\alpha\beta} \qquad (54)
$$

where $\tilde{\mathcal{F}}^{(g)}_{\alpha\beta}$ depends polynomially on λ, $u_\gamma(x, 1)$, $v(y, 1)$ and its first derivatives. So we can expect that the corresponding metric $g_{\alpha\beta}(x, \lambda)$ tends to the solution of General Relativity, if λ resp. μ and m tend to 0.

Now, we want to investigate the system of equations for the vector and the scalar field in the case of a flat metric $g = \eta = \text{diag}(-1, +1, +1, +1)$. The equations (51), (52) then take the form

$$
\begin{aligned}
\eta_\rho\, \partial^2_\rho\, u_\gamma + \tfrac{9}{2}\, \mu^2\, u_\gamma + \nu^2\, \eta_\rho\, \partial_\gamma \left[\partial_\rho u_\rho\, v^2\right] &= 0 \quad \text{for} \ \ \gamma = 0, .., 3, \qquad (55) \\
\eta_\gamma\, \partial^2_\gamma v - m^2\, v - \nu^2\, \eta_\gamma\, \eta_\rho\, \partial_\gamma u_\gamma\, \partial_\rho u_\rho\, v &= 0 \qquad (56)
\end{aligned}
$$

where for simplicity we wrote $\eta_\gamma := \eta_{\gamma\gamma}$.

The vector field $\mathcal{U}$ does not satisfy the condition of a nondynamical zero component imposed in [7], which means the equations for $\mathcal{U}$ should not include a term $\partial_0 u_0$. So we will have a closer look to this system. The equations (55) can be written in the form

$$\mathcal{P}(x,\partial)\,\mathcal{U} = p^{(2)}_{\alpha\beta}(x)\,\partial_\alpha\partial_\beta\,\mathcal{U} + p^{(1)}_{\gamma}(x)\,\partial_\gamma\,\mathcal{U} + p^{(0)}(x)\,\mathcal{U} = 0 \tag{57}$$

with 4×4 matrices $p^{(2)}_{\alpha\beta}$, $p^{(1)}_{\gamma}$, $p^{(0)}$. The character of this system is determined by its main part

$$\mathcal{P}^{(2)}(x,\partial) = p^{(2)}_{\alpha\beta}(x)\,\partial_\alpha\partial_\beta \tag{58}$$

and the corresponding main symbol $\mathcal{P}^{(2)}(x,\xi)$ with $\xi = \xi_\alpha\big|_{\alpha=0}^{3}$. The system is called hyperbolic, if for all x the equation

$$\det\mathcal{P}^{(2)}(x,\lambda,\hat{\xi}) = 0 \tag{59}$$

with $\hat{\xi} = \xi_i\big|_{i=1}^{3}$ has only real solutions λ and strictly hyperbolic, if additionally for $\hat{\xi} \neq 0$ all these solutions are different.

For (55) the main part is

$$\mathcal{P}^{(2)}(x,\partial)\,\mathcal{U} = \left[\eta_\rho\,\partial^2_\rho\,u_\gamma + \nu^2 v^2\,\eta_\rho\,\partial_\gamma\partial_\rho u_\rho\right]_{\gamma=0}^{3} \tag{60}$$

and the corresponding main symbol

$$\begin{aligned}\mathcal{P}^{(2)}(x,\xi) \quad &= \quad \left(\left|\hat{\xi}\right|^2 - \xi_0^2\right)\,\mathcal{I} \\ &\quad + \nu^2 v^2\begin{bmatrix} -\xi_0^2 & \xi_0\,\xi_1 & \xi_0\,\xi_2 & \xi_0\,\xi_3 \\ -\xi_0\,\xi_1 & \xi_1^2 & \xi_1\,\xi_2 & \xi_1\,\xi_3 \\ -\xi_0\,\xi_2 & \xi_1\,\xi_2 & \xi_2^2 & \xi_2\,\xi_3 \\ -\xi_0\,\xi_3 & \xi_1\,\xi_3 & \xi_2\,\xi_3 & \xi_3^2 \end{bmatrix}\end{aligned} \tag{61}$$

with the identical matrix $\mathcal{I}$. The equation

$$\det\mathcal{P}^{(2)}(x,\lambda,\hat{\xi}) = \left(\nu^2 v^2 + 1\right)\left(\left|\hat{\xi}\right|^2 - \lambda^2\right)^4 = 0 \tag{62}$$

has only real, but multiple solutions $\lambda = \left|\hat{\xi}\right|$ and $\lambda = -\left|\hat{\xi}\right|$. Hence, the system (55) is hyperbolic, but not strictly hyperbolic. For such systems there are a priori estimates showing that the solution does not grow exponential in case of bounded right hand side and initial conditions, for example in [8].

Next, we want to determine the number of degrees of freedom for the system (55), (56) using the Hamilton-Dirac analysis [9]. For this purpose we have to derive the Hamiltonian from the Lagrangian, which in the flat case $g = \eta$ is reduced to

$$\begin{aligned}\overline{\mathcal{L}} &= -\tfrac{1}{2}\,\eta_\alpha\eta_\beta\left[\partial_\alpha u_\beta\left(\partial_\alpha u_\beta - \partial_\beta u_\alpha\right) + \partial_\alpha u_\alpha\,\partial_\beta u_\beta\right] + \left(\tfrac{3}{2}\mu\right)^2\eta_\gamma u_\gamma^2 \\ &\quad -\tfrac{1}{2}\left[\eta_\alpha\left(\partial_\alpha v\right)^2 + \left(m^2 + \nu^2\eta_\alpha\eta_\beta\,\partial_\alpha u_\alpha\,\partial_\beta u_\beta\right)\,v^2\right] \end{aligned} \tag{63}$$

Here, we used equations (6), (7), (9), (16) – (20) and (36). In this expression we have to separate time derivatives $\partial_0 u_0 =: \dot{u}_0$, $\partial_0 u_i =: \dot{u}_i$, $i = 1,2,3$, $\partial_0 v =: \dot{v}$ from spatial derivatives $\partial_j u_0$, $\partial_j u_i$, $\partial_j v$, $i,j = 1,2,3$ and get

$$\begin{aligned}\overline{\mathcal{L}} &= \tfrac{1}{2}\sum_i\left(\partial_i u_0 - \dot{u}_i\right)^2 - \tfrac{1}{2}\sum_{i,j}\partial_i u_j\left(\partial_i u_j - \partial_j u_i\right) \\ &\quad -\tfrac{1}{2}\left(\dot{u}_0 - \sum_i\partial_i u_i\right)^2 + \left(\tfrac{3}{2}\,\mu\right)^2\left(u_0^2 - \sum_i u_i^2\right) \\ &\quad +\tfrac{1}{2}\,\dot{v}^2 - \tfrac{1}{2}\sum_i\left(\partial_i v\right)^2 - \tfrac{1}{2}\left[m^2 + \nu^2\left(\dot{u}_0 - \sum_i\partial_i u_i\right)^2\right]v^2 \end{aligned} \tag{64}$$

Determining the conjugate momenta

$$\pi_0 := \frac{\partial\overline{\mathcal{L}}}{\partial\dot{u}_0} = -\left(1 + \nu^2 v^2\right)\left(\dot{u}_0 - \sum_i\partial_i u_i\right) \tag{65}$$

$$\pi_i := \frac{\partial\overline{\mathcal{L}}}{\partial\dot{u}_i} = \dot{u}_i - \partial_i u_0 \tag{66}$$

$$\pi_4 := \frac{\partial\overline{\mathcal{L}}}{\partial\dot{v}} = \dot{v} \tag{67}$$

we obtain the base Hamiltonian defined by

$$\begin{aligned}\mathcal{H}_0 &:= \pi_0\,\dot{u}_0 + \sum_i\pi_i\,\dot{u}_i + \pi_4\,\dot{v} - \overline{\mathcal{L}} \\ &= -\tfrac{1}{2}\sum_i\left(\partial_i u_0 - \dot{u}_i\right)\left(\partial_i u_0 + \dot{u}_i\right) + \tfrac{1}{2}\sum_{i,j}\partial_i u_j\left(\partial_i u_j - \partial_j u_i\right) \\ &\quad + \left(\tfrac{3}{2}\,\mu\right)^2\left(u_0^2 - \sum_i u_i^2\right) + \tfrac{1}{2}\,\dot{v}^2 + \tfrac{1}{2}\sum_i\left(\partial_i v\right)^2 + \tfrac{1}{2}\,m^2 v^2 \\ &\quad -\tfrac{1}{2}\left(1 + \nu^2 v^2\right)\left(\dot{u}_0 - \sum_i\partial_i u_i\right)\left(\dot{u}_0 + \sum_i\partial_i u_i\right) \end{aligned} \tag{68}$$

The fact, that the matrix

$$\left.\frac{\partial^2\overline{\mathcal{L}}}{\partial\dot{u}_i\,\partial\dot{u}_j}\right|_{i,j=0}^{4} = \operatorname{diag}\left(-\left(1 + \nu^2 v^2\right), 1, 1, 1, 1\right) \tag{69}$$

with $\dot{u}_4 := \dot{v}$ is invertible implies, that there are no constraints. The base Hamiltonian is already the final Hamiltonian and can be expressed by spatial derivatives of the original fields and the conjugate momenta

$$\begin{aligned}\mathcal{H} = \mathcal{H}_0 &= -\tfrac{1}{2}\sum_i\pi_i\left(2\partial_i u_0 + \pi_i\right) + \tfrac{1}{2}\sum_{i,j}\partial_i u_j\left(\partial_i u_j - \partial_j u_i\right) \\ &\quad + \left(\tfrac{3}{2}\,\mu\right)^2\left(u_0^2 - \sum_i u_i^2\right) + \tfrac{1}{2}\pi_4^2 + \tfrac{1}{2}\sum_i\left(\partial_i v\right)^2 + \tfrac{1}{2}\,m^2 v^2 \\ &\quad +\tfrac{1}{2}\,\pi_0\left(2\sum_i\partial_i u_i - \frac{1}{1 + \nu^2 v^2}\,\pi_0\right) \end{aligned} \tag{70}$$

Without any constraints the number of degrees of freedom is equal to the number of field components, which is 5. This property, respectively the invertibility of matrix (69), should be stable under small perturbations and hence preserved for spacetime with small warping.

5 The spherically symmetric case

Usually, the spherically symmetric case is described by coordinates (t, r, ϕ, θ) which are defined by the relations

$$\tilde{x}_1 = r\,\cos\theta\,\cos\phi,\ \tilde{x}_2 = r\,\cos\theta\,\sin\phi,\ \tilde{x}_3 = r\,\sin\theta \tag{71}$$

to cartesian coordinates, and a metric tensor

$$\tilde{g} = \operatorname{diag}\left(g_0, g_1, r^2, r^2\cos^2\theta\right) \tag{72}$$

with functions $g_0 = g_0(r,t)$, $g_1 = g_1(r,t)$, But in our system of equations this is not suitable, since the spherical coordinates have an intrinsic warping. The problem can be seen in equation (51). Here, the right hand side should vanish in the flat case, but it does not in spherical coordinates. Therefore, we transform the metric (72) to Cartesian coordinates using the formula

$$g_{\alpha\beta} = \frac{\partial\tilde{x}^\mu}{\partial x^\alpha}\frac{\partial\tilde{x}^\nu}{\partial x^\beta}\,\tilde{g}_{\mu\nu} \tag{73}$$

and obtain

$$g_{00} = g_0,\ g_{0i} = g_{i0} = 0,\quad g^{00} = g_0^{-1},\ g^{0i} = g^{i0} = 0, \tag{74}$$

$$g_{ij} = \delta_{ij} + (g_1 - 1)\,\frac{x_i x_j}{r^2},\quad g^{ij} = \delta_{ij} + \left(g_1^{-1} - 1\right)\frac{x_i x_j}{r^2} \tag{75}$$

or

$$g = \begin{bmatrix} g_0 & 0 & 0 & 0 \\ 0 & (g_1-1)\,\frac{x_1^2}{r^2}+1 & (g_1-1)\,\frac{x_1x_2}{r^2} & (g_1-1)\,\frac{x_1x_3}{r^2} \\ 0 & (g_1-1)\,\frac{x_1x_2}{r^2} & (g_1-1)\,\frac{x_2^2}{r^2}+1 & (g_1-1)\,\frac{x_2x_3}{r^2} \\ 0 & (g_1-1)\,\frac{x_1x_3}{r^2} & (g_1-1)\,\frac{x_2x_3}{r^2} & (g_1-1)\,\frac{x_3^2}{r^2}+1 \end{bmatrix} \tag{76}$$

and a similar representation for g^{-1} with g_0 and g_1 replaced by its inverse g_0^{-1} and g_1^{-1}. A simple calculation shows

$$\det g = g_0\,g_1,\quad \det g^{-1} = g_0^{-1}\,g_1^{-1}. \tag{77}$$

In the following we will write a prime for differentiation with respect to r and a dot for differentiation with respect to t, i.e. for $a = a(r,t)$

$$\frac{\partial a}{\partial r} =: a', \quad \frac{\partial a}{\partial t} =: \dot{a}, \quad \frac{\partial^2 a}{\partial r^2} =: a'', \quad \frac{\partial^2 a}{\partial t^2} =: \ddot{a}, \quad \frac{\partial^2 a}{\partial r\, \partial t} =: \dot{a}'.$$

The Christoffel symbols (1) are

$$\Gamma^0_{00} = \tfrac{1}{2}\,\frac{\dot{g}_0}{g_0}, \quad \Gamma^0_{i0} = \tfrac{1}{2}\,\frac{g_0'}{g_0}\frac{x_i}{r}, \quad \Gamma^k_{00} = -\tfrac{1}{2}\,\frac{g_0'}{g_1}\frac{x_k}{r}, \quad \Gamma^k_{i0} = \tfrac{1}{2}\,\frac{\dot{g}_1}{g_1}\frac{x_i x_k}{r^2},$$
$$\Gamma^0_{ij} = -\tfrac{1}{2}\,\frac{\dot{g}_1}{g_0}\frac{x_i x_j}{r^2}, \quad \Gamma^k_{ij} = \left[\left(\frac{r}{2}\,\frac{g_1'}{g_1} + \frac{1}{g_1} - 1\right)\frac{x_i x_j}{r^2} + \delta_{ij}\left(1 - \frac{1}{g_1}\right)\right]\frac{x_k}{r^2} \quad (78)$$

Next, we can evaluate the Ricci tensor $\mathcal{R}_{\alpha\beta}$ (2) and the Ricci scalar R (4). First, for a simple representation we define an additional scalar $\tilde{R}$ by

$$\tilde{R} := \frac{1}{g_1}\left[-\frac{g_0''}{g_0} + \tfrac{1}{2}\,\frac{(g_0')^2}{g_0^2} + \tfrac{1}{2}\,\frac{g_0' g_1'}{g_0 g_1}\right] + \frac{1}{g_0}\left[-\frac{\ddot{g}_1}{g_1} + \tfrac{1}{2}\,\frac{(\dot{g}_1)^2}{g_1^2} + \tfrac{1}{2}\,\frac{\dot{g}_0 \dot{g}_1}{g_0 g_1}\right] \quad (79)$$

Then, we obtain

$$\mathcal{R}_{00} = \tfrac{1}{2}\, g_0\, \tilde{R} - \frac{1}{r}\,\frac{g_0'}{g_1}, \quad \mathcal{R}_{0i} = \mathcal{R}_{i0} = \frac{\dot{g}_1}{g_1}\frac{x_i}{r^2}, \quad (80)$$

$$\mathcal{R}_{ij} = \left[\tfrac{1}{2}\, g_1\, \tilde{R} + \frac{1}{r}\,\frac{g_1'}{g_1}\right]\frac{x_i x_j}{r^2} + \frac{1}{2 r g_1}\left[\frac{g_1'}{g_1} - \frac{g_0'}{g_0} + \frac{2}{r}\,(g_1 - 1)\right]\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right) \quad (81)$$

and

$$R = \tilde{R} + \frac{2}{r g_1}\left[\frac{g_1'}{g_1} - \frac{g_0'}{g_0} + \frac{1}{r}\,(g_1 - 1)\right] \quad (82)$$

The terms $\mathcal{R}_{\alpha\beta} - \frac{1}{2}\, R\, g_{\alpha\beta}$ with non-vanishing $g_{\alpha\beta}$ are

$$\mathcal{R}_{00} - \tfrac{1}{2}\, R\, g_{00} = \frac{1}{r}\,\frac{g_0}{g_1}\left[\frac{g_1'}{g_1} + \frac{1}{r}\,(g_1 - 1)\right] \quad (83)$$

$$\mathcal{R}_{ij} - \tfrac{1}{2}\, R\, g_{ij} = \frac{1}{r}\left[\frac{g_0'}{g_0} - \frac{1}{r}\,(g_1 - 1)\right]\frac{x_i x_j}{r^2} - \tfrac{1}{2}\left[\tilde{R} + \frac{1}{r g_1}\left(\frac{g_1'}{g_1} - \frac{g_0'}{g_0}\right)\right]\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right) \quad (84)$$

In the spherically symmetric case the vector and scalar fields can be represented in the form

$$u_0 = u_o(r,t), \quad u_i = u_e(r,t)\,\frac{x_i}{r}, \quad v = v(r,t) \quad (85)$$

with scalar functions u_o, u_e and v. Now, we may calculate some terms of the equations (50) - (52). Such containing no or only linear vector field components are

$$\mathcal{N}^{(1)}(\mathcal{U}) \;:=\; g^{\sigma\tau}\left[u_\gamma\,\Gamma^\gamma_{\sigma\tau}(\partial) - u_\sigma\,\Gamma^\gamma_{\tau\gamma}(\partial)\right] \tag{86}$$

$$\mathcal{N}^{(2)}_{\alpha\beta}(\mathcal{U}) \;:=\; u_\gamma\,\Gamma^\gamma_{\alpha\beta}(\partial) \tag{87}$$

$$\mathcal{N}^{(3)}_\alpha \;:=\; g_{\alpha\gamma}g^{\sigma\tau}\,\Gamma^\gamma_{\sigma\tau}(\partial) \tag{88}$$

$$\mathcal{N}^{(4)}_\gamma \;:=\; \Gamma^\rho_{\gamma\rho}(\partial) - g_{\gamma\rho}\,g^{\sigma\tau}\,\Gamma^\rho_{\sigma\tau}(\partial) \tag{89}$$

In this case we get

$$\mathcal{N}^{(1)}(\mathcal{U}) = -\frac{1}{g_0\,g_1}\left(\dot g_1 u_o + g_0' u_e\right) + \frac{2}{r}\left(1 - \frac{1}{g_1}\right)u_e \tag{90}$$

$$\mathcal{N}^{(2)}_{00}(\mathcal{U}) \;=\; \tfrac{1}{2}\left[\frac{\dot g_0}{g_0}u_o - \frac{g_0'}{g_0}u_e\right] \tag{91}$$

$$\mathcal{N}^{(2)}_{0i}(\mathcal{U}) = \mathcal{N}^{(2)}_{i0}(\mathcal{U}) \;=\; \tfrac{1}{2}\left[\frac{g_0'}{g_0}u_o + \frac{\dot g_1}{g_1}u_e\right]\frac{x_i}{r} \tag{92}$$

$$\begin{aligned}\mathcal{N}^{(2)}_{ij}(\mathcal{U}) \;=\;& \tfrac{1}{2}\left[-\frac{\dot g_1}{g_0}u_o + \frac{g_1'}{g_1}u_e\right]\frac{x_i x_j}{r^2} \\ &+\frac{1}{r}\left(1 - \frac{1}{g_1}\right)\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right)u_e\end{aligned} \tag{93}$$

$$\mathcal{N}^{(3)}_0 \;=\; \tfrac{1}{2}\left(\frac{\dot g_0}{g_0} - \frac{\dot g_1}{g_1}\right) \tag{94}$$

$$\mathcal{N}^{(3)}_i \;=\; \left[\tfrac{1}{2}\left(-\frac{g_0'}{g_0} + \frac{g_1'}{g_1}\right) + \frac{2}{r}\,(g_1 - 1)\right]\frac{x_i}{r} \tag{95}$$

$$\mathcal{N}^{(4)}_0 = \frac{\dot g_1}{g_1}\,, \quad \mathcal{N}^{(4)}_i = \left[\frac{g_0'}{g_0} - \frac{2}{r}\,(g_1 - 1)\right]\frac{x_i}{r} \tag{96}$$

The quadratic term is

$$\mathcal{N}^{(5)}(\mathcal{U}) := u_\gamma u^\gamma \;=\; \frac{1}{g_0}u_o^2 + \frac{1}{g_1}u_e^2 \tag{97}$$

For the terms with derivatives of first order

$$\mathcal{N}^{(6)}(\mathcal{U},\partial\mathcal{U}) \;:=\; \nabla_\gamma u^\gamma \tag{98}$$

$$\mathcal{N}^{(7)}_{\alpha\beta}(\mathcal{U},\partial\mathcal{U}) \;:=\; \nabla^\gamma u_\beta\left(\nabla_\gamma u_\alpha - \nabla_\alpha u_\gamma\right) \tag{99}$$

$$\mathcal{N}^{(8)}(\mathcal{U},\partial\mathcal{U}) \;:=\; \nabla^\gamma u^\rho\left(\nabla_\gamma u_\rho - \nabla_\rho u_\gamma\right) \tag{100}$$

$$\mathcal{N}^{(9)}(v,\partial v) \;:=\; \nabla_\gamma v\,\nabla^\gamma v \tag{101}$$

we obtain

$$\mathcal{N}^{(6)}(\mathcal{U},\partial\mathcal{U}) = \frac{\dot{u}_o}{g_0} + \frac{u_e'}{g_1} + \tfrac{1}{2}\left(-\frac{\dot{g}_0}{g_0} + \frac{\dot{g}_1}{g_1}\right)\frac{u_o}{g_0} + \tfrac{1}{2}\left(\frac{g_0'}{g_0} - \frac{g_1'}{g_1} + \frac{4}{r}\right)\frac{u_e}{g_1} \quad (102)$$

$$\begin{aligned}
\mathcal{N}^{(7)}_{00}(\mathcal{U},\partial\mathcal{U}) &= \frac{1}{g_1}\left[u_o' - \tfrac{1}{2}\left(\frac{g_0'}{g_0}u_o + \frac{\dot{g}_1}{g_1}u_e\right)\right](u_o' - \dot{u}_e) & (103)\\
\mathcal{N}^{(7)}_{i0}(\mathcal{U},\partial\mathcal{U}) &= \frac{1}{g_0}\left[-\dot{u}_o + \tfrac{1}{2}\left(\frac{\dot{g}_0}{g_0}u_o - \frac{g_0'}{g_1}u_e\right)\right](u_o' - \dot{u}_e)\frac{x_i}{r} & (104)\\
\mathcal{N}^{(7)}_{0j}(\mathcal{U},\partial\mathcal{U}) &= \frac{1}{g_1}\left[u_e' + \tfrac{1}{2}\left(\frac{\dot{g}_1}{g_0}u_o - \frac{g_1'}{g_1}u_e\right)\right](u_o' - \dot{u}_e)\frac{x_j}{r} & (105)\\
\mathcal{N}^{(7)}_{ij}(\mathcal{U},\partial\mathcal{U}) &= \frac{1}{g_0}\left[-\dot{u}_e + \tfrac{1}{2}\left(\frac{g_0'}{g_0}u_o + \frac{\dot{g}_1}{g_1}u_e\right)\right](u_o' - \dot{u}_e)\frac{x_i x_j}{r^2} & (106)
\end{aligned}$$

$$\mathcal{N}^{(8)}(\mathcal{U},\partial\mathcal{U}) = \frac{1}{g_0\,g_1}(u_o' - \dot{u}_e)^2\,, \quad \mathcal{N}^{(9)}(v,\partial v) \;=\; \frac{1}{g_0}(\dot{v})^2 + \frac{1}{g_1}(v')^2 \quad (107)$$

Terms with second order derivatives of the vector or the scalar field are

$$\mathcal{N}^{(10)}_\gamma(\mathcal{U},\partial\mathcal{U},\partial^2\mathcal{U}) := \nabla_\rho\nabla^\rho u_\gamma, \quad \mathcal{N}^{(11)}(v,\partial v,\partial^2 v) := \nabla_\gamma\nabla^\gamma v \quad (108)$$

We get

$$\begin{aligned}
\mathcal{N}^{(10)}_0(\mathcal{U},\partial\mathcal{U},\partial^2\mathcal{U}) &= \frac{1}{g_0}\ddot{u}_o + \frac{1}{g_1}u_o'' + \frac{1}{g_0}\left(-\tfrac{3}{2}\frac{\dot{g}_0}{g_0} + \tfrac{1}{2}\frac{\dot{g}_1}{g_1}\right)\dot{u}_o + \\
&\quad + \frac{1}{g_1}\left(-\tfrac{1}{2}\frac{g_0'}{g_0} - \tfrac{1}{2}\frac{g_1'}{g_1} + \frac{2}{r}\right)u_o' + \\
&\quad + \frac{1}{g_1}\left(\frac{g_0'}{g_0}\dot{u}_e - \frac{\dot{g}_1}{g_1}u_e'\right) + \mathcal{S}^{(o)}_0 u_o + \mathcal{S}^{(e)}_0 u_e & (109)\\
\mathcal{N}^{(10)}_i(\mathcal{U},\partial\mathcal{U},\partial^2\mathcal{U}) &= \left[\frac{1}{g_0}\ddot{u}_e + \frac{1}{g_1}u_e'' + \frac{1}{g_0}\left(-\tfrac{1}{2}\frac{\dot{g}_0}{g_0} - \tfrac{1}{2}\frac{\dot{g}_1}{g_1}\right)\dot{u}_e + \right.\\
&\quad + \frac{1}{g_1}\left(\tfrac{1}{2}\frac{g_0'}{g_0} - \tfrac{3}{2}\frac{g_1'}{g_1} + \frac{2}{r}\right)u_e' + \\
&\quad \left. + \frac{1}{g_0}\left(-\frac{g_0'}{g_0}\dot{u}_o + \frac{\dot{g}_1}{g_1}u_o'\right) + \mathcal{S}^{(o)}_1 u_o + \mathcal{S}^{(e)}_1 u_e\right]\frac{x_i}{r} & (110)\\
\mathcal{N}^{(11)}(v,\partial v,\partial^2 v) &= \frac{1}{g_0}\ddot{v} + \frac{1}{g_1}v'' + \frac{2}{r}v' & (111)
\end{aligned}$$

where

$$\mathcal{S}_0^{(o)} = \frac{1}{g_0}\left[-\tfrac{1}{2}\frac{\ddot g_0}{g_0} + \left(\frac{\dot g_0}{g_0}\right)^2 - \tfrac{1}{4}\frac{\dot g_0\,\dot g_1}{g_0\,g_1} - \tfrac{1}{4}\left(\frac{\dot g_1}{g_1}\right)^2\right] + \frac{1}{g_1}\left[-\tfrac{1}{2}\frac{g_0''}{g_0} + \tfrac{1}{4}\left(\frac{g_0'}{g_0}\right)^2 + \tfrac{1}{4}\frac{g_0'\,g_1'}{g_0\,g_1} - \frac{1}{r}\frac{g_0'}{g_0}\right] \tag{112}$$

$$\mathcal{S}_0^{(e)} = \frac{1}{g_1}\left[\tfrac{1}{2}\frac{\dot g_0'}{g_0} - \tfrac{1}{2}\frac{\dot g_1'}{g_1} - \tfrac{1}{2}\frac{\dot g_0\,g_0'}{g_0^2} - \tfrac{1}{2}\frac{g_0'\,\dot g_1}{g_0\,g_1} + \frac{\dot g_1\,g_1'}{g_1^2} - \frac{1}{r}\frac{\dot g_1}{g_1}\right] \tag{113}$$

$$\mathcal{S}_1^{(o)} = \frac{1}{g_0}\left[-\tfrac{1}{2}\frac{\dot g_0'}{g_0} + \tfrac{1}{2}\frac{\dot g_1'}{g_1} + \frac{\dot g_0\,g_0'}{g_0^2} - \tfrac{1}{2}\frac{g_0'\,\dot g_1}{g_0\,g_1} - \tfrac{1}{2}\frac{\dot g_1\,g_1'}{g_1^2} + \frac{1}{r}\frac{\dot g_1}{g_1}\right] \tag{114}$$

$$\mathcal{S}_1^{(e)} = \frac{1}{g_1}\left[-\tfrac{1}{2}\frac{g_1''}{g_1} - \tfrac{1}{4}\left(\frac{g_0'}{g_0}\right)^2 - \tfrac{1}{4}\frac{g_0'\,g_1'}{g_0\,g_1} + \left(\frac{g_1'}{g_1}\right)^2 - \frac{1}{r}\frac{g_1'}{g_1} - \frac{2}{r^2}\right] + \frac{1}{g_0}\left[-\tfrac{1}{2}\frac{\ddot g_1}{g_1} + \tfrac{1}{4}\left(\frac{\dot g_1}{g_1}\right)^2 + \tfrac{1}{4}\frac{\dot g_0\,\dot g_1}{g_0\,g_1}\right] \tag{115}$$

With a reduced Lagrangian, defined by

$$\tilde{\mathcal{L}} = \tfrac{1}{2}\mathcal{N}^{(8)}(\mathcal{U},\partial\mathcal{U}) - \mu\,\mathcal{N}^{(6)}(\mathcal{U},\partial\mathcal{U}) - \tfrac{1}{2}\mu\,\mathcal{N}^{(1)}(\mathcal{U}) - \tfrac{9}{4}\mu^2\mathcal{N}^{(5)}(\mathcal{U}) + \tfrac{1}{2}\left(\nu^2 v^2 + 1\right)\left[\mathcal{N}^{(6)}(\mathcal{U},\partial\mathcal{U})\right]^2 + \tfrac{1}{2}\mathcal{N}^{(9)}(v,\partial v) + \tfrac{1}{2}m^2 v^2 \tag{116}$$

$$= \tfrac{1}{2}\frac{1}{g_0\,g_1}\left(u_o' - \dot u_e\right)^2 + \tfrac{1}{2}\left[\frac{1}{g_0}(\dot v)^2 + \frac{1}{g_1}(v')^2\right] + \tfrac{1}{2}\left(\nu^2 v^2 + 1\right) * \\ * \left[\frac{1}{g_0}\dot u_o + \frac{1}{g_1}u_e' + \tfrac{1}{2}\left(-\frac{\dot g_0}{g_0} + \frac{\dot g_1}{g_1}\right)\frac{u_o}{g_0} + \tfrac{1}{2}\left(\frac{g_0'}{g_0} - \frac{g_1'}{g_1} + \frac{4}{r}\right)\frac{u_e}{g_1}\right]^2 \\ + \mu\left[-\frac{1}{g_0}\dot u_o - \frac{1}{g_1}u_e' + \tfrac{1}{2}\frac{\dot g_0}{g_0^2}u_o + \tfrac{1}{2}\frac{g_1'}{g_1^2}u_e - \frac{1}{r}\left(1 - \frac{1}{g_1}\right)u_e\right] \\ - \tfrac{9}{4}\mu^2\left(\frac{1}{g_0}u_o^2 + \frac{1}{g_1}u_e^2\right) + \tfrac{1}{2}m^2 v^2 \tag{117}$$

we can write equation (50) in the form

$$\mathcal{R}_{\alpha\beta}(\partial) - \tfrac{1}{2}R(\partial)\,g_{\alpha\beta} + \tilde{\mathcal{L}}\,g_{\alpha\beta} + \tfrac{1}{2}\left[\mathcal{N}^{(7)}_{\alpha\beta}(\mathcal{U},\partial\mathcal{U}) + \mathcal{N}^{(7)}_{\beta\alpha}(\mathcal{U},\partial\mathcal{U})\right] \\ - \nabla_\alpha v\,\nabla_\beta v + \mu\left[\mathcal{N}^{(2)}_{\alpha\beta}(\mathcal{U}) - \tfrac{1}{2}\left(u_\beta\,\mathcal{N}^{(3)}_\alpha + u_\alpha\,\mathcal{N}^{(3)}_\beta\right)\right] \\ + \tfrac{1}{2}\mu\left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha\right) - \tfrac{9}{2}\mu^2\,u_\alpha u_\beta = \kappa\,\mathcal{T}_{\alpha\beta} \tag{118}$$

For indices $(\alpha,\beta)=(0,0)$ we get

$$\begin{aligned}
&\frac{1}{r}\,\frac{g_0}{g_1}\left[\frac{g_1'}{g_1}+\frac{1}{r}\,(g_1-1)\right]+\tilde{\mathcal{L}}\,g_0+\frac{1}{g_1}\left[u_o'-\tfrac{1}{2}\left(\frac{g_0'}{g_0}u_o+\frac{\dot{g}_1}{g_1}u_e\right)\right]*\\
&*\,(u_o'-\dot{u}_e)+\mu\left[\dot{u}_o+\tfrac{1}{2}\left(-\frac{\dot{g}_0}{g_0}+\frac{\dot{g}_1}{g_1}\right)u_o+\tfrac{1}{2}\left(-\frac{g_0'}{g_0}+\frac{g_0'}{g_1}\right)u_e\right]\\
&-\tfrac{9}{2}\,\mu^2\,u_o^2-\dot{v}^2\;=\;\kappa\,\mathcal{T}_{00}\;,
\end{aligned}\tag{119}$$

for $(\alpha,\beta)=(i,0)$ or $(\alpha,\beta)=(0,i)$

$$\begin{aligned}
&\left\{\frac{1}{r}\,\frac{\dot{g}_1}{g_1}+\tfrac{1}{2}\left[-\frac{1}{g_0}\dot{u}_o+\frac{1}{g_1}u_e'+\tfrac{1}{2}\,\frac{1}{g_0}\left(\frac{\dot{g}_0}{g_0}+\frac{\dot{g}_1}{g_1}\right)u_o-\right.\right.\\
&-\tfrac{1}{2}\,\frac{1}{g_1}\left.\left(\frac{g_0'}{g_0}+\frac{g_1'}{g_1}\right)u_e\right](u_o'-\dot{u}_e)+\tfrac{1}{2}\,\mu\left[u_o'+\dot{u}_e+\right.\\
&+\left.\left(\tfrac{1}{2}\,\frac{g_0'}{g_0}-\tfrac{1}{2}\,\frac{g_1'}{g_1}-\frac{2}{r}\,(g_1-1)\right)u_o+\left(-\tfrac{1}{2}\,\frac{\dot{g}_0}{g_0}+\frac{\dot{g}_1}{g_1}\right)u_e\right]\\
&\left.-\tfrac{9}{2}\,\mu^2\,u_o\,u_e-\dot{v}\,v'\right\}\frac{x_i}{r}\;=\;\kappa\,\mathcal{T}_{i0}\;,
\end{aligned}\tag{120}$$

and for $(\alpha,\beta)=(i,j)$

$$\begin{aligned}
&\frac{1}{r}\left[\frac{g_0'}{g_0}-\frac{1}{r}\,(g_1-1)\right]\frac{x_ix_j}{r^2}-\tfrac{1}{2}\left[\tilde{R}+\frac{1}{rg_1}\left(\frac{g_1'}{g_1}-\frac{g_0'}{g_0}\right)\right]\left(\delta_{ij}-\frac{x_ix_j}{r^2}\right)\\
&+\tilde{\mathcal{L}}\left[\delta_{ij}+(g_1-1)\,\frac{x_ix_j}{r^2}\right]+\frac{1}{g_0}\left[-\dot{u}_e+\tfrac{1}{2}\left(\frac{g_0'}{g_0}u_o+\frac{\dot{g}_1}{g_1}u_e\right)\right]*\\
&*\,(u_o'-\dot{u}_e)\,\frac{x_ix_j}{r^2}+\mu\left[u_e'+\left(\tfrac{1}{2}\frac{g_0'}{g_0}-\tfrac{1}{2}\frac{g_1'}{g_1}-\frac{2}{r}\,(g_1-1)\right)u_e\right]\frac{x_ix_j}{r^2}\\
&+\mu\,\frac{2}{r}\,u_e\left(\delta_{ij}-\frac{x_ix_j}{r^2}\right)-\tfrac{9}{2}\,\mu^2\,u_e^2\,\frac{x_ix_j}{r^2}-(v')^2\,\frac{x_ix_j}{r^2}\;=\;\kappa\,\mathcal{T}_{ij}
\end{aligned}\tag{121}$$

For equation (51) in the case $\gamma=0$ we obtain

$$\begin{aligned}
&\frac{1}{g_0}\ddot{u}_o+\frac{1}{g_1}u_o''+\frac{1}{g_0}\left(-\tfrac{3}{2}\,\frac{\dot{g}_0}{g_0}+\tfrac{1}{2}\,\frac{\dot{g}_1}{g_1}\right)\dot{u}_o+\frac{1}{g_1}\left(-\tfrac{1}{2}\,\frac{g_0'}{g_0}-\tfrac{1}{2}\,\frac{g_1'}{g_1}+\frac{2}{r}\right)u_o'\\
&+\frac{1}{g_1}\left(\frac{g_0'}{g_0}\dot{u}_e-\frac{\dot{g}_1}{g_1}u_e'\right)+\mathcal{S}_0^{(o)}u_o+\mathcal{S}_0^{(e)}u_e+\tfrac{9}{2}\,\mu^2\,u_o+\nu^2\,*\\
&*\frac{\partial}{\partial t}\left\{\left[\frac{1}{g_0}\dot{u}_o+\frac{1}{g_1}u_e'+\tfrac{1}{2}\left(-\frac{\dot{g}_0}{g_0}+\frac{\dot{g}_1}{g_1}\right)\frac{u_o}{g_0}+\tfrac{1}{2}\left(\frac{g_0'}{g_0}-\frac{g_1'}{g_1}+\frac{4}{r}\right)\frac{u_e}{g_1}\right]v^2\right\}\\
&=\tfrac{1}{2}\,\mu\,\frac{\dot{g}_1}{g_1}
\end{aligned}\tag{122}$$

and for $\gamma = i$

$$
\begin{aligned}
&\frac{1}{g_0}\ddot{u}_e + \frac{1}{g_1}u_e'' - \frac{1}{g_0}\left(\tfrac{1}{2}\,\frac{\dot{g}_0}{g_0} + \tfrac{1}{2}\,\frac{\dot{g}_1}{g_1}\right)\dot{u}_e + \frac{1}{g_1}\left(\tfrac{1}{2}\,\frac{g_0'}{g_0} - \tfrac{3}{2}\,\frac{g_1'}{g_1} + \frac{2}{r}\right)u_e' \\
&+\frac{1}{g_0}\left(-\frac{g_0'}{g_0}\,\dot{u}_o + \frac{\dot{g}_1}{g_1}\,u_o'\right) + \mathcal{S}_1^{(o)}u_o + \mathcal{S}_1^{(e)}u_e + \tfrac{9}{2}\,\mu^2\,u_e + \nu^2 * \\
&*\frac{\partial}{\partial r}\left\{\left[\frac{1}{g_0}\dot{u}_o + \frac{1}{g_1}u_e' + \tfrac{1}{2}\left(-\frac{\dot{g}_0}{g_0} + \frac{\dot{g}_1}{g_1}\right)\frac{u_o}{g_0} + \tfrac{1}{2}\left(\frac{g_0'}{g_0} - \frac{g_1'}{g_1} + \frac{4}{r}\right)\frac{u_e}{g_1}\right]v^2\right\} \\
&= \tfrac{1}{2}\,\mu\left[\frac{g_0'}{g_0} - \frac{2}{r}\,(g_1 - 1)\right] \qquad (123)
\end{aligned}
$$

where we omitted the factor $\frac{x_i}{r}$ on both sides. Finally, equation (52) gives

$$
\begin{aligned}
&\frac{1}{g_0}\ddot{v} + \frac{1}{g_1}v'' + \frac{2}{r}v' - m^2\,v \\
&-\nu^2\left[\frac{1}{g_0}\dot{u}_o + \frac{1}{g_1}u_e' + \tfrac{1}{2}\,\left(-\frac{\dot{g}_0}{g_0} + \frac{\dot{g}_1}{g_1}\right)\frac{u_o}{g_0} + \tfrac{1}{2}\left(\frac{g_0'}{g_0} - \frac{g_1'}{g_1} + \frac{4}{r}\right)\frac{u_e}{g_1}\right]^2 v \\
&= 0 \qquad (124)
\end{aligned}
$$

Now, we consider the stationary case $g_0 = g_0(r)$, $g_1 = g_1(r)$, $u_o = u_o(r)$, $u_e = u_e(r)$ and $v = v(r)$. After multiplication with g_1 equations (122) - (124) are reduced to

$$
\begin{aligned}
u_o'' &+ \left(-\tfrac{1}{2}\frac{g_0'}{g_0} - \tfrac{1}{2}\frac{g_1'}{g_1} + \frac{2}{r}\right)u_o' \\
&+ \left[-\tfrac{1}{2}\frac{g_0''}{g_0} + \tfrac{1}{4}\left(\frac{g_0'}{g_0}\right)^2 + \tfrac{1}{4}\frac{g_0'\,g_1'}{g_0\,g_1} - \frac{1}{r}\frac{g_0'}{g_0}\right]u_o + \tfrac{9}{2}\,\mu^2\,g_1\,u_o = 0 \qquad (125)
\end{aligned}
$$

$$
\begin{aligned}
u_e'' &+ \left(\tfrac{1}{2}\frac{g_0'}{g_0} - \tfrac{3}{2}\frac{g_1'}{g_1} + \frac{2}{r}\right)u_e' \\
&+ \left[-\tfrac{1}{2}\frac{g_1''}{g_1} - \tfrac{1}{4}\left(\frac{g_0'}{g_0}\right)^2 - \tfrac{1}{4}\frac{g_0'\,g_1'}{g_0\,g_1} + \left(\frac{g_1'}{g_1}\right)^2 - \frac{1}{r}\frac{g_1'}{g_1} - \frac{2}{r^2}\right]u_e \\
&+\tfrac{9}{2}\,\mu^2\,g_1\,u_e + \nu^2 g_1\,\frac{\partial}{\partial r}\left\{\frac{1}{g_1}\left[u_e' + \tfrac{1}{2}\left(\frac{g_0'}{g_0} - \frac{g_1'}{g_1} + \frac{4}{r}\right)u_e\right]v^2\right\} \\
&= \tfrac{1}{2}\,\mu\,g_1\left[\frac{g_0'}{g_0} - \frac{2}{r}\,(g_1 - 1)\right] \qquad (126)
\end{aligned}
$$

$$
v'' + \frac{2}{r}\,g_1\,v' - m^2\,g_1\,v - \nu^2\frac{1}{g_1}\left[u_e' + \tfrac{1}{2}\left(\frac{g_0'}{g_0} - \frac{g_1'}{g_1} + \frac{4}{r}\right)u_e\right]^2 v = 0 \quad (127)
$$

We see that in the system (125) - (127) the function u_o is decoupled from u_e and v. So, from the vanishing right hand side of (125) we can derive $u_o = 0$.

To describe functions as power series with logarithmic factors by terms of highest order we will use the following notation. For two functions $f^{(0)}$, $f^{(1)}$ defined by converging series

$$f^{(i)}(r) = \sum_{k,l\geq 0} f^{(i)}_{kl}\, r^{p_i-k}\,(\ln r)^{q_i-l} \quad i = 1,2 \quad \text{we write}$$

$$f_1 \sim f_2 \quad \text{if} \quad p_1 = p_2,\ q_1 = q_2,\ f^{(1)}_{00} = f^{(2)}_{00} \tag{128}$$

Some simple properties are

$$f_1 + f_2 \sim \begin{cases} f^{(1)}_{00}\, r^{p_1}\,(\ln r)^{q_1} & \text{for } p_1 = p_2 \text{ and } q_1 > q_2 \\ & \text{or } p_1 > p_2 \\ \left(f^{(1)}_{00} + f^{(2)}_{00}\right) r^{p_1}\,(\ln r)^{q_1} & \text{for } p_1 = p_2 \text{ and } q_1 = q_2 \end{cases} \tag{129}$$

$$f_1 f_2 \sim f^{(1)}_{00} f^{(2)}_{00}\, r^{p_1+p_2}\,(\ln r)^{q_1+q_2}, \tag{130}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}r} f_i \sim \begin{cases} f^{(i)}_{00}\, p_i\, r^{p_i-1}\,(\ln r)^{q_i} & \text{for } p_i \neq 0 \text{ or } p_i = 0, q_i = 0 \\ f^{(i)}_{00}\, q_i\, r^{-1}\,(\ln r)^{q_i-1} & \text{for } p_i = 0 \text{ and } q_i \neq 0 \end{cases} \tag{131}$$

From (54) we obtain that for $\lambda = 0$ resp. $\mu = 0$ and $m = 0$ the equations (119) - (121) pass over to general relativity, giving the well known Schwarzschild solution for a central mass M. So, as a first approximation for g_0, g_1 we use the Schwarzschild metric

$$g_{0(1)} = -\left(1 - \frac{r_S}{r}\right), \quad g_{1(1)} = \left(1 - \frac{r_S}{r}\right)^{-1}, \quad \text{where} \quad r_S = 2\,\frac{MG}{c^2} \tag{132}$$

and derive the behavior of corresponding functions $u_{o(1)}$, $u_{e(1)}$, $v_{(1)}$ from equations (125) - (127). First, we observe

$$g_{0(1)} \sim -1, \quad g_{1(1)} \sim 1, \quad -\frac{g'_{0(1)}}{g_{0(1)}} \sim \frac{g'_{1(1)}}{g_{1(1)}} \sim \frac{r_S}{r^2}, \quad -\frac{g''_{0(1)}}{g_{0(1)}} \sim \frac{g''_{1(1)}}{g_{1(1)}} \sim -2\,\frac{r_S}{r^3} \tag{133}$$

We will now assume, that $r_S \ll r$ and $r \ll \frac{2}{3}\frac{1}{\mu}$, i.e. $\frac{9}{2}\mu^2 \ll \frac{2}{r^2}$. Furthermore, we set $m = 0$. Then, from (126) and (127) we get

$$u''_{e(1)} + \frac{2}{r}\,u'_{e(1)} - \frac{2}{r^2}\,u_{e(1)} + \nu^2 \frac{\mathrm{d}}{\mathrm{d}r}\left[\left(u'_{e(1)} + \frac{2}{r}\,u_{e(1)}\right) v^2_{(1)}\right] \sim -\tfrac{1}{2}\,\mu\,\frac{r_S}{r^2} \tag{134}$$

$$v''_{(1)} + \frac{2}{r}\,v'_{(1)} - \nu^2\left(u'_{e(1)} + \frac{2}{r}\,u_{e(1)}\right)^2 v_{(1)} \sim 0 \tag{135}$$

Defining for some $r_0 > r_S$ the variables

$$\tilde{r} := \frac{r}{r_0}, \quad \tilde{s} := \ln\left(\frac{r}{r_0}\right) \tag{136}$$

we use the Ansatz

$$u_{e(1)}(r) \sim \tilde{u}_{e(1)}\, \tilde{r}^{\,p_{1(1)}}\, \tilde{s}^{\,q_{1(1)}}, \quad v_{(1)}(r) \sim \tilde{v}_{(1)}\, \tilde{r}^{\,p_{2(1)}}\, \tilde{s}^{\,q_{2(1)}}, \tag{137}$$

To ensure that the coupling term in equation (135) has the same order of r resp. $\tilde{r}$ as the remaining terms, $u'_{e(1)}$ respectively $\frac{2}{r}\, u_{e(1)}$ should have order -1, hence $p_{1(1)} = 0$. In equation (134) the coupling term should have the same order as the right hand side, i.e. $u'_{e(1)} v^2_{(1)}$ must be of order -1, so that the derivative has order -2. Then follows $p_{2(1)} = 0$. Inserting now (137) with $p_{1(1)} = p_{2(1)} = 0$ into (134) and (135) we get

$$\begin{aligned}
&\frac{\tilde{u}_{e(1)}}{r^2}\left[-q_{1(1)}\, \tilde{s}^{\,q_{1(1)}-1} + 2\, q_{1(1)}\, \tilde{s}^{\,q_{1(1)}-1} - 2\, \tilde{s}^{\,q_{1(1)}}\right] \\
&\qquad + \nu^2\, \tilde{u}_{e(1)}\, \tilde{v}^2_{(1)}\, \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{2}{r}\, \tilde{s}^{\,q_{1(1)}+2\, q_{2(1)}}\right] \sim \\
&\qquad -\frac{\tilde{u}_{e(1)}}{r^2}\left[2\, \tilde{s}^{\,q_{1(1)}} + 2\, \nu^2\, \tilde{v}^2_{(1)}\, \tilde{s}^{\,q_{1(1)}+2\, q_{2(1)}}\right] \sim -\tfrac{1}{2}\, \mu\, \frac{r_S}{r^2}
\end{aligned} \tag{138}$$

$$\begin{aligned}
&\frac{\tilde{v}_{(1)}}{r^2}\left[-q_{2(1)}\, \tilde{s}^{\,q_{2(1)}-1} + 2 q_{2(1)}\, \tilde{s}^{\,q_{2(1)}-1}\right] - 4\, \frac{\nu^2}{r^2}\, \tilde{u}^2_{e(1)}\, \tilde{v}_{(1)}\, \tilde{s}^{\,2\, q_{1(1)}+q_{2(1)}} \sim \\
&\qquad \frac{\tilde{v}_{(1)}}{r^2}\left[q_{2(1)} \tilde{s}^{\,q_{2(1)}-1} - 4\, \nu^2\, \tilde{u}^2_{e(1)}\, \tilde{s}^{\,2\, q_{1(1)}+q_{2(1)}}\right] \sim 0
\end{aligned} \tag{139}$$

Equation (139) implies $q_{2(1)} - 1 = 2\, q_{1(1)} + q_{2(1)}$, i.e. $q_{1(1)} = -\frac{1}{2}$. Then, (138) gives $q_{1(1)} + 2\, q_{2(1)} = 0$, i.e. $q_{2(1)} = \frac{1}{4}$. The coefficients are obtained from

$$2\, \nu^2\, \tilde{u}_{e(1)}\, \tilde{v}^2_{(1)} = \tfrac{1}{2}\, \mu\, r_S \quad \text{and} \quad q_{2(1)} = 4\, \nu^2\, \tilde{u}^2_{e(1)}$$

We get

$$\tilde{u}_{e(1)} = \frac{1}{4\, \nu}, \quad \tilde{v}_{(1)} = \left(\frac{\mu}{\nu}\, r_S\right)^{\frac{1}{2}} \tag{140}$$

and

$$u_{e(1)}(r) \sim \frac{1}{4\, \nu}\left[\ln\left(\frac{r}{r_0}\right)\right]^{-\frac{1}{2}}, \quad v_{(1)}(r) \sim \left(\frac{\mu}{\nu}\, r_S\right)^{\frac{1}{2}}\left[\ln\left(\frac{r}{r_0}\right)\right]^{\frac{1}{4}} \tag{141}$$

The resulting gravity $-\mathrm{grad}\, v_{(1)}$, generated by the scalar field, has the amount

$$\mathcal{G}_{S(1)} \sim \frac{\mathrm{d}}{\mathrm{d}r} v_{(1)} \sim \frac{1}{2^{\frac{3}{2}}}\, \left(\frac{\mu}{\nu}\right)^{\frac{1}{2}}\, \frac{G^{\frac{1}{2}}}{c}\, M^{\frac{1}{2}}\, \frac{1}{r}\left[\ln\left(\frac{r}{r_0}\right)\right]^{-\frac{3}{4}} \tag{142}$$

This corresponds to the description of observed gravity $\mathcal{G}_O$ by Newtonian gravity $\mathcal{G}_N$ for the outer regions of galaxies in [10] and [11] by

$$\mathcal{G}_O \approx (\mathcal{G}_L\,\mathcal{G}_N)^{\frac{1}{2}} = (\mathcal{G}_L\,G)^{\frac{1}{2}}\,M^{\frac{1}{2}}\,\frac{1}{r} \tag{143}$$

up to the weak logarithmic term, when $\mathcal{G}_N < \mathcal{G}_L \approx 10^{-10}\mathrm{m\,s^{-2}}$.

Now, we can use a Robertson series [12] to get a second approximation of the metric:

$$\begin{aligned} g_{0(2)} &= -1 + \frac{2}{c^2}\,v_{(1)} = -1 + \hat{v}_{(1)}\,\tilde{s}^{\frac{1}{4}}, \\ g_{1(2)} &= \quad 1 + \frac{2}{c^2}\,v_{(1)} = \quad 1 + \hat{v}_{(1)}\,\tilde{s}^{\frac{1}{4}}, \end{aligned} \qquad \text{where} \quad \hat{v}_{(1)} = \frac{2}{c^2}\,\tilde{v}_{(1)} \tag{144}$$

assuming

$$\hat{v}_{(1)}\,\tilde{s}^{\frac{1}{4}} \ll 1 \quad \text{or} \quad r \ll r_0\,e^{\frac{1}{\hat{v}^4_{(1)}}} \quad \text{with} \quad \frac{1}{\hat{v}^4_{(1)}} = \frac{4\,c^{12}}{\left(\frac{\mu}{\nu}\,MG\right)^2} \tag{145}$$

For the derivatives of the metric elements we get

$$g'_{0(2)} \sim g'_{1(2)} \sim \frac{1}{4\,r}\,\hat{v}_{(1)}\,\tilde{s}^{-\frac{3}{4}}, \quad g''_{0(2)} \sim g''_{1(2)} \sim -\frac{1}{4\,r^2}\,\hat{v}_{(1)}\,\tilde{s}^{-\frac{3}{4}}, \tag{146}$$

while according to (145) we set

$$g_{0(2)} \approx -1, \quad g_{1(2)} \approx 1 \tag{147}$$

Again we assume $r_S \ll r$, $r \ll \frac{2}{3}\frac{1}{\mu}$ and $m = 0$. Then, for corresponding functions $u_{o(2)}$, $u_{e(2)}$, $v_{(2)}$ equations (122) - (124) give $u_{o(2)} = 0$ and

$$u''_{e(2)} + \frac{2}{r}\,u'_{e(2)} - \frac{2}{r^2}\,u_{e(2)} + \nu^2\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(u'_{e(2)} + \frac{2}{r}\,u_{e(2)}\right)v^2_{(2)}\right] \sim -\mu\,\hat{v}_{(1)}\,\frac{1}{r}\,\tilde{s}^{\frac{1}{4}} \tag{148}$$

$$v''_{(2)} + \frac{2}{r}\,v'_{(2)} - \nu^2\left(u'_{e(2)} + \frac{2}{r}\,u_{e(2)}\right)^2 v_{(2)} \sim 0 \tag{149}$$

For the Ansatz

$$u_{e(2)}(r) \sim \tilde{u}_{e(2)}\,\tilde{r}^{\,p_{1(2)}}\,\tilde{s}^{\,q_{1(2)}}, \quad v_{(2)}(r) \sim \tilde{v}_{(2)}\,\tilde{r}^{\,p_{2(2)}}\,\tilde{s}^{\,q_{2(2)}}, \tag{150}$$

we get again $p_{1(2)} = 0$ from equation (149) and for the coupling term in (148) in comparison with the right hand side $p_{1(2)} + 2\,p_{2(2)} - 2 = -1$, i.e. $p_{2(2)} = \frac{1}{2}$.

With these first results we insert (150) into (148) and (149) and get

$$\begin{aligned}&\frac{\tilde{u}_{e(2)}}{r^2}\left[-q_{1(2)}\,\tilde{s}^{\,q_{1(2)}-1}+2\,q_{1(2)}\,\tilde{s}^{\,q_{1(2)}-1}-2\,\tilde{s}^{\,q_{1(2)}}\right]\\ &\qquad\qquad+2\,\nu^2\,\tilde{u}_{e(2)}\,\tilde{v}_{(2)}^2\,\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\tilde{r}}{r}\,\tilde{s}^{\,q_{1(2)}+2\,q_{2(2)}}\right]\ \sim\\ &\frac{2}{r_0\,r}\,\nu^2\left(q_{1(2)}+2\,q_{2(2)}\right)\tilde{u}_{e(2)}\,\tilde{v}_{(2)}^2\,\tilde{s}^{\,q_{1(2)}+2\,q_{2(2)}-1}\ \sim\ -\mu\,\hat{v}_{(1)}\,\frac{1}{r}\,\tilde{s}^{\frac{1}{4}}\end{aligned}\tag{151}$$

$$\begin{aligned}&\frac{\tilde{r}^{\frac{1}{2}}}{r^2}\tilde{v}_{(2)}\left[-\tfrac{1}{4}\,\tilde{s}^{\,q_{2(2)}}+\tilde{s}^{\,q_{2(2)}}-4\,\nu^2\,\tilde{u}_{e(2)}^2\,\tilde{s}^{\,2\,q_{1(2)}+q_{2(2)}}\right]\ \sim\\ &\qquad r^{-\frac{3}{2}}\,r_0^{-\frac{1}{2}}\,\tilde{v}_{(2)}\left[\tfrac{3}{4}\,\tilde{s}^{\,q_{2(2)}}-4\,\nu^2\,\tilde{u}_{e(2)}^2\,\tilde{s}^{\,2\,q_{1(2)}+q_{2(2)}}\right]\ \sim\ 0\end{aligned}\tag{152}$$

Now, equation (152) implies $q_{2(2)} = 2\,q_{1(2)} + q_{2(2)}$, i.e. $q_{1(2)} = 0$. Then, (151) gives $q_{1(2)} + 2\,q_{2(2)} - 1 = \frac{1}{4}$, i.e. $q_{2(2)} = \frac{5}{8}$. The coefficients are obtained from

$$\frac{2}{r_0}\,\nu^2\left(q_{1(2)}+2\,q_{2(2)}\right)\tilde{u}_{e(2)}\,\tilde{v}_{(2)}^2=-\mu\,\hat{v}_{(1)}\quad\text{and}\quad\tfrac{3}{4}=4\,\nu^2\,\tilde{u}_{e(2)}^2$$

We get

$$\tilde{u}_{e(2)}=-\frac{3^{\frac{1}{2}}}{4}\frac{1}{\nu},\quad \tilde{v}_{(2)}=\frac{2^{\frac{3}{2}}}{3^{\frac{1}{4}}5^{\frac{1}{2}}}\left(\frac{\mu}{\nu}\right)^{\frac{1}{2}}r_0^{\frac{1}{2}}\,\hat{v}_{(1)}=\frac{4}{3^{\frac{1}{4}}5^{\frac{1}{2}}}\left(\frac{\mu}{\nu}\right)^{\frac{3}{4}}\frac{1}{c}\,r_0^{\frac{1}{2}}\,r_S^{\frac{1}{4}}\tag{153}$$

and

$$u_{e(2)}(r)\sim-\frac{3^{\frac{1}{2}}}{4}\,\frac{1}{\nu},\quad v_{(2)}(r)\sim\frac{4}{3^{\frac{1}{4}}5^{\frac{1}{2}}}\left(\frac{\mu}{\nu}\right)^{\frac{3}{4}}\frac{1}{c}\,r_S^{\frac{1}{4}}\,r^{\frac{1}{2}}\left[\ln\left(\frac{r}{r_0}\right)\right]^{\frac{5}{8}}\tag{154}$$

The resulting gravity, generated by the scalar field, has the amount

$$\mathcal{G}_{S(2)}\sim\frac{\mathrm{d}}{\mathrm{d}r}v_{(2)}\sim\frac{2^{\frac{5}{4}}}{3^{\frac{1}{4}}5^{\frac{1}{2}}}\left(\frac{\mu}{\nu}\right)^{\frac{3}{4}}\frac{G^{\frac{1}{4}}}{c^{\frac{3}{2}}}\,M^{\frac{1}{4}}\,\frac{1}{r^{\frac{1}{2}}}\left[\ln\left(\frac{r}{r_0}\right)\right]^{\frac{5}{8}}\tag{155}$$

and is stronger than $\mathcal{G}_{S(1)}$ in the sense that the exponent in $r^{-\frac{1}{2}}$ is greater than that in r^{-1}. Since the coefficient is lower, the higher gravity would act on longer distances and could describe the cohesion of galaxy clusters, which can not be sufficiently explained by gravity with the strength of $\mathcal{G}_{S(1)}$. A comparison to the p-Laplacian, which is used in [5] to describe galaxy clusters shows, that the dependence on M and r in (155) up to the logarithmic term coincides with solutions for $p = 5$.

Finally, we will consider the case of very large $r \gg \frac{2}{3}\frac{1}{\mu}$, so that we have to take into account the term $\frac{9}{2}\,\mu^2\,g_1\,u_e$ in equation (126). For the searched functions, in this case denoted by $u_{o(3)}$, $u_{e(3)}$, $v_{(3)}$ we get $u_{o(3)} = 0$ and

$$u_{e(3)} \sim \tfrac{1}{9}\,\frac{1}{\mu}\left[\frac{g_0'}{g_0} - \frac{2}{r}\,(g_1 - 1)\right] \tag{156}$$

Then, for the Ansatz

$$u_{e(3)}(r) \sim \tilde{u}_{e(3)}\,\tilde{r}^{\,p_{1(3)}}\,\tilde{s}^{\,q_{1(3)}}, \quad v_{(3)}(r) \sim \tilde{v}_{(3)}\,\tilde{r}^{\,p_{2(3)}}\,\tilde{s}^{\,q_{2(3)}}, \tag{157}$$

we can assume $p_{1(3)} \leq -1$ and $g_0 \approx 1$ for large r. For $m = 0$ equation (127) implies that

$$\begin{aligned} v_{(3)}'' + \frac{2}{r}\,v_{(3)}' \;=\; & \tilde{v}_{(3)}\frac{\tilde{r}^{\,p_{2(3)}}}{r^2}\left[p_{2(3)}\left(p_{2(3)}+1\right)\tilde{s}^{\,q_{2(3)}} + \right. \\ & \left.\left(2\,p_{2(3)}+1\right)q_{2(3)}\,\tilde{s}^{\,q_{2(3)}-1} + q_{2(3)}\left(q_{2(3)}-1\right)\tilde{s}^{\,q_{2(3)}-2}\right] \end{aligned} \tag{158}$$

must have an order $p_{2(3)} + 2\,p_{1(3)} - 2 \leq p_{2(3)} - 4$ with respect to r resp. $\tilde{r}$. This is satisfied only for $q_{2(3)} = 0$ and $p_{2(3)} = 0$ or $p_{2(3)} = -1$. For $p_{2(3)} = 0$ the function $v_{(3)}$ would be constant and the gradient gives no contribution to gravity. For $p_{2(3)} = -1$ we get

$$v_{(3)}(r) \sim \tilde{v}_{(3)}\,\tilde{r}^{-1} \quad \text{and} \quad \mathcal{G}_{S(3)} \sim \frac{\mathrm{d}}{\mathrm{d}r}v_{(3)} \sim -\tilde{v}_{(3)}\frac{r_0}{r^2} \tag{159}$$

i.e. for very large distances the gravity generated by the scalar field has the order of Newtonian gravity. This behavior corresponds to results about gravity on cosmological scales [13].

6 Cosmology

Next, we shall describe the dynamics of a homogeneous, isotropic, flat universe. The corresponding metric should be diagonal with identical components for the three spatial dimensions, depending only on time, i.e.

$$g = \mathrm{diag}\,(g_0, g_1, g_1, g_1) \quad \text{with} \quad g_0 = g_0(t),\; g_1 = g_1(t), \tag{160}$$

The only non-vanishing Christoffel symbols (1) are

$$\Gamma^0_{00} = \tfrac{1}{2}\,\frac{\dot{g}_0}{g_0}, \;\; \Gamma^i_{i0} = \tfrac{1}{2}\,\frac{\dot{g}_1}{g_1}, \;\; \Gamma^0_{ii} = -\tfrac{1}{2}\,\frac{\dot{g}_1}{g_0} \tag{161}$$

Using again the scalar $\tilde{R}$, defined in (79)

$$\tilde{R} := \frac{1}{g_0}\left[-\frac{\ddot{g}_1}{g_1} + \tfrac{1}{2}\,\frac{(\dot{g}_1)^2}{g_1^2} + \tfrac{1}{2}\,\frac{\dot{g}_0\dot{g}_1}{g_0 g_1}\right] \tag{162}$$

for the Ricci tensor and the Ricci scalar we get

$$\mathcal{R}_{00} = \tfrac{3}{2}\, g_0\, \tilde{R}, \quad \mathcal{R}_{ii} = \tfrac{1}{2}\, g_1 \tilde{R} - \tfrac{1}{2}\frac{(\dot{g}_1)^2}{g_0 g_1}, \quad R = 3\,\tilde{R} - \tfrac{3}{2}\frac{(\dot{g}_1)^2}{g_0 g_1^2},$$
$$\mathcal{R}_{00} - \tfrac{1}{2}\, R\, g_{00} = \tfrac{3}{4}\,\frac{(\dot{g}_1)^2}{g_1^2}, \qquad \mathcal{R}_{ii} - \tfrac{1}{2}\, R\, g_{ii} = -\, g_1\, \tilde{R} + \tfrac{1}{4}\frac{(\dot{g}_1)^2}{g_0 g_1} \tag{163}$$

Because of the isotropy the spatial components u_i of the vector field vanish. Hence, the fields additional to the metric are $u_0 = u_0(t)$ and $v = v(t)$.

We may now calculate the terms $\mathcal{N}^{(1)}$ - $\mathcal{N}^{(11)}$, defined in (86) - (89), (97) - (101) and (108).

$$\mathcal{N}^{(1)}(\mathcal{U}) \;=\; \frac{1}{g_0}\left(\tfrac{1}{2}\,\frac{\dot{g}_0}{g_0} - 3\,\frac{\dot{g}_1}{g_1}\right) u_o \tag{164}$$

$$\mathcal{N}^{(2)}_{00}(\mathcal{U}) = \tfrac{1}{2}\,\frac{\dot{g}_0}{g_0}\, u_o, \;\; \mathcal{N}^{(2)}_{ii}(\mathcal{U}) \;=\; -\tfrac{1}{2}\,\frac{\dot{g}_1}{g_0} u_o, \tag{165}$$

$$\mathcal{N}^{(2)}_{i0}(\mathcal{U}) \;=\; \mathcal{N}^{(2)}_{0i}(\mathcal{U}) \;=\; \mathcal{N}^{(2)}_{ij}(\mathcal{U}) \;=\; 0 \;\;(i \neq j) \tag{166}$$

$$\mathcal{N}^{(3)}_{0} \;=\; \tfrac{1}{2}\,\frac{\dot{g}_0}{g_0} - \tfrac{3}{2}\,\frac{\dot{g}_1}{g_1}, \;\; \mathcal{N}^{(3)}_{i} \;=\; 0 \tag{167}$$

$$\mathcal{N}^{(4)}_{0} \;=\; 3\,\frac{\dot{g}_1}{g_1}, \;\; \mathcal{N}^{(4)}_{i} \;=\; 0 \tag{168}$$

$$\mathcal{N}^{(5)}(\mathcal{U}) \;=\; \frac{1}{g_0} u_o^2 \tag{169}$$

$$\mathcal{N}^{(6)}(\mathcal{U}, \partial\mathcal{U}) \;=\; \frac{1}{g_0}\left[\dot{u}_o + \left(-\tfrac{1}{2}\,\frac{\dot{g}_0}{g_0} + \tfrac{3}{2}\,\frac{\dot{g}_1}{g_1}\right) u_o\right] \tag{170}$$

$$\mathcal{N}^{(7)}_{\alpha\beta}(\mathcal{U}, \partial\mathcal{U}) \;=\; \mathcal{N}^{(8)}(\mathcal{U}, \partial\mathcal{U}) \;=\; 0 \tag{171}$$

$$\mathcal{N}^{(9)}(v, \partial v) \;=\; \frac{1}{g_0}\dot{v}^2 \tag{172}$$

$$\mathcal{N}^{(10)}_{0}(\mathcal{U}, \partial\mathcal{U}, \partial^2\mathcal{U}) \;=\; \frac{1}{g_0}\left\{\ddot{u}_o + \left[-\tfrac{3}{2}\,\frac{\dot{g}_0}{g_0} + \tfrac{3}{2}\,\frac{\dot{g}_1}{g_1}\right]\dot{u}_o + \right.$$
$$\left. + \left[-\tfrac{1}{2}\,\frac{\ddot{g}_0}{g_0} + \left(\frac{\dot{g}_0}{g_0}\right)^2 - \tfrac{3}{4}\,\frac{\dot{g}_0\,\dot{g}_1}{g_0\, g_1} - \tfrac{3}{4}\left(\frac{\dot{g}_1}{g_1}\right)^2\right]\right\} \tag{173}$$

$$\mathcal{N}^{(10)}_{i}(\mathcal{U}, \partial\mathcal{U}, \partial^2\mathcal{U}) \;=\; 0 \tag{174}$$

$$\mathcal{N}^{(11)}(v, \partial v, \partial^2 v) \;=\; \frac{1}{g_0}\ddot{v} \tag{175}$$

For the reduced Lagrangian, defined in (117) we get

$$\begin{aligned} \tilde{\mathcal{L}} &= \frac{1}{g_0}\left\{\tfrac{1}{2}\left(\nu^2 v^2+1\right)\frac{1}{g_0}\left[\dot{u}_o+\left(-\tfrac{1}{2}\frac{\dot{g}_0}{g_0}+\tfrac{3}{2}\frac{\dot{g}_1}{g_1}\right)u_o\right]^2\right. \\ &\quad \left. -\mu\left(\dot{u}_o-\tfrac{1}{2}\frac{\dot{g}_0}{g_0}u_o\right)-\tfrac{9}{4}\mu^2 u_o^2+\tfrac{1}{2}\dot{v}^2+\tfrac{1}{2}m^2 g_0 v^2\right\} \end{aligned} \qquad (176)$$

Then, equation (118) gives

$$\begin{aligned} &\mu\left[\dot{u}_o+\left(-\tfrac{1}{2}\frac{\dot{g}_0}{g_0}+\tfrac{3}{2}\frac{\dot{g}_1}{g_1}\right)u_o\right]-\tfrac{9}{2}\mu^2 u_o^2+\tfrac{1}{2}\dot{v}^2 \\ &\qquad +\tfrac{3}{4}\frac{(\dot{g}_1)^2}{g_1^2}+g_0\tilde{\mathcal{L}} = \kappa\,\mathcal{T}_{00} \end{aligned} \qquad (177)$$

$$-g_1\tilde{R}+\tfrac{1}{4}\frac{(\dot{g}_1)^2}{g_0 g_1}+g_1\tilde{\mathcal{L}} = \kappa\,\mathcal{T}_{ii} \qquad (178)$$

for $(\alpha,\beta)=(0,0)$ and $(\alpha,\beta)=(i,i)$, while equation (51) is relevant only in the case $\gamma=0$

$$\begin{aligned} &\frac{1}{g_0}\ddot{u}_o+\frac{1}{g_0}\left(-\tfrac{3}{2}\frac{\dot{g}_0}{g_0}+\tfrac{3}{2}\frac{\dot{g}_1}{g_1}\right)\dot{u}_o \\ &+\frac{1}{g_0}\left[-\tfrac{1}{2}\frac{\ddot{g}_0}{g_0}+\left(\frac{\dot{g}_0}{g_0}\right)^2-\tfrac{3}{4}\frac{\dot{g}_0\,\dot{g}_1}{g_0\,g_1}-\tfrac{3}{4}\left(\frac{\dot{g}_1}{g_1}\right)^2\right]u_o+\tfrac{9}{2}\mu^2 u_o \\ &+\nu^2\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{g_0}\left[\dot{u}_o+\left(-\tfrac{1}{2}\frac{\dot{g}_0}{g_0}+\tfrac{3}{2}\frac{\dot{g}_1}{g_1}\right)u_o\right]v^2\right\} = \tfrac{3}{2}\mu\frac{\dot{g}_1}{g_1} \end{aligned} \qquad (179)$$

Equation (52) gives

$$\frac{1}{g_0}\ddot{v}-m^2 v-\nu^2\frac{1}{g_0^2}\left[\dot{u}_o+\left(-\tfrac{1}{2}\frac{\dot{g}_0}{g_0}+\tfrac{3}{2}\frac{\dot{g}_1}{g_1}\right)u_o\right]^2 v = 0 \qquad (180)$$

We consider the universe as matter dominated, i.e.

$$\mathcal{T}_{\alpha\beta}=\rho\,\phi_\alpha\,\phi_\beta \quad \text{with} \quad \phi_\gamma\,\phi^\gamma=g^{\alpha\beta}\phi_\alpha\,\phi_\beta=c^2, \qquad (181)$$

where ρ is the matter density and ϕ_γ the four-velocity of the matter. In the isotropic case we have $\phi_i=0$ for $i>0$ and hence $g_0^{-1}\phi_0^2=c^2$. So we get

$$\mathcal{T}_{\alpha\beta}=\mathrm{diag}\left(c^2 g_0\,\rho,0,0,0\right) \qquad (182)$$

with $\rho=\rho(t)$. Neglecting radiation the preservation of mass gives

$$\rho(t)\ [a(t)]^3=\rho(t_0)\ [a(t_0)]^3 =: M_0 \qquad (183)$$

for a starting point t_0, where $a(t)$ describes the expansion of the universe and can be defined by $a^2 = g_1$. Now, the system (177), (178) is equivalent to

$$\mu \left[\dot{u}_o + \left(-\tfrac{1}{2}\,\frac{\dot{g}_0}{g_0} + \tfrac{3}{2}\,\frac{\dot{g}_1}{g_1} \right) u_o \right] - \tfrac{9}{2}\,\mu^2 u_o^2 + \tfrac{1}{2}\,\dot{v}^2 + g_0\,\tilde{R} + \tfrac{1}{2}\,\frac{(\dot{g}_1)^2}{g_1^2} = \kappa\, c^2 M_0\, g_0\, g_1^{-\frac{3}{2}} \tag{184}$$

$$g_0\,\tilde{\mathcal{L}} = g_0\,\tilde{R} - \tfrac{1}{4}\,\frac{(\dot{g}_1)^2}{g_1^2} \tag{185}$$

To find the behavior of the unknown functions for large times we use again an Ansatz similar to (137) with the variables

$$\tilde{t} := \frac{t}{t_0}, \quad \tilde{w} := \ln\left(\frac{t}{t_0}\right) \tag{186}$$

$$\begin{aligned} g_0(t) &\sim \overline{g}_0\,\tilde{t}^{\,p_0}\,\tilde{w}^{\,q_0}, \quad g_1(t) \sim \overline{g}_1\,\tilde{t}^{\,p_1}\,\tilde{w}^{\,q_1}, \\ u_o(t) &\sim \overline{u}_o\,\tilde{t}^{\,p_2}\,\tilde{w}^{\,q_2}, \quad v(t) \sim \overline{v}\,\tilde{t}^{\,p_3}\,\tilde{w}^{\,q_3}. \end{aligned} \tag{187}$$

For a first approximation we search for a solution without logarithmic terms, i.e. $q_0 = q_1 = q_2 = q_3 = 0$. Then, we get

$$g_0\,\tilde{R} \sim p_1\,\tilde{p}\,t^{-2} \quad \text{with} \quad \tilde{p} := 1 - \tfrac{1}{2}p_1 + \tfrac{1}{2}p_0 \tag{188}$$

and a coupling term

$$\dot{u}_o + \left(-\tfrac{1}{2}\,\frac{\dot{g}_0}{g_0} + \tfrac{3}{2}\,\frac{\dot{g}_1}{g_1} \right) u_o = \overline{p}\,\overline{u}_o\,\tilde{t}^{\,p_2}\,t^{-1} \quad \text{with} \quad \overline{p} := p_2 + \tfrac{3}{2}\,p_1 - \tfrac{1}{2}\,p_0 \tag{189}$$

Equations (184), (185) give

$$\begin{aligned} &\mu\,\overline{p}\,\overline{u}_o\,t_0^{-p_2}\,t^{\,p_2-1} - \tfrac{9}{2}\,\mu^2\,\overline{u}_o^2\,t_0^{-2p_2}\,t^{\,2p_2} \\ &+\tfrac{1}{2}\,p_3^2\,\overline{v}^2\,t_0^{-2p_3}\,t^{\,2p_3-2} + p_1\left(1 + \tfrac{1}{2}p_0\right)t^{-2} \\ \sim\ &\kappa\,c^2 M_0\,\overline{g}_0\,\overline{g}_1^{\,-\frac{3}{2}}\,t_0^{-p_0+\frac{3}{2}p_1}\,t^{\,p_0-\frac{3}{2}p_1} \end{aligned} \tag{190}$$

$$\begin{aligned} &\tfrac{1}{2}\left(\nu^2\overline{v}^2\,t_0^{-2p_3}\,t^{\,2p_3} + 1\right)\overline{p}^2\,\overline{g}_0^{\,-1}\overline{u}_o^2\,t_0^{\,p_0-2p_2}\,t^{\,2p_2-p_0-2} \\ &-\mu\,\left(p_2 - \tfrac{1}{2}\,p_0\right)\overline{u}_o\,t_0^{-p_2}\,t^{\,p_2-1} - \tfrac{9}{4}\,\mu^2\,\overline{u}_o^2\,t_0^{-2p_2}\,t^{\,2p_2} \\ &+\tfrac{1}{2}\,p_3^2\,\overline{v}^2\,t_0^{-2p_3}\,t^{\,2p_3-2} + \tfrac{1}{2}\,m^2\,\overline{g}_0\,\overline{v}^2\,t_0^{-p_0-2p_3}\,t^{\,p_0+2p_3} \\ \sim\ &p_1\left(\tilde{p} - \tfrac{1}{4}p_1\right)t^{-2} \end{aligned} \tag{191}$$

Equations (179), (180) multiplied by g_0 imply

$$\begin{aligned} & \hat{p}\,\overline{u}_o\,t_0^{-p_2}\,t^{\,p_2-2} + \tfrac{9}{2}\,\mu^2\,\overline{g}_0\,\overline{u}_o\,t_0^{-p_0-p_2}\,t^{\,p_0+p_2} \\ & +\nu^2\,(p_2+2p_3-p_0-1)\,\overline{p}\,\overline{u}_o\,\overline{v}^2\,t_0^{-p_2-2p_3}\,t^{\,p_2+2p_3-2} \\ \sim\;\; & \tfrac{3}{2}\,\mu\,p_1\,\overline{g}_0\,t_0^{-p_0}\,t^{\,p_0-1} \end{aligned} \qquad (192)$$

$$\begin{aligned} & \Big[p_3\,(p_3-1)\,t^{-2} - m^2\,\overline{g}_0\,t_0^{-p_0}\,t^{\,p_0} - \\ & -\nu^2\overline{p}^2\,\overline{g}_0^{-1}\,\overline{u}_o^2\,t_0^{p_0-2p_2}\,t^{\,2p_2-p_0-2}\Big]\,\overline{v}\,t_0^{-p_3}\,t^{\,p_3} \sim 0 \end{aligned} \qquad (193)$$

with

$$\hat{p} := p_2\,(p_2-1) + \tfrac{3}{2}\,p_2\,(p_1-p_0) + \tfrac{1}{2}\,p_0\,(p_0+1) - \tfrac{3}{4}\,p_1\,(p_0+p_1) \qquad (194)$$

First, we want to evaluate the exponents p_i so, that in each equation (190), (191) and (192), (193) all terms have the same order with respect to t. From (190) and (191) we can derive $p_2 = -1$. Then, equation (192) gives $p_2 - 2 = p_2 + 2p_3 - 2 = p_0 - 1$, hence $p_0 = -2$ and $p_3 = 0$. Finally, from equation (190) we derive $-2 = p_0 - \frac{3}{2}\,p_1$ and get $p_1 = 0$. With these exponents we find $\tilde{p} = \overline{p} = \hat{p} = 0$ and hence only the trivial solution $\overline{v} = \overline{u}_o = \overline{g}_0 = 0$.

Now, we only demand, that in each equation at least two terms have the same order with respect to t, while all other terms are of lower order. This is satisfied by a small deviation from the values of p_i found above

$$(p_0, p_1, p_2, p_3) = (-2-\epsilon, 0, -1-\epsilon, 0) \quad \text{with} \quad \epsilon > 0 \qquad (195)$$

Equation (193) can only be fulfilled, if $\overline{v} = 0$. The comparison of coefficients for the terms of highest order with respect to t gives no condition for (192), since $\hat{p} = p_1 = 0$, while equations (190) and (191) lead to the relations

$$-\tfrac{1}{2}\,\epsilon\,\mu\,\overline{u}_o\,t_0^{1+\epsilon} = \kappa\,c^2 M_0\,\overline{g}_0\,\overline{g}_1^{-\frac{3}{2}}\,t_0^{2+\epsilon}, \quad \tfrac{1}{8}\,\epsilon^2\,\overline{g}_0^{-1}\,\overline{u}_o^2\,t_0^{\epsilon} + \tfrac{1}{2}\epsilon\,\mu\,\overline{u}_o\,t_0^{1+\epsilon} = 0 \qquad (196)$$

from which we can derive a relation between $\overline{g}_0$ and $\overline{u}_o$, while $\overline{g}_1$ can be determined explicitly. With $\tilde{g}_0 := -\overline{g}_0\,t_0^{2+\epsilon}$ as a first approximation we get

$$\begin{aligned} & g_{0(1,\epsilon)}(t) \sim -\tilde{g}_0\,\frac{1}{t^{2+\epsilon}}, \qquad g_{1(1,\epsilon)}(t) \sim \left(\frac{1}{2\mu^2}\,\kappa\,c^2 M_0\right)^{\frac{2}{3}}, \\ & u_{o(1,\epsilon)}(t) \sim \frac{4\,\mu}{\epsilon}\,\tilde{g}_0\,\frac{1}{t^{1+\epsilon}}, \qquad v_{(1,\epsilon)} \sim 0 \end{aligned} \qquad (197)$$

To find a solution with a warping of time as small as possible we return to the general Ansatz (187), where the exponents of $\tilde{t}$ satisfy (195) with the smallest value $\epsilon = 0$, i.e. $p_0 = -2$, $p_1 = 0$, $p_2 = -1$, $p_3 = 0$. Using the additional values

$$\begin{aligned} \tilde{q} &:= 1 - \tfrac{1}{2} q_1 + \tfrac{1}{2} q_0, \quad \overline{q} := q_2 + \tfrac{3}{2}\, q_1 - \tfrac{1}{2}\, q_0 \\ \hat{q} &:= q_2\,(q_2 - 1) + \tfrac{3}{2}\, q_2\,(q_1 - q_0) + \tfrac{1}{2}\, q_0\,(q_0 + 1) - \tfrac{3}{4}\, q_1\,(q_0 + q_1) \end{aligned} \tag{198}$$

we find

$$g_0\, \tilde{R} \sim q_1\, \tilde{q}\, t^{-2}\, \tilde{w}^{-2}, \quad \dot{u}_o + \left(-\tfrac{1}{2}\, \frac{\dot{g}_0}{g_0} + \tfrac{3}{2}\, \frac{\dot{g}_1}{g_1} \right) u_o \sim \overline{q}\, \overline{u}_o\, t^{-2}\, \tilde{w}^{\,q_2 - 1} \tag{199}$$

For some terms we have to take into account lower order terms, since the terms of highest order vanish. The equations (190) - (193) multiplied by t^2 or t^3, respectively, are exchanged by

$$\begin{aligned} &\mu\, \overline{q}\, \overline{u}_o\, t_0\, \tilde{w}^{\,q_2 - 1} - \tfrac{9}{2}\, \mu^2\, \overline{u}_o^2\, t_0^2\, \tilde{w}^{\,2q_2} + \tfrac{1}{2}\, q_3^2\, \overline{v}^2\, \tilde{w}^{\,2q_3 - 2} + q_1 \left(1 + \tfrac{1}{2} q_0\right)\, \tilde{w}^{\,-2} \\ &\sim \kappa\, c^2 M_0\, \overline{g}_0\, \overline{g}_1^{\,-\frac{3}{2}}\, t_0^2\, \tilde{w}^{\,q_0 - \frac{3}{2} q_1} \end{aligned} \tag{200}$$

$$\begin{aligned} &\tfrac{1}{2} \left(\nu^2 \overline{v}^2\, \tilde{w}^{\,2q_3} + 1\right) \overline{q}^2\, \overline{g}_0^{\,-1} \overline{u}_o^2\, \tilde{w}^{\,-q_0 + 2q_2 - 2} - \mu\, \left(q_2 - \tfrac{1}{2}\, q_0\right) \overline{u}_o\, t_0\, \tilde{w}^{\,q_2 - 1} \\ &- \tfrac{9}{4}\, \mu^2\, \overline{u}_o^2\, t_0^2\, \tilde{w}^{\,2q_2} + \tfrac{1}{2}\, q_3^2\, \overline{v}^2\, \tilde{w}^{\,2q_3 - 2} + \tfrac{1}{2}\, m^2\, \overline{g}_0\, \overline{v}^2\, t_0^2\, \tilde{w}^{\,q_0 + 2q_3} \\ &\sim q_1 \left(\tilde{q} - \tfrac{1}{4} q_1\right) \tilde{w}^{\,-2} \end{aligned} \tag{201}$$

$$\begin{aligned} &\hat{q}\, \overline{u}_o\, t_0\, \tilde{w}^{\,q_2 - 2} + \tfrac{9}{2}\, \mu^2\, \overline{g}_0\, \overline{u}_o\, t_0^3\, \tilde{w}^{\,q_0 + q_2} \\ &+ \nu^2\, (q_2 + 2q_3 - q_0 - 1)\, \overline{q}\, \overline{u}_o\, \overline{v}^2\, t_0\, \tilde{w}^{\,q_2 + 2q_3 - 2} \\ &\sim \tfrac{3}{2}\, \mu\, q_1\, \overline{g}_0\, t_0^2\, \tilde{w}^{\,q_0 - 1} \end{aligned} \tag{202}$$

$$\left[-q_3\, \tilde{w}^{\,-1} - m^2\, \overline{g}_0\, t_0^2\, \tilde{w}^{\,q_0} - \nu^2\, \overline{q}^2\; \overline{g}_0^{\,-1}\, \overline{u}_o^2\, \tilde{w}^{\,2q_2 - q_0 - 2} \right] \overline{v}\, \tilde{w}^{\,q_3} \sim 0 \tag{203}$$

Equations (200) – (203) are very similar to (190) – (193). So, we get a similar solution for the values q_i in the sense of at least two terms having the same exponents with respect to $\tilde{w}$ in each equation, while all other exponents are lower. Similar to (195) we find

$$(q_0, q_1, q_2, q_3) = (-2 - \delta, 0, -1 - \delta, 0) \quad \text{with} \quad \delta > 0 \tag{204}$$

Again we get a solution for g_0, g_1 and u_o only, if $\overline{v} = 0$. The comparison of coefficients for the terms of highest order with respect to $\tilde{w}$ gives

$$-\tfrac{1}{2}\, \delta\, \mu\, \overline{u}_o\, t_0 = \kappa\, c^2 M_0\, \overline{g}_0\, \overline{g}_1^{\,-\frac{3}{2}}\, t_0^2, \quad \tfrac{1}{8}\, \delta^2\, \overline{g}_0^{\,-1}\, \overline{u}_o^2 + \tfrac{1}{2} \delta\, \mu\, \overline{u}_o\, t_0 = 0 \tag{205}$$

Using again the notion $\tilde{g}_0 := -\overline{g}_0\, t_0^2$ as a second approximation we get

$$g_{0(2,\delta)}(t) \sim -\tilde{g}_0\, \frac{1}{t^2} \left[\ln\left(\frac{t}{t_0}\right)\right]^{-2-\delta}, \qquad g_{1(2,\delta)}(t) \sim \left(\frac{1}{2\mu^2}\,\kappa\, c^2 M_0\right)^{\frac{2}{3}},$$
$$u_{o(2,\delta)}(t) \sim \frac{4\,\mu}{\delta}\,\tilde{g}_0\,\frac{1}{t}\left[\ln\left(\frac{t}{t_0}\right)\right]^{-1-\delta}, \quad v_{(2,\delta)} \sim 0 \tag{206}$$

We may now discuss the meaning of this result for the expansion of the universe in the long term and the observable redshift in comparison with the Robertson-Walker metric using the representation of the line element

$$(\mathrm{d}s)^2 = g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = a^2\,(\mathrm{d}x)^2 - b^2\,(\mathrm{d}t)^2 \tag{207}$$

In our case the factors are $b^2(t) = -g_0(t)$ and $a^2(t) = g_1(t)$, while for the Robertson-Walker metric $b^{(RW)} = 1$.

What we actually observe is the redshift of light, from which we conclude to the velocity of the emitter. Since light moves with velocity c, it satisfies $\mathrm{d}s = 0$ and (207) gives

$$\mathrm{d}x = \frac{b}{a}\,\mathrm{d}t \tag{208}$$

Considering light sent from a point P_0 with wavelength λ_0 at time $t_0 + \Delta t_0$ to P_1 with wavelength λ_1 at time $t_1 + \Delta t_1$ we can assume that the distance between P_0 and P_1 does not change during short times Δt_0 resp. Δt_1. Integration of (208) for $\Delta t_i = 0$ and for $\Delta t_i = \lambda_i$ $(i = 0, 1)$ gives

$$\int_{P_0}^{P_1} \mathrm{d}x = \int_{t_0}^{t_1} \frac{b(t)}{a(t)}\,\mathrm{d}t = \int_{t_0+\lambda_0}^{t_1+\lambda_1} \frac{b(t)}{a(t)}\,\mathrm{d}t \tag{209}$$

Subtracting the integral from $t_0 + \lambda_0$ to t_1 we get

$$\lambda_0\,\frac{b(t_0)}{a(t_0)} \approx \int_{t_0}^{t_0+\lambda_0} \frac{b(t)}{a(t)}\,\mathrm{d}t = \int_{t_1}^{t_1+\lambda_1} \frac{b(t)}{a(t)}\,\mathrm{d}t \approx \lambda_1\,\frac{b(t_1)}{a(t_1)} \tag{210}$$

and finally

$$\frac{\lambda_1}{\lambda_0} \approx \frac{b(t_0)}{a(t_0)}\frac{a(t_1)}{b(t_1)} \tag{211}$$

The redshift of observed light should be independent of the used metric and is determined by the function

$$L(t) := \frac{a(t)}{b(t)} = \frac{a^{(RW)}(t)}{b^{(RW)}(t)}, \tag{212}$$

i.e. $a^{(RW)}(t) = L(t)$, since $b^{(RW)} = 1$.

For the two approximations (197) and (206) we get

$$a^{(RW)}_{(1,\epsilon)}(t) \sim \overline{a}_{(1,\epsilon)}\, t^{1+\frac{\epsilon}{2}}, \quad a^{(RW)}_{(2,\delta)}(t) \sim \overline{a}_{(2,\delta)}\, t\, \left[\ln\left(\frac{t}{t_0}\right)\right]^{1+\frac{\delta}{2}} \tag{213}$$

with some constants $\overline{a}_{(1,\epsilon)}$, $\overline{a}_{(2,\delta)}$. Since the second derivative of $a^{(RW)}_{(1,\epsilon)}$ resp. $a^{(RW)}_{(2,\delta)}$ is positive, this interpretation suggests an asymptotically accelerated expansion of the universe in the Robertson-Walker metric, but without any Dark Energy, because the redshift is generated by time contraction. This virtually shortening of time in the past could also explain the observations of objects in the early universe, which seem to have developed much faster than expected.

Reminding the identification of t with x^0 to get a 'real' time one has to replace t by $\frac{1}{c}t$.

7 Discussion

The stability of the theory was examined in the flat case for the system of vector and scalar fields, which can be generalized to a small warping of spacetime. A further investigation of spherically symmetric perturbations in the sense of [14], [15] was not possible, since the simplifications derived there could not be achieved. At least, in the two considered cases we got solutions for the highest order terms of a power series Ansatz with logarithmic terms. Therefore, we expect no exponential growth for any of the participated fields.

In the case of a central mass a first approximation of the scalar field generates gravity for great distances from the center which very good coincides with observations in galaxies, while a second approximation gives a stronger gravity for larger distances, which could describe galaxy clusters without the need of Dark Matter. On cosmological scales the resulting gravity shows a Newtonian behavior.

In cosmology the long term development shows no expansion of the universe, but a time contraction in the past, which results in redshift of observations, that can be interpreted as an accelerated expansion in the standard metric without the need of Dark Energy.

For the verification of the theory in galaxies or galaxy clusters and the determination of the coupling constants μ, ν the system (119) - (124) with at least four unknown functions or more generally (50) - (52) has to be solved numerically and compared to observations. The coupling constant μ should be sufficiently small such that in the solar system the deviations from general relativity are negligible in the sense of (54).

Since matter is coupled only to the tensor field the theory should not be falsified by observations of gravity waves [16].

The results about solutions in the central symmetric case and in cosmology only describe the behavior of gravity for sufficiently large distances from the center or a long term, respectively. So it would be important to find a description of gravity near great masses and in the early universe. Furthermore, rotating objects should be investigated.

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