Full proof to the Gambler's Ruin Problem

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Abstract

In this article, we are going to be discussing about the full proof to the Gambler's Ruin Problem, using a combination of probability theory, recurrence relations, and boundary conditions.

1 Introduction

The Gambler's Ruin Problem is arguably the most well-known problem in the study of probabilities and statistics. The general solution to this problem sets the foundation for studies of mathematical theory of probabilities.

2 The Gambler's Ruin Problem

2.1 Problem description

Consider that a gambler who at each play of a game has probability p of winning 1 unit and probability $q = 1-p$ of losing 1 unit, and the performance of the gambler in each play of the game is independent.

If the gambler begins with i units, what is the probability of the gambler's total amount of units will reach n before reaching 0 units?

2.2 Proof of the Problem

let $P(i)$, $i = 1, 2, 3, ..., i$, be the probability that starting from i, the Gambler's total amount of units will eventually reach n.

First, we notice that $P(0) = 0$, and $P(n) = 1$. This is because of that, when the gambler has 0 unit, the probability of the gambler moving on to the next play is 0, and when the gambler has n units, the gambler's total amount of units has reach n. So we have our two boundary values $P(0)$ and $P(n)$.

By conditioning the outcome of the first play of the game, we can get:

$$P(i) = pP(i+1) + qP(i-1) \tag{1}$$

This means that, starting from i units, the probability of the gambler winning 1 unit is p, which moves i to $i+1$, and the probability of the gambler losing 1 unit is q, which moves i to $i-1$.

The equation above is equivalent to:

$$1 \times P(i) = pP(i+1) + qP(i-1) \tag{2}$$

Since $q = 1 - p$, so $1 = p + q$. We can replace the 1 in the equation with $p + q$. Hence:

$$(p+q)P(i) = pP(i+1) + qP(i-1) \tag{3}$$

Rearranging the equation, we get:

$$p[P(i+1) - P(i)] = q[P(i) - P(i-1)] \tag{4}$$

$$P(i+1) - P(i) = (q/p)[P(i) - P(i-1)] \tag{5}$$

From the structure of the equation, we can clearly see that there exists a recursion pattern. We can plug in some values of i to test it out:

For $i = 1, 2, 3$, we have:

$$P(2) - P(1) = (q/p)[P(1) - P(0)] \tag{6}$$

$$P(3) - P(2) = (q/p)[P(2) - P(1)] \tag{7}$$

$$P(4) - P(3) = (q/p)[P(3) - P(2)] \tag{8}$$

Since the left-hand side of (6) is equivalent to the $P(2) - P(1)$ in the right-hand side of (7). We can replace the $[P(2) - P(1)]$ in the right-hand side of (7) with $(q/p)[P(1) - P(0)]$. Hence we have:

$$P(3) - P(2) = (q/p)^2[P(1) - P(0)] \tag{9}$$

Since $P(0) = 0$, we have:

$$P(3) - P(2) = (q/p)^2 P(1) \tag{10}$$

Summarizing this pattern, for $i = 1, 2, 3, ..., n$, we have:

$$P(i) - P(i-1) = (q/p)^{i-1} P(1) \tag{11}$$

Since the equation above is true, we have:

$$\sum_{k=1}^{i} P(k) - P(k-1) = \sum_{k=1}^{i} (q/p)^{k-1} P(1) \tag{12}$$

But we noticed that, since $P(i) - P(i-1)$ follows a recursion pattern, for $i = 1, 2, 3, ..., n$:

$$\sum_{k=1}^{i} P(k) - P(k-1) = P(i) \tag{13}$$

$$P(i) = \sum_{k=1}^{i} (q/p)^{k-1} P(1) \tag{14}$$

$$P(i) = P(1)[1 + (q/p) + (q/p)^2 + ... + (q/p)^{i-1}] \tag{15}$$

We noticed that $[1 + (q/p) + (q/p)^2 + ... + (q/p)^{i-1}]$ is a finite geometric series. So using the geometric sum formula, we can see that:

$$P(i) = P(1)[1 - (q/p)^i]/[1 - (q/p)] \quad (16)$$

Keep in mind this formula only works when $p \neq q$. If $p = q = 1/2$:

$$P(i) = iP(1) \quad (17)$$

Now, we can plug in the value $i = n$:
For $p \neq q$:

$$P(n) = P(1)[1 - (q/p)^n]/[1 - (q/p)] \quad (18)$$

For $p = q = 1/2$:

$$P(n) = nP(1) \quad (19)$$

Since $P(n)$ is one of our boundary values, and $P(n) = 1$, we can solve for the value of P(1):
For $p \neq q$:

$$P(1) = P(n)[1 - (q/p)]/[1 - (q/p)^n] \quad (20)$$

For $p = q = 1/2$:

$$P(1) = 1/n \quad (21)$$

As such, we have found a general solution for $P(i)$:
For $p \neq q$:

$$P(i) = [1 - (q/p)^i]/[1 - (q/p)^n] \tag{22}$$

For $p = q = 1/2$:

$$P(i) = i/n \tag{23}$$

Hence we have found the solution to this problem.