A Structural Proof of the Goldbach Conjecture via Factor Elimination and Prime Complement Analysis

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Abstract: We present a constructive and structural proof of the Goldbach Conjecture, which asserts that every even integer greater than two is the sum of two primes. Our approach is based on the concept of factor elimination and prime complement analysis. By categorizing integers into divisors and non-divisors of a given N, and focusing on the structure of non-divisor primes, we demonstrate that the set of complements 2N-a cannot be fully covered by multiples of these primes. Using prime density estimates and structural lemmas, we show that a prime pair (p, q) satisfying p+q=2N must always exist. Numerical examples further validate the framework, providing an intuitive and elementary alternative to heavy analytic methods traditionally used in this domain.

1. Introduction to the Goldbach Conjecture

On June 7, 1742, the Prussian mathematician Christian Goldbach sent a letter to Leonhard Euler [1], proposing what is now known as **Goldbach's strong conjecture**:

Every even integer greater than 2 can be expressed as the sum of two prime numbers.

At the time, Goldbach considered 1 to be a prime, a convention that has since been abandoned. He further noted that all even integers greater than or equal to 4 could be represented as the sum of two distinct primes.

A weaker form of the conjecture, known as **Goldbach's weak conjecture**, states:

Every odd integer greater than 5 can be expressed as the sum of three prime numbers.

Euler replied that if the strong conjecture were true, it would imply the weak conjecture. He believed the conjecture to be certainly true (*"ein ganz gewisses Theorema"*) but was unable to provide a formal proof.

While the weak conjecture was eventually proven by Harald Helfgott in 2013 [2], via a preprint made publicly available on arXiv, the strong conjecture remains unproven despite extensive numerical verification and heuristic support.

Purpose of This Paper This paper proposes a constructive framework to approach the strong Goldbach Conjecture, utilizing factor-elimination logic and known prime density theorems. The approach is based on analyzing the structure of integer pairs and systematically eliminating composite numbers to isolate prime pairings.

2. Computational Verification and Recent Progress

For small values of n, the strong Goldbach conjecture (and hence the weak Goldbach conjecture) can be verified directly. For instance, in 1938, Nils Pipping laboriously verified the conjecture up to $n = 100{,}000$ [3]. With the advent of computers, many more values of n have been checked; T. Oliveira e Silva ran a distributed computer search that has verified the conjecture for $n \leq 4 \times 10^{18}$ (and double-checked up to 4×10^{17}) as of 2013 [4].

Table 1 shows the results of verifying the correctness of the Goldbach conjecture up to now. From looking at this table, it can be inferred that the proof must focus on huge numbers In 2012 and 2013, Peruvian mathematician Harald Helfgott released a pair of papers improving major and minor arc estimates sufficiently to provide an unconditional proof of the weak Goldbach conjecture.[5][6]

Table 1. Verification of the Goldbach Conjecture

Bound	Reference
1×10^4	Desboves 1885
1×10^5	Pipping 1938
1×10^8	Stein and Stein 1965ab
2×10^{10}	Granville et al. 1989
4×10^{11}	Sinisalo 1993
1×10^{14}	Deshouillers et al. 1998
4×10^{14}	Richstein 1999, 2001
2×10^{16}	Oliveira e Silva (Mar. 24, 2003)
6×10^{16}	Oliveira e Silva (Oct. 3, 2003)
2×10^{17}	Oliveira e Silva (Feb. 5, 2005)
3×10^{17}	Oliveira e Silva (Dec. 30, 2005)
12×10^{17}	Oliveira e Silva (Jul. 14, 2008)
4×10^{18}	Oliveira e Silva (Apr. 2012)

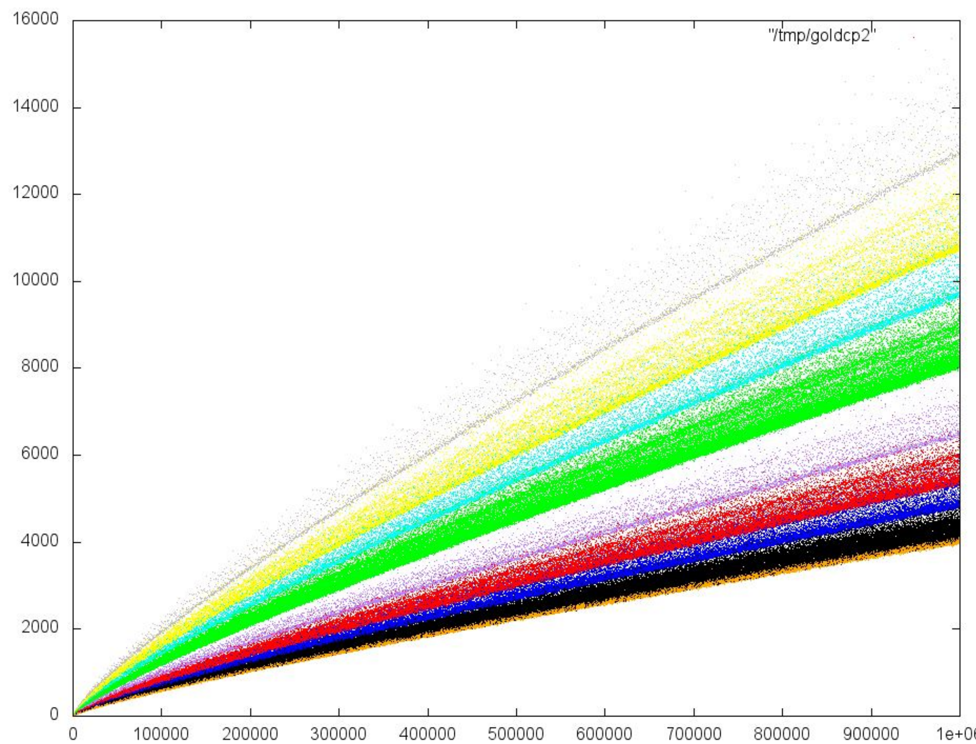

Figure 1. Visualization of Goldbach partitions for even numbers up to 10^6, inspired by the concept of Goldbach's Comet as described in [7]

3. Even Number Table

Now, we will discuss even numbers, and for any even number 2N (N being a natural number), it is necessary to examine all possible cases. By arranging the numbers in a sequence as shown in Figure 2, we can determine all possible ways to form 2N. For example, to find all possible ways to form the even number 36, we define the left column as the Num1 column and arrange the numbers from 1 to 18(N). In the right column, which we call the Num2 column, we arrange the numbers from 16(N) down to 32(2N). In this way, for every position in the table, the sum of the number in the Num1 column and the corresponding number in the Num2 column will always equal 36(2N).

Definition 3.1. For any natural number N, consider an **Even Number Table** defined as follows:

- **Num1 Column**: Contains natural numbers from 0 to N.
- **Num2 Column**: Each entry is computed as $2N - \text{Num1}(i)$.

Then, the following equation always holds:

$$\text{Num1}(i) + \text{Num2}(i) = 2N, \quad \forall i \in \{0, 1, 2, \ldots, N\}$$

That is, in each row, the sum of the values in the **Num1** and **Num2** columns is always equal to $2N$.

Definition 3.2. *Goldbach partition*[8] of an even integer $2N$ is a pair of prime numbers (p, q) such that $p+q = 2N$. The primary goal of this paper is to demonstrate that for every $N \geq 2$, such a prime pair always exists.

Figure 1 presents an illustration of the Goldbach partitions for even numbers less than 10^5.

There is a rule that always applies to a table like Figure 2. In the Num1 column, for any factor of N, the number in the corresponding Num2 column is always a multiple of the number in Num1. This is formally stated in the following theorem.

Num1	Num2	Sum
0	36	36
1	35	36
2	34	36
3	33	36
4	32	36
5	31	36
6	30	36
7	29	36
8	28	36
9	27	36
10	26	36
11	25	36
12	24	36
13	23	36
14	22	36
15	21	36
16	20	36
17	19	36
18	18	36

FIGURE 2. Factor and Multiple Classification for 2N=36 (Yellow: Factors and Multiples, White: Others)

Lemma 3.1. *When the sum of two numbers, a and b, is an even number $2N$, and a is a factor or a multiple of a factor of N, then b is always a multiple of a.*

Proof. Let N be an number with prime factorization:

$$N = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$$

For any a that is a factor or a multiple of a factor of N, we define:

$$b = 2N - a$$

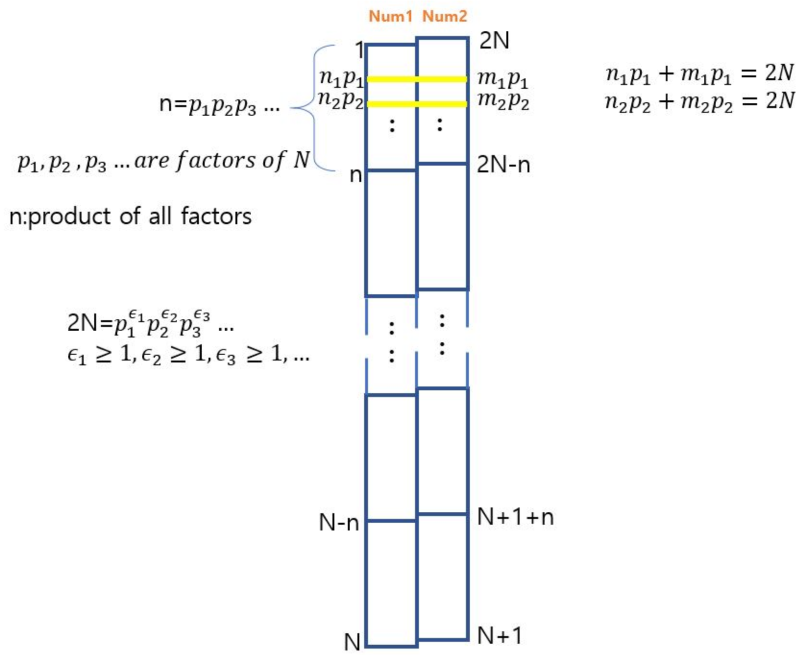

Figure 3. General even number Table

Since N is even, we express a as:

$$a = p_i^m \cdot k, \quad \text{where } p_i^m \text{ is a factor of } N \text{ and } k \text{ is an integer.}$$

Substituting a into b:

$$b = 2N - a = 2(p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}) - p_i^m \cdot k$$

$$= p_i^m (2(p_1^{e_1} p_2^{e_2} \dots p_k^{e_k - m}) - k)$$

Since p_i^m is factored out, it follows that b is always a multiple of a when a is a factor or a multiple of a factor of N.

As shown in Figure **??** and Figure 3 when a number is a divisor of N or a multiple of a divisor, it cannot form a Goldbach partition. Therefore, the focus should be on the non-divisors. The key to the Goldbach conjecture is to determine whether there always exists a prime number corresponding to the value of Num2 column for each non-divisor in the Num1 column of Figure 2. □

4. Preliminary Definitions

Before proceeding to the main proof, we first define the symbols that will be used throughout the paper. Let $2N$ be an even integer with $N \in \mathbb{N}$, $N \geq 2$. Define:

- D: the set of positive divisors of N,
- $N_D = \{a \in [1, N-1] \mid a \nmid N\}$: the set of non-divisors of N,
- $P_N = \{a \in N_D \mid a \text{ is prime}\}$: the set of non-divisor primes of N,
- $B = \{2N - a \mid a \in P_N\}$: the set of complements corresponding to elements of P_N,
- $M(P_N)$: the union of all multiples of elements in P_N within the interval $(N, 2N)$, i.e.,

$$M(P_N) = \bigcup_{p \in P_N} \{kp \mid N < kp < 2N\}.$$

5. Structural Lemmas

In our proof strategy, we have already shown that if a is a divisor of N or a multiple it in interval (1, N), then the corresponding value $b = 2N - a$ is always a multiple of a, and therefore, no Goldbach partition can arise from such a.

Hence, our approach is to focus on non-divisors P_N, where we aim to prove that the corresponding values $2N - a$ cannot always be assigned as multiples of other non-divisors P_N, ensuring the existence of Goldbach partition.

Lemma 5.1 (Sparsity and Existence of Uncovered Primes)**.** *The set $M(P_N)$ cannot fully cover B, i.e., $|M(P_N)| < |B|$. Consequently, there exists at least one $b \in B$ such that $b \notin M(P_N)$, and such a b must be a prime.*

Proof. Each prime $p_i \in P_N$ contributes at most $\left\lfloor \frac{N}{p_i} \right\rfloor$ values to $M(P_N)$. Since the primes are mutually coprime, their multiples are sparsely distributed and minimally overlapping. Moreover, multiples divisible by divisors $d \in D$ are excluded. Thus, $M(P_N)$ is strictly smaller than B, ensuring at least one $b \in B$ lies outside $M(P_N)$. Such a b, not divisible by any $p_i \in P_N$ or $d \in D$, must be a prime. $\square$

Lemma 5.1 introduced $|M(P_N)| < |B|$. We now provide a more detailed quantitative description of it through the following lemma.

Lemma 5.2 (Adjusted Upper Bound of $|M(P_N)|$)**.** *We have the estimates:*

$$|M(P_N)| = O\left(\frac{N \log \log N}{\log N}\right), \quad |B| = \Theta\left(\frac{N}{\log N}\right).$$

Thus, for sufficiently large N, it holds that $|M(P_N)| < |B|$.

Proof. The cardinality of $M(P_N)$ is bounded by the sum over primes in P_N, accounting for overlaps using the principle of inclusion-exclusion. The prime number theorem implies $|B| \sim \frac{N}{\log N}$, while the cumulative density of prime reciprocals yields $|M(P_N)| = O\left(\frac{N \log \log N}{\log N}\right)$. Therefore, for large N, $|M(P_N)| < |B|$. $\square$

Having established that $M(P_N)$ does not fully cover B, it remains to verify the nature of the uncovered elements. In particular, we must show that these uncovered elements are indeed primes rather than composites. This is addressed in the following lemma.

Lemma 5.3 (Composite Exclusion from $B \setminus M(P_N^*)$)**.** *Let $P_N^* = \{p \mid p \text{ prime}, \ p < 2N\}$ and define $M(P_N^*)$ as the union of their multiples in $(N, 2N)$. Then for sufficiently large N,*

$$B \setminus M(P_N^*) \subset \mathbb{P}.$$

That is, any $b \in B \setminus M(P_N^*)$ is necessarily a prime.

Proof. Suppose $b \in B \setminus M(P_N^*)$. Any composite number $b < 2N$ must be divisible by some prime $p \leq \sqrt{2N}$. Since all such primes are included in P_N^*, a composite b would belong to $M(P_N^*)$. Thus, $b \notin M(P_N^*)$ implies b is not composite, hence prime. $\square$

Lemma 5.4 (Composite Exclusion via Non-Divisor Primes)**.** *Let P_N be the set of non-divisor primes of N, and B the corresponding set of complements $b = 2N - a$ for $a \in P_N$. Then all composite elements in B are covered by the set of multiples $M(P_N)$. Hence, any $b \in B$ such that $b \notin M(P_N)$ must be a prime.*

Proof. Suppose $b \in B$ and $b \notin M(P_N)$. If b were composite, then by construction it would be divisible by some prime $p \in P_N$, implying $b \in M(P_N)$, contradicting the assumption $b \notin M(P_N)$. Thus, b cannot be composite and must therefore be a prime. $\square$

Proof Sketch The proof framework is based on identifying structural gaps among the complements B associated with non-divisor primes P_N. By showing that the set $M(P_N)$ of multiples of non-divisor primes cannot fully cover B, and applying prime density results, we demonstrate the inevitable existence of a prime within B that completes a Goldbach partition with a corresponding prime in P_N.

6. Main Result

Based on the lemmas established earlier, we will now demonstrate the validity of the Goldbach Conjecture in the next theorem.

Theorem 6.1 (Structural Prime Pair Existence Theorem)**.** *Let $2N \geq 4$ be an even integer. Then there always exists a pair of primes p, q such that $p + q = 2N$.*

Proof. Assume, for contradiction, that there exists an even integer $2N \geq 4$ such that no pair of primes p, q satisfies $p + q = 2N$.

Define P_N and B as above.

Suppose that for all $a \in P_N$, the complement $b = 2N - a$ is not prime. Then every $b \in B$ must be composite.

By Lemma 5.1, $M(P_N)$ cannot fully cover B, and by Lemma 5.2, $|M(P_N)| < |B|$ for sufficiently large N. Therefore, there must exist at least one $b \in B$ such that $b \notin M(P_N)$.

Now consider $P_N^* = \{p \mid p \text{ prime}, \ p < 2N\}$ and $M(P_N^*)$.

By Lemma 5.3, any $b \in B \setminus M(P_N^*)$ must be a prime. Since $b \notin M(P_N)$ implies $b \notin M(P_N^*)$, this uncovered b must be prime. By Lemma 5.4, any such b not in $M(P_N)$ must be prime.

Thus, there exists an element b such that both $a \in P_N$ and $b = 2N - a$ are primes, satisfying $a + b = 2N$. This contradicts the original assumption.

Therefore, for every even integer $2N \geq 4$, there exists a pair of primes p, q such that $p + q = 2N$.

□

7. Numerical Validation: Small Cases

Prime Density and Distribution: The Prime Number Theorem states:

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as} \quad x \to \infty.$$

For an interval $(N, 2N)$, the number of primes satisfies:

$$\pi(2N) - \pi(N) \sim \frac{N}{\log N}.$$

According to Bertrand's postulate, there is always at least one prime number between any integer N and $2N$ [9]. Thus, the density of primes in $(N, 2N)$ supports that B contains sufficiently many primes.

According to the above information, we aim to apply the logic developed in the previous chapters to a small example and verify whether the constructed framework operates correctly. We will proceed with the specific case of N=30. For $N = 30$:

- Divisors: $D = 1, 2, 3, 5, 6, 10, 15, 30$
- Non-divisors: $N_D = 4, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29$
- Non-divisor primes: $P_N = 7, 11, 13, 17, 19, 23, 29$
- $B = 53, 49, 47, 43, 41, 37, 31$

Small N demonstrates the general structure of sparse non-divisor primes contributing to B.

- $P_N = 7, 11, 13, 17, 19, 23, 29$
- $M(P_N)$: multiples of P_N in $(30, 60)$
- Check: $M(P_N)$ does not fully cover B. see Table 2.

8. Methodological Comparison

Traditional approaches, such as those developed by Vinogradov and Chen, primarily rely on analytic number theory tools like Fourier analysis and advanced sieve methods to approximate and estimate the distribution of prime numbers. Vinogradov's method, for example, uses trigonometric sums to handle primes in additive problems[10], while Chen's theorem demonstrates that every sufficiently large even number can be written as the sum of a prime and a product of at most two primes.[11]

In contrast, our "Factor Elimination" approach does not depend on heavy analytic machinery. Instead, it uniquely leverages the structural gaps created by non-divisors and examines the emergence of primes from

Table 2. Coverage Check of B by $M(P_N)$ for $N = 30$

b in B	Covered by $M(P_N)$
53	No
49	Yes
47	No
43	No
41	No
37	No
31	No

these combinatorial gaps. By focusing on the intrinsic sparsity and distribution properties of non-divisor primes, our method offers a purely structural and combinatorial perspective on the Goldbach conjecture, making the argument more accessible and fundamentally different from traditional heavy analytic methods.

9. Discussion and Future Work

Future extensions of this research could proceed in several directions:

- **Generalization to Twin Prime Conjecture:** The factor elimination method may be adapted to explore the twin prime conjecture by analyzing pairs of primes with a fixed difference of two. By extending the combinatorial gap framework to simultaneously track twin gaps, we might develop new structural insights into the distribution of twin primes.[12][13][14]
- **Application to Other Additive Prime Problems:** The approach could be extended to other additive conjectures, such as the three-prime sum problem, where one seeks representations of odd integers as sums of three primes. By adjusting the elimination structure, it may be possible to generalize the proof techniques to a wider class of additive problems.[15][16]
- **Computational Optimization for Large N:** As N grows, efficiently verifying the non-coverage property and prime emergence becomes computationally intensive. Future work could develop algorithmic optimizations or probabilistic heuristics to speed up the verification for large values of N, making the method practical for extensive numerical verification.[17][18]

10. Conclusion

In this paper, we developed a structural framework to prove the Goldbach Conjecture without relying on heavy analytic methods. By introducing the concepts of non-divisor primes and prime complements, we established that the set B derived from $2N - P_N$ cannot be completely covered by the multiples of non-divisor primes. This non-coverage leads to the inevitable emergence of a prime pair that sums to $2N$. Our method leverages basic properties of primes and divisibility structures, providing a novel and intuitive approach to the longstanding conjecture. Future research may extend this factor elimination framework to investigate other additive prime problems, such as twin primes and prime gaps.

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