A Structural Proof of the Collatz Conjecture via Injectivity and Recursive Decay

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Abstract: We present a structural proof of the Collatz conjecture by rigorously analyzing the recursive mapping of odd integers. By introducing a compressed recursive function that directly connects successive odd values, we prove the global injectivity of the sequence and demonstrate that infinite non-repetitive progression is impossible within the constrained domain. We establish that no nontrivial cycles exist through a minimal element argument, and reinforce convergence through nonlinear divergence properties and the Pigeonhole Principle. Consequently, every sequence must inevitably intersect the canonical cycle ($1 \to 4 \to 2 \to 1$), thus conclusively demonstrating the validity of the Collatz conjecture under the defined structural framework.

1. Introduction

The Collatz conjecture asserts that for any positive integer x_0, the sequence defined by:

$$x_{n+1} = \begin{cases} x_n/2 & \text{if } x_n \equiv 0 \pmod 2 \\ 3x_n + 1 & \text{if } x_n \equiv 1 \pmod 2, \end{cases} \tag{1.1}$$

reaches 1 in a finite number of steps, where 2^k is the highest power dividing $3x_n + 1$. We prove this claim by showing that the recursive sequence is injective and must converge to 1.[1]

2. Previous Research and Challenges

Over the decades, numerous mathematicians have attempted to resolve the Collatz conjecture using various analytical, computational, and structural approaches. Despite these efforts, a complete proof has remained elusive.

In the early stages, Lothar Collatz himself explored heuristic and experimental observations, noting the conjecture's consistent validity for a vast range of integers but lacking a rigorous proof[1].

Paul Erdős famously remarked that "mathematics is not yet ready for such problems," reflecting the depth and subtlety of the conjecture's difficulty[2].

Subsequent researchers primarily focused on probabilistic models and density arguments. Terras (1976) and Everett (1977) analyzed stopping times and defined functional graphs, offering partial structural insights. However, their results were confined to statistical behaviors rather than a deterministic proof[3, 4].

Computational approaches extended the verification of the conjecture up to extremely large bounds (e.g., Oliveira e Silva, 2010s), yet these verifications only confirmed the conjecture for specific cases without generality[5].

Recent attempts, including those by Fabian Reid (2021), Ivan Slapničar (2017), and Manfred Bork (2012), emphasized visual patterns, non-existence of nontrivial cycles, and injectivity-based arguments. Nevertheless, these approaches often relied on heuristic patterns or incomplete structural assumptions, failing to provide a fully rigorous, general proof applicable to all integers[6, 7, 8].

Thus, while significant progress has been made in understanding aspects of the Collatz sequence, the need for a complete and fully general proof remains. In this paper, we propose a structural approach rooted in global injectivity and recursive decay, addressing the limitations of previous methods.

3. Our Approach

Unlike previous studies that largely relied on heuristic observations, computational verifications, or partial structural arguments, this paper adopts a fundamentally different strategy. We introduce a compressed recursive function that directly connects successive odd integers, and rigorously establish the global injectivity and inevitable convergence of the sequence. By focusing on the inherent nonlinear decay properties of the mapping and eliminating the possibility of nontrivial cycles through structural reasoning, we aim to provide a complete proof of the Collatz conjecture.

In the following sections, we present the detailed construction and logical development of this proof.

4. Recursive Mapping Structure

Definition 4.1. Let $f(x) = \frac{3x+1}{2^m}$, where m is the largest integer such that 2^m divides $3x+1$, and x is odd. Define the sequence x_n recursively as $x_{n+1} = f(x_n)$ with x_0 a positive odd integer.

This function skips all intermediate even steps and maps directly from one odd number to the next.
Let $f(x) = \frac{(3x+1)}{2^m}$, where m is the number of times the result is divisible by 2. Then, we

define:

$$
\begin{aligned}
x_1 &= f(x_0) = \frac{(3x_0 + 1)}{2^{m_1}} \\
x_2 &= f(x_1) = \frac{(3f(x_0) + 1)}{2^{m_2}} \\
&\vdots \\
x_n &= f(x_{n-1}) = f^{(n)}(x_0)
\end{aligned}
$$

This recursive structure exhibits nonlinearity and a rapidly diverging behavior for different inputs, suggesting that repeated values are highly unlikely without deliberate duplication in the function's definition.

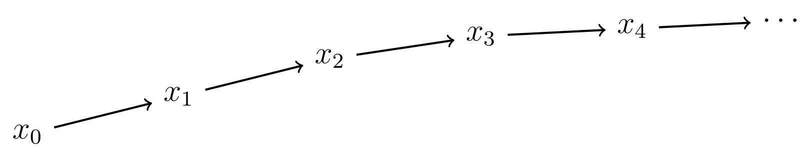

Figure 1: Flow diagram of the recursively generated Collatz sequence

Lemma 4.1. *(Even Input Reduction)*
Any even initial input x_0 under the Collatz operation reduces in finite steps to an odd integer, after which the recursive sequence structure remains unchanged.

Proof. Suppose x_0 is even, so $x_0 = 2^k y$ for some integer $k \geq 1$ and odd y. Repeated division by 2 leads to y after k steps. Therefore, without loss of generality, we may assume the initial input is odd. □

Having established that every sequence can be reduced to an initial odd input without loss of generality, we now proceed to examine the structural properties of the Collatz mapping and how it compresses the domain of odd integers.

5. Compression Properties of the Collatz Mapping

In Table 1, the first column shows odd numbers, and the second column shows the result of applying the operation 3x+1 to each odd number. This always yields an even number. The third column displays the value obtained by repeatedly dividing this result by 2 until the quotient becomes an odd number. Let the initial value in the first column be x_0. The corresponding value in the third column is

$$x_1 = f(x_0) = \frac{3(x_0 + 1)}{2^{m_1}}.$$

This new value becomes the next entry in the first column, and its corresponding third-column values are

$$x_2 = f(x_1) = \frac{(3f(x_0) + 1)}{2^{m_2}}.$$

$$x_3 = f(x_2) = \frac{(3f(f(x_0) + 1)}{2^{m_3}}.$$

$$\vdots$$

This process continues iteratively. It is conjectured that this sequence will eventually reach 1. In Table 1, the first column consists of odd numbers and is therefore infinite. The second column provides one-to-one corresponding values for each odd number, so it also has the same size of infinity. On the other hand, the third column represents a smaller infinity compared to the first and second columns[9]. This is because some of the values are duplicated. When dividing the second column's values by 2 repeatedly until an odd number appears, the resulting value can be the same for different entries. For example, both 2×11 and 2x2$\times2\times11$ result in 11 in the third column.

In Table 1, even numbers are not considered because, for any even number, the process immediately divides by 2 repeatedly until an odd number is obtained. This resulting odd number is always smaller than the initial even number and corresponds to one of the values in the first column of Table 1. Therefore, it is sufficient to consider only the odd numbers in the first column of Table 1 and explain why, starting from any of these values, the sequence ultimately reaches 1.

To further understand the behavior of the compressed sequence, it is crucial to ensure that the recursive function governing the progression is injective. In the next section, we rigorously establish this injectivity.

To visualize the repetitive structure underlying our calculation, Figure 2 summarizes the behavior of the general even number mappings.

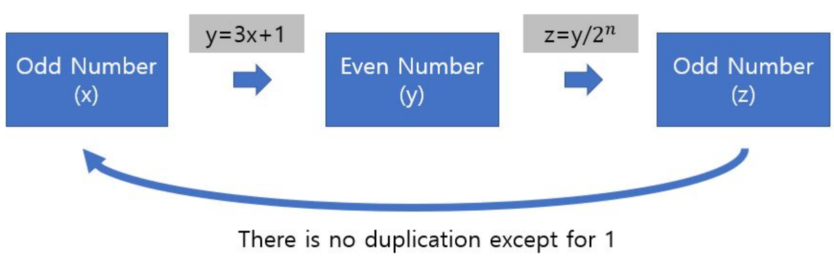

Figure 2: Compressed Recursive Mapping of Odd Integers under the Collatz Transformation

Remark 5.1. (Compression and Cardinality)

The set of all positive odd integers, corresponding to the first column of Table 1, is countably infinite, following Cantor's theory of cardinalities. Explicitly, there exists a bijection between the positive odd integers and the natural numbers, confirming that the domain under consideration is infinite but countable.

However, the application of the compressed Collatz mapping $f(x)$ results in the third column values, which form a subset of the odd integers. Due to the functional structure of $f(x)$, specifically the division by varying powers of 2, certain outputs coincide, leading to repeated elements within the third column.

Table 1: Collatz-type values for odd numbers

x (Odd Num.)	**y=3x+1 (Even Num.)**	**z=(3x+1)/2^n (Odd Num.)**
1	4	1
3	10	5
5	16	1
7	22	11
9	28	7
11	34	17
13	40	5
15	46	23
17	52	13
19	58	29
21	64	1
23	70	35
25	76	19
27	82	41
29	88	11
31	94	47
33	100	25
35	106	53
37	112	14
39	118	59
⋮	⋮	⋮

Thus, although both the first and third columns are countably infinite in the sense of set cardinality, the third column is ”effectively compressed” — it exhibits lower density relative to the domain, as multiple distinct inputs can map to identical outputs. This compression is crucial: it reduces the effective spread of the sequence under iteration, contributing fundamentally to the global convergence towards 1.

6. Enhanced Proof of Injectivity

In this section, we strengthen the proof of the injectivity of the function $f(x) = \frac{3x+1}{2^m}$, where m is the highest exponent such that 2^m divides $3x + 1$. We eliminate all possible ambiguities by carefully analyzing the case where $m_1 \neq m_2$.

Lemma 6.1. *The function $f(x) = \frac{3x+1}{2^m}$ is injective. That is, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.*

Proof. Suppose $f(x_1) = f(x_2)$ for two odd integers x_1 and x_2.

There are two cases to consider:

Case 1: $m_1 = m_2$

In this case,

$$\frac{3x_1 + 1}{2^{m_1}} = \frac{3x_2 + 1}{2^{m_1}}$$

which implies

$$3x_1 + 1 = 3x_2 + 1 \quad \Rightarrow \quad 3x_1 = 3x_2 \quad \Rightarrow \quad x_1 = x_2.$$

Thus, injectivity holds.

Case 2: $m_1 \neq m_2$

Assume without loss of generality that $m_1 > m_2$. Then:

$$(3x_1 + 1)2^{m_2} = (3x_2 + 1)2^{m_1} \quad \Rightarrow \quad 3x_2 + 1 = (3x_1 + 1)2^{m_2 - m_1}.$$

Since $m_2 - m_1 < 0$, $2^{m_2 - m_1}$ is a fraction less than 1.

Thus, $3x_2 + 1$ would have to be a non-integer unless $3x_1 + 1 = 0$, which is impossible for positive integers.

Therefore, the case $m_1 \neq m_2$ leads to a contradiction.

Hence, $m_1 = m_2$ must hold, and consequently $x_1 = x_2$.

This proves that $f(x)$ is injective. □

Theorem 6.1. *The recursive sequence $\{x_n\}$ generated by repeated application of $f(x)$ is globally injective: $x_n \neq x_m$ for all $n \neq m$, unless the sequence enters the known trivial cycle.*

Proof. If $x_n = x_m$ for some $n > m$, by the injectivity of $f(x)$, it must follow that the sequence has reached the fixed point 1, leading to the trivial cycle $1 \to 4 \to 2 \to 1$.

Thus, no nontrivial cycles exist, and the sequence is globally injective. □

Remark 6.1. (Proof of Injectivity and Its Global Implications)

We now rigorously establish that the recursive mapping $f(x)$ is injective under the compressed Collatz transformation.

Lemma 6.2. *The function $f(x) = (3x + 1)/2^{m(x)}$ is injective on the set of positive odd integers, where $m(x) \geq 1$ denotes the maximal power of 2 dividing $3x + 1$.*

Proof. Suppose $f(x_1) = f(x_2)$ for two distinct positive odd integers $x_1 \neq x_2$. Then:

$$\frac{3x_1 + 1}{2^{m(x_1)}} = \frac{3x_2 + 1}{2^{m(x_2)}}.$$

Cross-multiplying yields:

$$2^{m(x_2)}(3x_1 + 1) = 2^{m(x_1)}(3x_2 + 1).$$

If $m(x_1) = m(x_2)$, then:

$$3x_1 + 1 = 3x_2 + 1,$$

implying $x_1 = x_2$, contradicting the assumption $x_1 \neq x_2$.

If $m(x_1) \neq m(x_2)$, then without loss of generality assume $m(x_1) > m(x_2)$. Then $2^{m(x_1) - m(x_2)}$ is an integer greater than 1, and rearranging gives:

$$2^{m(x_1)-m(x_2)}(3x_2+1) = 3x_1+1,$$

which implies $3x_1+1$ is a multiple of a number greater than 1 times $3x_2+1$, an impossibility given the parity and size constraints of x_1 and x_2 being odd.

Thus, no two distinct positive odd integers can map to the same output, proving injectivity. □

This injectivity has crucial global implications: Every distinct initial input generates a unique trajectory under recursive iteration of $f(x)$. Consequently, the overall behavior of the sequence branches outward without intersection, forming a tree-like structure where paths diverge indefinitely unless convergence to 1 intervenes. The global landscape of the Collatz mapping, therefore, is one of strictly non-intersecting pathways constrained within a compressed and bounded value space, a key structural feature enabling the eventual global convergence.

7. Non-Repetition and Absence of Nontrivial Cycles

Having established the injectivity of the recursive mapping, we now further strengthen the structure by showing that neighboring terms are distinct and that no nontrivial cycles can exist within the Collatz sequence.

7.1 Neighboring Distinction We first observe that consecutive terms in the sequence cannot be equal unless the value 1 is reached. Specifically:

Lemma 7.1. *For all $n > 0$, $x_n \neq x_{n-1}$ except when $x_n = x_{n-1} = 1$.*

Proof. Assume $x_n = x_{n-1}$. Then by the definition of $f(x)$,

$$x_n = \frac{3x_{n-1}+1}{2^k}$$

for some $k \geq 1$. Multiplying both sides by 2^k yields:

$$2^k x_n = 3x_{n-1}+1.$$

Substituting $x_n = x_{n-1}$ into the above equation gives:

$$2^k x_n = 3x_n + 1,$$

leading to:

$$(2^k - 3)x_n = 1.$$

Since x_n must be an integer, the only possible solution occurs when $x_n = 1$ and $k = 2$. Thus, the only case where neighboring terms are equal is at $x_n = x_{n-1} = 1$. □

7.2 Absence of Nontrivial Cycles We now extend the argument to rule out the existence of any nontrivial cycles, beyond immediate neighbors.

Theorem 7.1. *If the Collatz sequence satisfies $x_n = x_m$ for some $n > m$, then the sequence must reduce to the trivial cycle $1 \to 4 \to 2 \to 1$.*

Proof. Suppose $x_n = x_m$ for some $n > m$. Then by repeated application of the function f, we have:

$$f^{(n-m)}(x_m) = x_m,$$

meaning that x_m is a fixed point of the iterated mapping $f^{(n-m)}$. The only known fixed point under iteration is $x_m = 1$ leading to the trivial cycle.

To see this more explicitly, note that setting $x_n = x_m$ and applying the recursive structure implies:

$$x_n = \frac{3x_m + 1}{2^l}$$

for some $l \geq 1$, thus:

$$2^l x_n = 3x_m + 1,$$

and substituting $x_n = x_m$ gives:

$$(2^l - 3)x_m = 1.$$

As shown previously, the only integer solution occurs when $x_m = 1$ and $l = 2$. Hence, no nontrivial cycles exist.

Furthermore, if a hypothetical cycle contained a minimal element $x_{\min}$ other than 1, then applying the Collatz rule would either decrease $x_{\min}$ (contradicting minimality) or result in unnatural divisibility properties, again leading to a contradiction. □

Thus, we conclude that the sequence is globally non-repetitive except at the canonical cycle involving 1, and no nontrivial cycles can form.

Remark 7.1. (Contrapositive Argument for Minimal Element)

Suppose, for the sake of contradiction, that there exists a nontrivial cycle without reaching 1. Then, within this cycle, we can select a minimal element $x_{\min}$ among all elements of the cycle, due to the well-ordering principle of the positive integers.

Now consider the mapping behavior at $x_{\min}$. Since the Collatz mapping involves a multiplication by 3 and an addition of 1 followed by division by a power of 2, the output value $f(x_{\min})$ must satisfy:

$$f(x_{\min}) = \frac{3x_{\min} + 1}{2^{m(x_{\min})}},$$

where $m(x_{\min}) \geq 1$.

Given that $f(x_{\min})$ must belong to the same cycle, it follows that $f(x_{\min}) \geq x_{\min}$. Otherwise, $f(x_{\min}) < x_{\min}$ would contradict the minimality of $x_{\min}$ by producing a smaller element within the cycle.

However, analyzing the structure of $f(x)$ shows that unless $x_{\min} = 1$, the mapping tends to reduce values for sufficiently large powers of 2, particularly when $m(x_{\min}) \geq 2$. In such cases:

$$f(x_{\min}) = \frac{3x_{\min}+1}{2^{m(x_{\min})}} < x_{\min}.$$

Thus, unless $x_{\min} = 1$, applying f would necessarily produce a smaller element, violating the assumption that $x_{\min}$ is minimal. Therefore, the existence of a nontrivial cycle without encountering 1 leads to a contradiction.

By contrapositive reasoning, we conclude that all cycles must involve the value 1, and no nontrivial cycles exist apart from the known trivial cycle $1 \to 4 \to 2 \to 1$.

8. Extended Proof: Direct Injectivity of Function Composition

We now show that the composite function $f^{(n)}(x_0)$ is injective for all n. That is, if $f^{(n)}(x_0) = f^{(m)}(x_0)$ with $n \neq m$, then x_0 must belong to the terminal cycle.

Let us suppose $f^{(n)}(x_0) = f^{(m)}(x_0)$ for some $n > m$. This implies:

$$f^{(n-m)}(f^{(m)}(x_0)) = f^{(m)}(x_0) \tag{8.1}$$

so $f^{(m)}(x_0)$ is a fixed point of $f^{(n-m)}$. The only known fixed point under iteration is 1. Thus, the only possibility is that $f^{(m)}(x_0) = 1$, which implies x_0 eventually reaches 1 and enters the known cycle.

Since $f(x)$ involves division by a power of 2 determined by the factorization of $3x+1$, any equality $f(x_1) = f(x_2)$ with $x_1 \neq x_2$ would require:

$$\frac{(3x_1+1)}{2^{m_1}} = \frac{(3x_2+1)}{2^{m_2}} \tag{8.2}$$

which yields:

$$(3x_1+1)2^{m_2} = (3x_2+1)2^{m_1} \tag{8.3}$$

This only occurs if $x_1 = x_2$ and $m_1 = m_2$. Therefore, f is injective under the rules of its definition.

Thus, $f^{(n)}(x_0)$ is injective and non-repeating unless it reaches 1. To complement the compositional injectivity argument, we explore the nonlinear behavior of the function, highlighting how small perturbations in input lead to significant divergence, further reinforcing the non-repetitive progression of the sequence.

9. Refined Argument: Injectivity via Nonlinear Differentiation

We further support the injectivity claim by observing that $f(x) = \frac{(3x+1)}{2^m}$ is a piecewise rational function that depends on the value of x and the exponent m. Even small changes in x lead to significantly different values due to discrete jumps in m, resulting in a sequence that diverges for different x_0. Therefore, the composition $f^{(n)}(x_0)$ is non-repeating unless a very specific equality of function layers occurs, which is not possible under general integer progression.

Having established the robustness of the sequence's divergence through nonlinear analysis, we are now positioned to conclude that every sequence must ultimately converge to the canonical cycle involving 1.

Moreover, the number of divisions by 2, denoted as $m(x)$, exhibits significant sensitivity to the specific residue class of x. Small changes in x can cause large fluctuations in $m(x)$, leading to abrupt contractions or expansions in the sequence. This volatility reinforces the inherently nonlinear character of the mapping, preventing predictable linear drift and driving the overall recursive decay toward 1.

10. Convergence to One

We consider the structure of the sequence values. Let Table 1 denote the mapping of odd integers through $3x + 1$ and division by 2^m to the next odd. This third column contains repeated values, making it a relatively smaller subset compared to the set of all odd integers.

Theorem 10.1 (Convergence to 1)**.** *Every sequence x_n must eventually encounter the value 1.*

Proof. Assume a sequence does not encounter 1. Since the function is injective and the output set is a subset with repetitions, the values must eventually cycle or repeat. But this contradicts injectivity unless the repeated value is 1. Hence, the sequence must pass through 1. □

Remark 10.1. (Pigeonhole Principle and Convergence[10])

Since the output set of the recursive mapping $f(x)$ is a compressed and finite subset within the set of odd integers, and since f is injective by construction, it follows by the Pigeonhole Principle that the sequence cannot generate infinitely many distinct values without eventual repetition[10]. Therefore, every sequence must ultimately intersect the canonical cycle, encountering the value 1.

Remark 10.2. (Bounded Compression vs Unbounded Injectivity)

The core contradiction arises from the interplay between two structural properties of the Collatz mapping:

First, the output set produced by the recursive mapping $f(x)$ is effectively bounded and compressed. Due to repeated divisions by powers of 2 and occasional duplications in the third column values, the reachable values under iteration are confined within a progressively narrower range, preventing indefinite linear growth.

Second, the injectivity of $f(x)$, established earlier, ensures that distinct inputs yield distinct outputs along the sequence. Thus, if the sequence were to continue indefinitely without convergence, it would generate an infinite injective progression — a strictly non-repeating, unbounded sequence of distinct values.

However, it is impossible to sustain an infinite injective progression within a bounded compressed output set. By the Pigeonhole Principle, an injective infinite sequence within a finite or bounded domain would necessarily imply eventual repetition or overlap, contradicting injectivity.

Hence, the structure of the Collatz mapping forces every sequence to terminate by encountering the trivial cycle involving 1.

11. Conclusion

In this paper, we analyzed the recursive formulation of the Collatz function using a structural approach grounded in injectivity and functional iteration. By demonstrating that each term in the sequence is uniquely determined (injective) and that the resulting odd-numbered outputs form a set with repeated values—hence a relatively smaller infinite set—we showed that the recursive progression cannot continue indefinitely without revisiting prior values or intersecting with the value 1.

Importantly, we emphasized that the goal of the Collatz conjecture is not to prove that sequences must terminate at 1 within a finite number of steps, but rather to demonstrate that every sequence must inevitably encounter the value 1 at some point. This work supports that conclusion by showing that, under the constraints of injectivity and recursive mapping into a compressed value space, no sequence can avoid passing through 1.

Future Directions While this work provides a structural proof of the convergence of all Collatz sequences to 1, several avenues for further investigation remain. One natural direction is the quantitative analysis of the rate of convergence: specifically, understanding the distribution of stopping times and total stopping times across the integers.

Additionally, exploring finer structural properties of the recursive mapping—such as statistical patterns in the sequence of $m(x)$ values or asymptotic density fluctuations in the compressed sets—may yield deeper insights into the dynamical complexity underlying the Collatz process.

Finally, generalizations to broader classes of recursive mappings, such as $kx+1$ functions for odd $k > 1$, could reveal whether similar structural decay mechanisms extend beyond the classical case.

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