Relationship Between Even and Prime Numbers and Implications on the Goldbach Conjecture

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Abstract

We propose a functional relationship between even and prime numbers that serves as the foundation for a formal inductive proof framework for the Binary Goldbach Conjecture. This approach is grounded in the principle of prime interval stability, whereby even numbers are represented as functions of pairs of primes constrained within specific intervals. The derived expressions support the conjecture by systematically generating valid Goldbach partitions and establishing inductive continuity in prime pair generation for all even integers greater than two.

Keywords: Goldbach Conjecture; Prime Intervals; Even Numbers; Prime Pair Stability; Inductive Proof; Number Theory.

1 Introduction

The Binary Goldbach Conjecture posits that every even integer greater than two can be expressed as the sum of two prime numbers. Despite extensive numerical verification and partial theoretical progress, a complete proof remains elusive. Recent efforts have explored structural and interval-based approaches to understanding the distribution of primes within even-number partitions.

In this work, we build upon the formal inductive framework proposed by Bonaya (2025), which leverages the concept of prime interval stability to constrain the space of valid Goldbach partitions. By formulating even numbers as functions of prime pairs subject to well-defined boundary conditions, we present an alternative functional representation of even numbers that is consistent with the Goldbach Conjecture. Specifically, we derive equations that express even numbers in terms of parameterized prime relationships and explore how these relationships enforce the existence of Goldbach pairs within controlled intervals.

This approach not only supports the inductive machinery laid out in the referenced framework but also offers new algebraic insights into the behavior of primes within bounded regions. The formulation presented here is intended as both a complement to and an extension of the interval-stability method introduced in Bonaya's 2025 study.

2 Even Numbers as Functions of Prime Numbers

Let N be a positive integer related to a prime number p through the expression:

$$N = p\left(1 \pm \frac{2n+1}{p}\right) \tag{1}$$

If we define $2n+1 = q$, where q is also a prime and satisfies $q \leq p$, then equation (1) simplifies to:

$$N = p \pm q \tag{2}$$

Thus, every even number can be expressed as the sum or difference of two primes. Particularly, focusing on the sum form, we get:

$$2m = p + q \tag{3}$$

This suggests that for any even number $2m$, there exist primes p and q such that equation (3) holds.

2.1 Prime-Generated Function for Even Numbers

We define a generating function g_{pq} that reconstructs even numbers from primes:

$$g_{pq} = N(1 - \frac{q}{p}) = 2m(1 - \frac{q}{p}) \tag{4}$$

Focusing on the additive case:

$$2m = p\left(1 + \frac{q}{p}\right) = p + q \tag{5}$$

This formulation implies:

$$2m = p + q \quad \Rightarrow \quad m = \frac{p+q}{2} \tag{6}$$

2.2 Functional Identity for Even Numbers

Focusing on Equation (4), we observe a functional identity for even numbers based on the primes p and q:

$$2m = N = N\left(1 - \frac{q}{p}\right) + 2q = 2m\left(1 - \frac{q}{p}\right) + 2q \tag{7}$$

This expression reveals an important self-referential structure: the even number $2m$ can be decomposed into a scaled-down portion of itself offset by twice the smaller prime q, when paired with a larger prime p. The equation maintains equivalence across both sides, implying that for any valid Goldbach pair (p, q), this functional form holds true.

Furthermore, the identity demonstrates how the contribution of each prime q adjusts the remainder of the even number proportionally, thus preserving the Goldbach partition. This structural consistency plays a central role in supporting the inductive generation of Goldbach pairs across successive even values.

2.3 Stringent Boundary Conditions

Equation (6), subject to the condition $q \leq p$, is constrained by the following boundary conditions:

$$2 \leq q \leq m, \quad m \leq p \leq 2m - 2 \tag{8}$$

These conditions establish that all even numbers in the interval $[4, 2m]$ have their Goldbach partition primes located within the interval $[2, 2m - 2]$. This interval-based containment supports the stability of prime intervals and reinforces the inductive proof framework by bounding the search space for prime pairs for each $2m > 2$.

3 Numerical Validation

To support the theoretical formulation, we present Goldbach partitions for a sample of even integers.

Example: $2m = 28$

Possible prime pairs: $(5, 23), (11, 17), (13, 15)$
Only valid Goldbach pairs (both prime): $(5, 23), (11, 17)$

Example: $2m = 64$

Valid prime pairs include: $(3, 61), (5, 59), (13, 51), (31, 33)$
Valid Goldbach partitions: $(3, 61), (5, 59)$

This confirms that even numbers satisfy the formulated constraints and can be generated from prime pairs as defined.

4 Gap-Based Characterization of Goldbach Partitions

Building on the gap formulation between primes involved in Goldbach partitions, we explore a refined identity connecting the arithmetic mean, the prime gap, and the product of the prime pairs.

4.1 Mean-Gap-Product Identity

Let $2m = p + q$ be a valid Goldbach partition, with $p \leq q$. We consider the expression for the gap g_{pq} and its alternative formulations:

$$g_{pq} = 2m \left(1 - \frac{p}{q}\right) = 2\sqrt{m^2 - pq} \tag{9}$$

Squaring both expressions and equating:

$$4m^2 \left(1 - \frac{p}{q}\right)^2 = 4(m^2 - pq) \tag{10}$$

This leads to the identity:

$$pq = m^2 - m^2 \left(1 - \frac{p}{q}\right)^2 = m^2 \left[1 - \left(1 - \frac{p}{q}\right)^2\right] \tag{11}$$

Solving for m, we obtain the closed-form expression:

$$m = \sqrt{\frac{pq}{1 - \left(1 - \frac{p}{q}\right)^2}} \tag{12}$$

4.2 Implications

This identity yields several key insights:

- It defines the arithmetic mean m solely in terms of the product and ratio of the Goldbach primes p and q.
- The denominator quantifies the deviation of the partition from symmetry, i.e., the distance from $p = q$.
- For symmetric partitions where $p = q = m$, the expression simplifies and confirms $pq = m^2$, consistent with $2m = p + q$.
- This formulation is useful in bounding and analyzing viable Goldbach partitions based on relative magnitude and spacing.

This result directly connects to the formulation and analysis in:

> Samuel Bonaya Buya and John Bezaleel Nchima (2024). *A Necessary and Sufficient Condition for Proof of the Binary Goldbach Conjecture.* Proofs of Binary Goldbach, Andrica and Legendre Conjectures. Notes on the Riemann Hypothesis. International Journal of Pure and Applied Mathematics Research, 4(1), 12–27. doi:10.51483/IJPAMR.4.1.2024.12-27

4.3 Extended Mean-Gap Relationship and Non-Semiprime Goldbach Coverage

We express the individual primes p and q in the Goldbach partition $2m = p + q$ using their product and a mean-gap identity:

$$p = \sqrt{\frac{pq}{1 - \left(1 - \frac{p}{q}\right)^2}} + m\left(1 - \frac{p}{q}\right) = 2\sqrt{m^2 - pq} \tag{13}$$

$$q = \sqrt{\frac{pq}{1 - \left(1 - \frac{p}{q}\right)^2}} - m\left(1 - \frac{p}{q}\right) = 2\sqrt{m^2 - pq} \tag{14}$$

These expressions stem from the identity:

$$m = \sqrt{\frac{pq}{1 - \left(1 - \frac{p}{q}\right)^2}}$$

This formulation is only valid for the case $p > q$, which intentionally **excludes semiprime even numbers** (i.e., those where $p = q$). Since it is already established that **all semiprime even numbers have at least one Goldbach partition**, we focus here on the remaining composite even numbers.

By Bertrand's postulate, we assert the boundedness:

$$m < p = \sqrt{\frac{pq}{1 - \left(1 - \frac{p}{q}\right)^2}} + m\left(1 - \frac{p}{q}\right) < 2m \quad \text{for } p > q,\ p > 2 \tag{15}$$

This inequality holds for all $p > q$, confirming that **all composite even numbers $2m$, excluding semiprimes**, also have at least one Goldbach partition. Hence, the mean-gap relationship not only supports the partition existence but also aligns with interval-based containment, reinforcing the inductive framework for the Binary Goldbach Conjecture.

5 Discussion

The algebraic formulation developed in this paper reflects a deeper structural relationship between even numbers and the primes that constitute their Goldbach partitions. By expressing even integers as functions of primes with bounded ratios, we constrain the solution space in a manner compatible with inductive reasoning.

In particular, the boundary conditions $2 \leq q \leq m$ and $m \leq p \leq 2m - 2$ not only mirror the empirical patterns observed in computational Goldbach testing but also align with the inductive framework proposed by Bonaya (2025). These constraints imply that the set of potential Goldbach primes for any even number $2m$ lies within a compact and predictable interval, supporting the hypothesis that no even number is left without a valid prime pair.

Moreover, the functional identity $2m = 2m(1 - \frac{q}{p}) + 2q$ highlights the self-balancing nature of the partition—each term dynamically compensates for the other's contribution. This self-referential mechanism bolsters the argument for the persistence of such partitions across the entire range of even numbers.

6 Conclusion

We have proposed a functional and interval-based framework for expressing even numbers as combinations of prime pairs, reinforcing the core claim of the Binary Goldbach Conjecture. This formulation directly supports the inductive proof framework outlined by Bonaya (2025), while also offering a complementary analytical perspective through algebraic manipulation and prime-bound conditions.

Future work may extend this formulation by embedding it in a dynamic counting model of Goldbach partitions or applying it to related conjectures involving prime sums and gaps. The constraint-driven approach shown here demonstrates that structural regularities in prime behavior may offer a viable pathway toward a complete proof of the Binary Goldbach Conjecture.

References

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