Considerations on the stopping time of the 3n+1 problem

V. Barbera

Abstract
This paper presents some considerations on the stopping time of the 3n+1 problem. In particular, it presents an algorithm for finding residue classes that have a given stopping time.

3n + 1 problem (or conjecture)

In the $3\cdot n+1$ problem[1] it is possible to define the function:

$$f(n)=\begin{cases} 3\cdot n+1 & if\ n\equiv 1(mod\,2) \\ \frac{n}{2} & if\ n\equiv 0(mod\,2) \end{cases}$$

the sequence obtained using the function $f(n)$ is as follows:

$$a_i=\begin{cases} n & for\ i=0 \\ f(a_{i-1}) & for\ i>0 \end{cases}$$

the smallest $i=k$ such that $a_i<a_0$ is called the stopping time of n .

If n is even $n\equiv 0(mod\,2)$ then the stopping time $k=1$ and $a_1=\frac{n}{2}$.

If $n\equiv 1(mod\,2^2)$ then the stopping time $k=3$ and $a_3=\frac{3\cdot n+1}{2^2}$.

Fixed an integer x with $x>1$ if n odd and $n\equiv r(mod\,2^{k-x})$ then for some residue classes with $r>1$ it is possible to observe that $a_k=\frac{3^x\cdot n+s}{2^{k-x}}$

with $s\geq 3^x-2^x$ and $s=3^{x-1}+\sum_{j=0}^{x-2}3^j\cdot 2^{\sum_{i=1}^{x-1-j}c_i}$ and with $y=k-x=\lfloor 1+x\cdot\log_2(3)\rfloor$.

As it is possible to analyze also in OEIS A177789[2] and in OEIS A020914[3] the following results can be obtained:

- stopping time 1 :

numbers 0 (mod 2)

- stopping time 3 :

numbers 1 (mod 2^2)

- stopping time 6 :

numbers 3 (mod 2^4)

- stopping time 8 :

numbers $11,23$ (mod 2^5)

- stopping time 11 :

numbers $7,15,59$ (mod 2^7)

- stopping time 13 :

numbers $39,79,95,123,175,199,219$ (mod 2^8)

From an analysis of the classes of remainder represented above it is possible to observe that the numbers $3,23,15,95,575$ have values of $s=3^x-2^x$ or $s=5,19,65,211,665$ these values of s can also be obtained using the sequence $s_i=3\cdot s_{i-1}+2^{x_{i-1}}$ with x_{i-1} the value of x associated with s_{i-1} ;

and the numbers $11,59,219$ have values of $s=23,85,287$ these values of s can also be obtained using the sequence $s_i=3\cdot s_{i-1}+2^{x_{i-1}+1}$.
Knowing the values of s generated by the sequences described above, it is possible to find an algorithm to list the residue classes based on the value of x and therefore on the equivalent stopping time.

From an analysis of the obtained sequences of type $s_i=3\cdot s_{i-1}+2^{x_{i-1}+j}$ it is possible to observe that, given an order x , if a starting index j_{min} is associated to each residue class associated with x it is possible to obtain the residue classes associated with $x+1$.
The residue class are obtained by the inverse module from s_i for $j_{min}\leq j<\lfloor 1+x\cdot\log_2(3)\rfloor-x$.

Example, for $x=1$ we have $r=1$, $s=1$ and $j_{min}=0$ then

for $x=2$ we have $r=3$, $s=5=3\cdot 1+2$ and $j_{min}=0$

the residue classes associated to $x+1=3$ since $0\leq j<\lfloor 1+2\cdot\log_2(3)\rfloor-2=2$ are obtained from $s=3\cdot s_{i-1}+2^{x_{i-1}}=3\cdot 5+2^2=19$ and $s=3\cdot s_{i-1}+2^{x_{i-1}+1}=3\cdot 5+2^3=23$

from which using the modular multiplicative inverse

$$r\equiv\frac{1}{3^{(x+1)}}\left(mod\, 2^{\lfloor 1+(x+1)\cdot\log_2(3)\rfloor}\right)$$

then are obtained $r=23$ with associated $j_{min}=0$ and $r=11$ with associated $j_{min}=1$

For $x=3$ we have $r=23$, $s=19$ with associated $j_{min}=0$

and $r=11$, $s=23$ with associated $j_{min}=1$

the residue classes associated to $x+1=4$ are obtained from $s_i=3*s_{i-1}+2^{x_{i-1}+j}$ then

$s=3\cdot 19+2^3=65$ and $s=3\cdot 19+2^4=73$ since $0\leq j<\lfloor 1+3\cdot\log_2(3)\rfloor-3=2$

from which are obtained $r=15$ with associated $j_{min}=0$

and $r=7$ with associated $j_{min}=1$

and from $s=3*23+2^4=85$ since $1\leq j<2$ you get $r=59$ with associated $j_{min}=1$.
For $x=4$ we have $r=15$, $s=65$ with associated $j_{min}=0$;

$r=7$, $s=73$ with associated $j_{min}=1$ and $r=59$, $s=85$ with associated $j_{min}=1$

the residue classes associated to $x+1=5$ are obtained from $s_i=3*s_{i-1}+2^{x_{i-1}+j}$ then from

$s=3*65+2^4=211$, $s=3*65+2^5=227$ and $s=3*65+2^6=259$

since $0\leq j<\lfloor 1+4*\log_2(3)\rfloor-4=3$

are obtained $r=95$ with associated $j_{min}=0$, $r=175$ with associated $j_{min}=1$ and $r=79$ with associated $j_{min}=2$

from $s=3*73+2^5=251$ and $s=3*73+2^6=283$ since $1\leq j<3$ you get $r=39$ with associated $j_{min}=1$ and $r=199$ with associated $j_{min}=2$
from $s=3*85+2^5=287$ and $s=3*85+2^6=319$ since $1\leq j<3$ you get $r=219$ with associated $j_{min}=1$ and $r=123$ with associated $j_{min}=2$.
Below is the algorithm code to generate the residue classes for $x>2$:

```
get_dim(n)={my(log2_3=log(3)/log(2),X1=matrix(2,floor(1+n*log2_3)-n),
d=0,dim=1);X1[1,1]=0;if(n>=3,for(x=3,n,y=floor(1+(x-1)*log2_3);for(i=1,y-x,for(j=i,y-x,
if(i==1,X1[1+(d+1)%2,j]=1);X1[1+(d+1)%2,j]=X1[1+(d+1)%2,j]+X1[d+1,i]));
d=(d+1)%2);for(i=1,floor(1+n*log2_3)-n,dim=dim+X1[d+1,i]));dim;}
{x_max=9;dim_max=get_dim(x_max);R=vector(dim_max);S=matrix(2,dim_max);
I1=matrix(2,dim_max);d=0;log2_3=log(3)/log(2);c=1;S[1,1]=1;I1[1,1]=1;
for(x=1,x_max-1, y=floor(1+x*log2_3);x1=x+1;y1=floor(1+x1*log2_3);
print1("x = ",x1,"    -    stopping time: ",y1+x1,"\nif n== ");c1=0;
for(i=1,c,for(i2=I1[1+d,i],y-x,s1=3*S[1+d,i]+2^(x+i2-1); c1=c1+1;
R[c1]=((2^y1-s1)*lift(Mod(1/3^x1,2^y1)))%2^y1;S[1+(1+d)%2,c1]=s1;I1[1+(1+d)%2,c1]=i2));
c=c1;R1=vector(c);R1=vecsort(R[1..c]);for(i2=1,c-1,print1(R1[i2],", "));
print1(R1[c]);d=(1+d)%2;print(" (mod 2^",y1,")\ntotal residue classes:  ",c,"\n\n"))}
```

PARI/GP code of the algorithm

x = 2 - stopping time: 6
if n== 3 (mod 2^4)
total residue classes: 1

x = 3 - stopping time: 8
if n== 11, 23 (mod 2^5)
total residue classes: 2

x = 4 - stopping time: 11
if n== 7, 15, 59 (mod 2^7)
total residue classes: 3

x = 5 - stopping time: 13
if n== 39, 79, 95, 123, 175, 199, 219 (mod 2^8)
total residue classes: 7

x = 6 - stopping time: 16
if n== 287, 347, 367, 423, 507, 575, 583, 735, 815, 923, 975, 999 (mod 2^10)
total residue classes: 12

x = 7 - stopping time: 19
if n== 231, 383, 463, 615, 879, 935, 1019, 1087, 1231, 1435, 1647, 1703, 1787, 1823, 1855, 2031, 2203, 2239, 2351, 2587, 2591, 2907, 2975, 3119, 3143, 3295, 3559, 3675, 3911, 4063 (mod 2^12)
total residue classes: 30

x = 8 - stopping time: 21
if n== 191, 207, 255, 303, 539, 543, 623, 679, 719, 799, 1071, 1135, 1191, 1215, 1247, 1327, 1563, 1567, 1727, 1983, 2015, 2075, 2079, 2095, 2271, 2331, 2431, 2607, 2663, 3039, 3067, 3135, 3455, 3483, 3551, 3687, 3835, 3903, 3967, 4079, 4091, 4159, 4199, 4223, 4251, 4455, 4507, 4859, 4927, 4955, 5023, 5103, 5191, 5275, 5371, 5439, 5607, 5615, 5723, 5787, 5871, 5959, 5979, 6047, 6215, 6375, 6559, 6607, 6631, 6747, 6815, 6983, 7023, 7079, 7259, 7375, 7399, 7495, 7631, 7791, 7847, 7911, 7967, 8047, 8103 (mod 2^13)
total residue classes: 85

x = 9 - stopping time: 24
if n== 127, 411, 415, 831, 839, 1095, 1151, 1275, 1775, 1903, 2119, 2279, 2299, 2303, 2719, 2727, 2767, 2799, 2847, 2983, 3163, 3303, 3611, 3743, 4007, 4031, 4187, 4287, 4655, 5231, 5311, 5599, 5631, 6175, 6255, 6503, 6759, 6783, 6907, 7163, 7199, 7487, 7783, 8063, 8187, 8347, 8431, 8795, 9051, 9087, 9371, 9375, 9679, 9711, 9959, 10055, 10075, 10655, 10735, 10863, 11079, 11119, 11567, 11679, 11807, 11943, 11967, 12063, 12143, 12511, 12543, 12571, 12827, 12967, 13007, 13087, 13567, 13695, 13851, 14031, 14271, 14399, 14439, 14895, 15295, 15343, 15839, 15919, 16027, 16123, 16287, 16743, 16863, 16871, 17147, 17727, 17735, 17767, 18011, 18639, 18751, 18895, 19035, 19199, 19623, 19919, 20079, 20199, 20507, 20527, 20783, 20927, 21023, 21103, 21223, 21471, 21727, 21807, 22047, 22207, 22655, 22751, 22811, 22911, 22939, 23231, 23359, 23399, 23615, 23803, 23835, 23935, 24303, 24559, 24639, 24647, 24679, 25247, 25503, 25583, 25691, 25703, 25831, 26087, 26267, 26527, 26535, 27111, 27291, 27759, 27839, 27855, 27975, 28703, 28879, 28999, 29467, 29743, 29863, 30311, 30591, 30687, 30715, 30747, 30767, 30887, 31711, 31771, 31899, 32155, 32239, 32575, 32603 (mod 2^15)
total residue classes: 173

References

[1] https://en.wikipedia.org/wiki/Collatz_conjecture

[2] https://oeis.org/A177789

[3] https://oeis.org/A020914