A universal expression of prime numbers

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Abstract

We found that all prime numbers can be expressed in the form:

$$p = \sum_{t=0}^{k} r_t p_{t-1}!^p + 2$$

where $p_{t-1}!^p = \prod_{k=1}^{t-1} p_t$ is the primorial of the (t-1)-th prime, and r_t are coefficients satisfying $0 \le r_t \le p_t - 1$. Which serves as a necessary but not sufficient condition for a number to be prime.

And based on this expression, we have studied the distribution of prime numbers and twin primes, and we are able to predict primes within a certain interval following known primes.

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1. Introduction

1.1. *Overview.*

This paper partitions all positive integers according to the Euclidean numbers $p_k!^p + 1$ into intervals

$$(p_{k-1}!^p + 1, p_k!^p + 1]$$

Based on the computational rule

$$M(k) = \begin{bmatrix} 1 \times p_{k-1}!^p \\ 2 \times p_{k-1}!^p \\ \vdots \\ (p_k - 1) \times p_{k-1}!^p \end{bmatrix} + \begin{bmatrix} F(k) \\ F(k) \\ \vdots \\ F(k) \end{bmatrix}$$

the calculation result matrix $M(k)$ for each interval is derived. The sequence M is then formed by taking $M(k)$ as its elements.

Iterating $F(k)$ down to 2 yields the expression for an individual element in $M(k)$:

$$m(k,i,j) = \sum_{i=1}^{k} r_{i,j} p_{i-1}!^p + 2$$

Using mathematical induction, descent, and proof by contradiction, it is proven that the set of primes $\mathbb{P}$ satisfies $\mathbb{P} \subseteq M$, leading to the necessary but not sufficient universal expression for primes:

$$p = \sum_{t=0}^{k} r_t p_{i-1}!^p + 2$$

Consequently, the study of primes is transformed into the study of the sequence M, i.e., the study of the matrix $M(k)$.

1.2. ***Major Achievements***.

(1) Proof $\mathbb{P} \subseteq M$ and derived a necessary but not sufficient universal expression for primes:

$$p = \sum_{t=0}^{k} r_t p_{t-1}!^p + 2$$

(2) If the product of two Mo numbers is still a Mo number, then it will follow the following operational properties:

$$M_{T5} \times M_{T5} \to M_{T7}$$

$$M_{T5} \times M_{T7} \to M_{T5}$$

$$M_{T7} \times M_{T7} \to M_{T7}$$

Corollary 1:
For any prime $p \geq 5$, if $p^n \in M$, then:

$$\begin{cases} p^n \in M_{T5} & \text{if } p \in M_{T5} \text{ and } n \text{ is odd,} \\ p^n \in M_{T7} & \text{if } p \in M_{T7} \text{ or } n \text{ is even.} \end{cases}$$

Corollary 2:
For any twin prime pair $p \geq 5$ and $p+2$, we have $p \in M_{T5}$ and $p+2 \in M_{T7}$. If $p(p+2) \in M$, then $p(p+2) \in M_{T5}$.

(3) For any Sophie Germain primes $p \geq 5$ must satisfy:
- $p \in M_{T5}$ (belongs to Type-5 Mo numbers);
- $p \not\equiv 7 \pmod{10}$ (the units digit cannot be 7);
- $2p+1 \in M_{T5}$ (belongs to Type-5 Mo numbers);
- $2p+1 \not\equiv 1 \pmod{10}$ ($p \geq 11$ the units digit cannot be 1).

(4) For any prime q, the set of Type-q Mo numbers M_{Tq} contains infinitely many primes. That is:

$$\forall \text{ prime } q, |\{p \in M_{Tq} : p \text{ is prime}\}| = \infty$$

(5) Conjecture: For any even spacing $2h$ (where $h \in \mathbb{N}^+$), and for any initial prime pair $\{(q, q+2h) \mid q \geq 5\}$, the set of type-$(q, q+2h)$ Mo number pairs $M_{T(q,q+2h)}$ contains infinitely many prime pairs spaced exactly $2h$ apart.

(6) Theorem 4.1.1: When $k \geq 2$, the column count C_k of the matrix $M(k)$, the element count $|M(k-1)|$ of the matrix $M(k-1)$, and the number of elements in M whose smallest prime factor is p_k, denoted by $|F^m_{min}(p_k)|$, satisfy the equality:

$$C_k = |M(k-1)| = |F^m_{min}(p_k)| = \prod_{t=1}^{k-1}(p_t - 1)$$

(7) The number of primes not exceeding a given number x:

$$\pi(x) = \sum_{t=0}^{k-1} |P(t)| + |M(k,x)| - \left|M^{'}(k,x)\right|$$

where:

- $P(t)$ is the set of primes in $M(t)$;
- $M(k,x)$ is the set of elements in $M(k)$ that do not exceed x;
- $M^{'}(k,x)$ is the set of composite numbers in $M(k)$ that do not exceed x.

(8) Method for generating primes in the matrix $M(k)$ based on $M(0)$ through $M(k-1)$.

(9) Theorem 5.2.1: When $k \geq 3$, the column count T_k of the matrix $M_2(k)$ and the number of twin number pairs in M whose smallest prime factor is p_k, denoted by $|F^m_{2min}(p_k)|$, satisfy the following equality:

$$T_k = \frac{1}{2}\left|F^m_{2min}(p_k)\right| = \prod_{t=2}^{k-1}(p_t - 2)$$

(10) An Alternative Method for Generating Twin Primes in the Matrix $M(k)$.

1.3. *Organization*.

This paper is organized as follows:

In Chapter 2, we provided the computational rules for the matrix $M(k)$ and proved that all primes satisfy $\mathbb{P} \subseteq M$. Thus we obtain a necessary but not sufficient expression for primes:

$$p = \sum_{t=0}^{k} r_t p_{t-1}!^p + 2$$

In Chapter 3, we classified the elements in M and derived several research findings.

In Chapter 4, we investigated the composition, properties, and distribution of primes within the matrix $M(k)$, and presented a method for obtaining primes in $M(k+1)$ based on $M(0)$ through $M(k)$. This is a relatively efficient method for generating prime tables.

In Chapter 5, we studied the reasons for the formation of twin prime pairs in the matrix $M(k)$ (which also explains the existence of twin primes), as well as their composition and properties. Additionally, we provided an alternative method for generating twin primes in $M(k+1)$ (distinct from the prime generation method described in Chapter 4).

2. Deduction and proof of expressions

2.1. *Definition*.

Let p_k denote the k-th prime number, for example: $p_1 = 2, p_2 = 3, p_3 = 5$.

Using the symbol $!^p$ to denote the primordial (Because I think the symbol # will disrupt the representation of the entire factorial), $p_k!^p$ represents the primorial of the prime number p_k. Then,

$$p_k!^p = p_k\# = \prod_{t=1}^{k} p_t$$

Specify $p_0!^p = 1! = 1, p_{-1}!^p = 0! = 1$.

Let $f^p_{min}(n)$ denote the smallest prime factor of n, for example:

$$f^p_{min}(15) = 3$$

$$f^p_{min}(31) = 31$$

Therefore, if $f^p_{min}(n) = n$ and $n \geq 2$, then n is a prime number.

Let $f^p_{2min}(m, m+2)$ denote the smallest prime factor of twin numbers $(m, m+2)$, for example:

$$f^p_{2min}(23, 25) = 5$$

$$f^p_{2min}(41, 43) = 41$$

Therefore, if $f^p_{2min}(m, m+2) = m$ and $m \geq 3$, then $(m, m+2)$ is twin primes.

2.2. *Calculation Rules*.

For ease of expression, we will temporarily refer to the calculated numbers as "Mo numbers" denoted as m. The computed numbers m are divided based on the Euclidean numbers $p_k!^p + 1$ for $k \geq 1$, the k-th interval is defined as $(p_{k-1}!^p + 1, p_k!^p + 1]$ for $k \geq 1$, The matrix composed of the numbers m in each interval is denoted as $M(k)$, with the stipulation that $M(0) = [2]$. The sequence formed by the matrices $M(k)$ for $k \geq 0$ as elements is denoted as M. Thus,

$$M = \{M(0), M(1), M(2), \ldots, M(k), \ldots\}$$

Denote the computational base number for the matrix $M(k)$ as b_k,

$$b_k = p_{k-1}!^p$$

The computation of the matrix $M(k)$ involves using the Mo numbers from $M(0)$ to $M(k-1)$ whose smallest prime factors are greater than or equal to p_k. These numbers are collected as the computation factors for $M(k)$ and represented as a row vector $F(k)$. Given the row vector $F(k)$, the computational base number b_k, and the constraint $r \in \mathbb{Z}$ with $r \in [1, p_k - 1]$, the r-th row of the matrix $M(k)$ is computed using the expression:

$$M(k, r) = r \times b_k + F(k), r \in \mathbb{Z}, r \in [1, p_k - 1]$$

For example:

(1) For $k = 1$:
- The interval is $(p_0!^p + 1, p_1!^p + 1] = (1 + 1, 2 + 1] = (2, 3]$
- $F(1) = [2], b_1 = p_0!^p = 1$
- $M(1) = [1 \times b_1] + [F(1)] = [1] + [2] = [3]$

(2) For $k = 2$:
- The interval is $(p_1!^p + 1, p_2!^p + 1] = (2 + 1, 6 + 1] = (3, 7]$
- $F(2) = [3], b_2 = p_1!^p = 2$
- $M(2) = \begin{bmatrix} 1 \times b_2 \\ 2 \times b_2 \end{bmatrix} + \begin{bmatrix} F(2) \\ F(2) \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

(3) For $k = 3$:
- The interval is $(p_2!^p + 1, p_3!^p + 1] = (6 + 1, 30 + 1] = (7, 31]$
- $F(3) = \begin{bmatrix} 5 & 7 \end{bmatrix}, b_3 = p_2!^p = 6$
- $M(3) = \begin{bmatrix} 1 \times b_3 \\ 2 \times b_3 \\ 3 \times b_3 \\ 4 \times b_3 \end{bmatrix} + \begin{bmatrix} F(3) \\ F(3) \\ F(3) \\ F(3) \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 18 \\ 24 \end{bmatrix} + \begin{bmatrix} 5 & 7 \\ 5 & 7 \\ 5 & 7 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 11 & 13 \\ 17 & 19 \\ 23 & 25 \\ 29 & 31 \end{bmatrix}$

(4) For $k = 4$:
- The interval is $(p_3!^p + 1, p_4!^p + 1] = (30 + 1, 210 + 1] = (31, 211]$
- $F(4) = \begin{bmatrix} 7 & 11 & 13 & 17 & 19 & 23 & 29 & 31 \end{bmatrix}, b_4 = p_3!^p = 30$
- Thus,

$$M(4) = \begin{bmatrix} 1 \times b_4 \\ 2 \times b_4 \\ 3 \times b_4 \\ 4 \times b_4 \\ 5 \times b_4 \\ 6 \times b_4 \end{bmatrix} + \begin{bmatrix} F(4) \\ F(4) \\ F(4) \\ F(4) \\ F(4) \\ F(4) \end{bmatrix}$$
$$= \begin{bmatrix} 37 & 41 & 43 & 47 & 49 & 53 & 59 & 61 \\ 67 & 71 & 73 & 77 & 79 & 83 & 89 & 91 \\ 97 & 101 & 103 & 107 & 109 & 113 & 119 & 121 \\ 127 & 131 & 133 & 137 & 139 & 143 & 149 & 151 \\ 157 & 161 & 163 & 167 & 169 & 173 & 178 & 181 \\ 187 & 191 & 193 & 197 & 199 & 203 & 209 & 211 \end{bmatrix}$$

Because $f^p_{min}(25) = 5 < p_4 = 7$, the number 25 does not satisfy the condition $f^p_{min}(m) \geq p_k$. Therefore, when computing $F(4)$, 25 will not be included in $F(4)$.

Therefore,

$$M(k) = \begin{bmatrix} 1 \times b_k \\ 2 \times b_k \\ \vdots \\ (p_k - 1) \times b_k \end{bmatrix} + \begin{bmatrix} F(k) \\ F(k) \\ \vdots \\ F(k) \end{bmatrix}$$
$$= \begin{bmatrix} 1 \times p_{k-1}!^p \\ 2 \times p_{k-1}!^p \\ \vdots \\ (p_k - 1) \times p_{k-1}!^p \end{bmatrix} + \begin{bmatrix} F(k) \\ F(k) \\ \vdots \\ F(k) \end{bmatrix}$$

The general expression for $M(k)$, when $F(k)$ is iteratively computed down to $F(1)$, is as follows:

$$M(k) = \begin{bmatrix} b_1 & \dots & 1 \times b_k \\ b_1 & \dots & 2 \times b_k \\ \vdots & \ddots & \vdots \\ b_1 & \dots & (p_k - 1) \times b_k \end{bmatrix} \times \begin{bmatrix} r_{1,1} & r_{1,2} & \dots & r_{1,C_k} \\ r_{2,1} & r_{2,2} & \dots & r_{2,C_k} \\ \vdots & \vdots & \ddots & \vdots \\ r_{k-1,1} & r_{k-1,2} & \dots & r_{k-1,C_k} \\ r_{k,1} & r_{k,2} & \dots & r_{k,C_k} \end{bmatrix} + \begin{bmatrix} F(1) \\ F(1) \\ \vdots \\ F(1) \end{bmatrix}$$
$$= \begin{bmatrix} b_1 & \dots & 1 \times b_k \\ b_1 & \dots & 2 \times b_k \\ \vdots & \ddots & \vdots \\ b_1 & \dots & (p_k - 1) \times b_k \end{bmatrix} \times \begin{bmatrix} r_{1,1} & r_{1,2} & \dots & r_{1,C_k} \\ r_{2,1} & r_{2,2} & \dots & r_{2,C_k} \\ \vdots & \vdots & \ddots & \vdots \\ r_{k-1,1} & r_{k-1,2} & \dots & r_{k-1,C_k} \\ r_{k,1} & r_{k,2} & \dots & r_{k,C_k} \end{bmatrix} + 2$$

Constraints:

- $k \geq 1$
- The coefficients $r_{i,j}$ are constrained as follows:
  - $r_{1,j} = 1$ for all j.
  - $r_{2,j} \in [1, 2]$ for all j.
  - $r_{k,j} = 1$ for all j.
  - For $i \in [3, k-1]$,$r_{i,j} \in [0, p_{i-1}]$, for all j.
  - $C_k = \prod_{t=1}^{k-1}(p_t - 1)$ is the number of columns in matrix $M(k)$.

For example:

$$
\begin{aligned}
M(1) &= \begin{bmatrix} 1 \times b_1 \end{bmatrix} \times \begin{bmatrix} r_{1,1} \end{bmatrix} + 2 \\
&= \begin{bmatrix} 1 \times 1 \end{bmatrix} \times \begin{bmatrix} 1 \end{bmatrix} + 2 \\
&= \begin{bmatrix} 3 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
M(2) &= \begin{bmatrix} b_1 & 1 \times b_2 \\ b_1 & 2 \times b_2 \end{bmatrix} \times \begin{bmatrix} r_{1,1} \\ r_{2,1} \end{bmatrix} + 2 \\
&= \begin{bmatrix} 1 & 1 \times 2 \\ 1 & 2 \times 2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \\
&= \begin{bmatrix} 5 \\ 7 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
M(3) &= \begin{bmatrix} b_1 & b_2 & 1 \times b_3 \\ b_1 & b_2 & 2 \times b_3 \\ b_1 & b_2 & 3 \times b_3 \\ b_1 & b_2 & 4 \times b_3 \end{bmatrix} \times \begin{bmatrix} r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2} \\ r_{3,1} & r_{3,2} \end{bmatrix} + 2 \\
&= \begin{bmatrix} 1 & 2 & 1 \times 6 \\ 1 & 2 & 2 \times 6 \\ 1 & 2 & 3 \times 6 \\ 1 & 2 & 4 \times 6 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} + 2 \\
&= \begin{bmatrix} 11 & 13 \\ 17 & 19 \\ 23 & 25 \\ 29 & 31 \end{bmatrix}
\end{aligned}
$$

$$M(4) = \begin{bmatrix} b_1 & b_2 & b_3 & 1\times b_4 \\ b_1 & b_2 & b_3 & 2\times b_4 \\ b_1 & b_2 & b_3 & 3\times b_4 \\ b_1 & b_2 & b_3 & 4\times b_4 \\ b_1 & b_2 & b_3 & 5\times b_4 \\ b_1 & b_2 & b_3 & 6\times b_4 \end{bmatrix} \times \begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} & r_{1,4} & r_{1,5} & r_{1,6} & r_{1,7} & r_{1,8} \\ r_{2,1} & r_{2,2} & r_{2,3} & r_{2,4} & r_{2,5} & r_{2,6} & r_{2,7} & r_{2,8} \\ r_{3,1} & r_{3,2} & r_{3,3} & r_{3,4} & r_{3,5} & r_{3,6} & r_{3,7} & r_{3,8} \\ r_{4,1} & r_{4,2} & r_{4,3} & r_{4,4} & r_{4,5} & r_{4,6} & r_{4,7} & r_{4,8} \end{bmatrix} + 2$$

$$= \begin{bmatrix} 1 & 2 & 6 & 1\times 30 \\ 1 & 2 & 6 & 2\times 30 \\ 1 & 2 & 6 & 3\times 30 \\ 1 & 2 & 6 & 4\times 30 \\ 1 & 2 & 6 & 5\times 30 \\ 1 & 2 & 6 & 6\times 30 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 2 & 3 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} + 2$$

$$= \begin{bmatrix} 37 & 41 & 43 & 47 & 49 & 53 & 59 & 61 \\ 67 & 71 & 73 & 77 & 79 & 83 & 89 & 91 \\ 97 & 101 & 103 & 107 & 109 & 113 & 119 & 121 \\ 127 & 131 & 133 & 137 & 139 & 143 & 149 & 151 \\ 157 & 161 & 163 & 167 & 169 & 173 & 178 & 181 \\ 187 & 191 & 193 & 197 & 199 & 203 & 209 & 211 \end{bmatrix}$$

We refer to the matrix

$$B(k) = \begin{bmatrix} b_1 & \dots & 1\times b_k \\ b_1 & \dots & 2\times b_k \\ \vdots & \ddots & \vdots \\ b_1 & \dots & (p_k-1)\times b_k \end{bmatrix}$$

as the base matrix, denoted as $B(k)$.

We refer to the matrix

$$R(k) = \begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,C_k} \\ r_{2,1} & r_{2,2} & \cdots & r_{2,C_k} \\ \vdots & \vdots & \ddots & \vdots \\ r_{k-1,1} & r_{k-1,2} & \cdots & r_{k-1,C_k} \\ r_{k,1} & r_{k,2} & \cdots & r_{k,C_k} \end{bmatrix}$$

as the coefficient matrix, denoted as $R(k)$.

Therefore, the expression for $M(k)$ can be simplified as:

$$M(k) = B(k) \times R(k) + 2$$

In fact, the last column of the base matrix $B(k)$ consists of the coefficients of the base b_k, so the last row of the coefficient matrix $R(k)$ satisfies $r_{k,j} = 1$ for all j.

Therefore, the Mo number $m(k,i,j)$ in $M(k)$ can be expressed as:

$$m(k,i,j) = \sum_{i=1}^{k} r_{i,j} b_i + 2 = \sum_{i=1}^{k} r_{i,j} p_{i-1}!^p + 2$$

where:

- When $i = k, r_{k,j} \in [1, p_i - 1]$.
- Other conditions are consistent with those defined in the expression for $M(k)$.

For example:

$$\begin{aligned} m(4,1,1) &= \sum_{i=1}^{4} r_{i,1} p_{i-1}!^p + 2 \\ &= 1 \times p_3!^p + 0 \times p_2!^p + 2 \times p_1!^p + 1 \times p_0!^p + 2 \\ &= 1 \times 30 + 0 \times 6 + 2 \times 2 + 1 \times 1 + 2 \\ &= 37 \end{aligned}$$

2.3. ***Proof***: $\mathbb{P} \subseteq M$.

The prime numbers p are divided based on the Euclidean numbers $p_k!^p + 1$ for $k \geq 1$, the k-th interval is defined as $(p_{k-1}!^p + 1, p_k!^p + 1]$ for $k \geq 1$, The set composed of the numbers p in each interval is denoted as $P(k)$, with the stipulation that $P(0) = 2$. The sequence formed by the sets $P(k)$ for $k \geq 0$ as elements is denoted as $\mathbb{P}$. Thus,

$$\mathbb{P} = \{P(0), P(1), P(2), \ldots, P(k), \ldots\}$$

Proof: $\mathbb{P} \subseteq M$.

Base Cases:

(1) For $k = 0$:
  - $P(0) = [2] = M(0)$, so $P(0) \subseteq M(0)$.

(2) For $k = 1$:
  - $P(1) = [3] = M(1)$, so $P(1) \subseteq M(1)$.

(3) For $k = 2$:
  - $P(2) = \begin{bmatrix} 5 \\ 7 \end{bmatrix} = M(2)$, so $P(2) \subseteq M(2)$.

(4) For $k = 3$:
  - $P(3) = \begin{bmatrix} 11 & 13 \\ 17 & 19 \\ 23 & \\ 29 & 31 \end{bmatrix}$, $M(3) = \begin{bmatrix} 11 & 13 \\ 17 & 19 \\ 23 & 25 \\ 29 & 31 \end{bmatrix}$, so $P(3) \subseteq M(3)$.

Inductive Hypothesis:

Assume that for $k = n$, where $n \geq 3$, $P(n) \subseteq M(n)$.

Inductive Step:

We need to prove that for $k = n+1$, $P(n+1) \subseteq M(n+1)$.

Assume for contradiction that there exists a prime $p \in P(n+1)$ such that $p \notin M(n+1)$.

- Let $t = p \mod p_n!^p$. Then, $t < p_n!^p$.
- Since p is a prime and $p > 2$, p is odd.
- Since $p_n!^p$ is even, t must be odd, and $t \in [1, p_n!^p)$.

Case 1: $t = 1$

- Then, $p = r \times p_n!^p + t = (r-1) \times p_n!^p + p_n!^p + 1$, where $2 \leq r \leq p_{n+1} - 1$.
- Since $p_n!^p + 1 \in M(n)$ (a Euclidean number), $p \in M(n+1)$, which contradicts $p \notin M(n+1)$.

Case 2: $t \in [3, p_n!^p)$ and t is odd

- Subcase 2.1: If $f^p_{min}(t) \leq p_n$, then $p \notin P$, which contradicts $p \in P(n+1)$.
- Subcase 2.2: If $f^p_{min}(t) \geq p_{n+1}$:
  - Subcase 2.2.1: If $t \in M$, then $p \in M$, which contradicts $p \notin M(n+1)$.
  - Subcase 2.2.2: If $t \notin M$, let $t \in (p_{i-1}!^p + 1, p_i!^p + 1]$, and define $t_1 = t \mod p_{i-1}!^p$. Then:$p = r \times p_n!^p + t = r \times p_n!^p + r_1 \times p_{i-1}!^p + t_1$
    - If $t_1 \in M$, then $t \in M$, which contradicts $t \notin M$.
    - If $t_1 \notin M$, repeat the process by defining $t_2 = t_1 \mod p_{i_1-1}!^p$, where $t_1 \in (p_{i_1-1}!^p + 1, p_{i_1}!^p + 1]$.
    - Continue this process until $t_j \in (p_0!^p + 1, p_2!^p + 1] = (2, 7]$.
    - Since $t_j \in M$, it follows that $t_{j-1} \in M$, which contradicts $t_{j-1} \notin M$.

Conclusion:

- For $k = n+1, p \in M(n+1)$.
- Therefore, $\mathbb{P} \subseteq M$.

Q.E.D.

Therefore, The set of prime numbers is a subset of the Mo numbers. the elements p in the set of prime numbers $P(k)$ can also be expressed in the following form:

$$p = \sum_{t=0}^{k} r_t b_t + 2 = \sum_{t=0}^{k} r_t p_{t-1}!^p + 2$$

The set of composite numbers in matrix $M(k)$ is denoted as $M^{'}(k)$. Then

$$M(k) = P(k) \cup M^{'}(k)$$

The set $M^{'}(k)$ consists of composite Mo numbers in the interval

$$(p_{k-1}!^p + 1, p_k!^p + 1]$$

whose smallest prime factor is at least p_k. That is,

$$M^{'}(k) = \{m \mid m \in (p_{k-1}!^p + 1, p_k!^p + 1], f^p_{min}(m) \geq p_k\}$$

3. Classification of Mo Numbers

The Mo numbers are systematically classified into distinct types based on their algebraic form and residue properties. The classification follows a hierarchical structure and exhibits specific multiplicative closure properties.

3.1. *Type Definitions*.

We define the Type-q Mo numbers, denoted M_{Tq}, as those Mo numbers expressible in the form:

$$m(k,i,j) = \sum_{t=n(q)}^{k} r_{i,j} p_{t-1}!^p + q$$

where:

- p_{t-1} denotes the (t-1)-th prime.
- $r_{i,j}$ are integer coefficients, $0 \leq r_{i,j} \leq p_t - 1$.
- $q \in M$ is a constant determining the type, if $q \in M(s)$ then $n(q) = s$.
- $t \geq n(q)$.

For example:

- Type-2 (M_{T2}):

$$m(k,i,j) = \sum_{t=0}^{k} r_{i,j} p_{t-1}!^p + 2$$

- Type-3 (M_{T3}):

$$m(k,i,j) = \sum_{t=1}^{k} r_{i,j} p_{t-1}!^p + 3$$

- Type-5 (M_{T5}):

$$m(k,i,j) = \sum_{t=2}^{k} r_{i,j} p_{t-1}!^p + 5$$

- Type-7 (M_{T7}):

$$m(k,i,j) = \sum_{t=2}^{k} r_{i,j} p_{t-1}!^p + 7$$

Similarly, we can obtain M_{T11}, M_{T13}, M_{T17}, M_{T19}, etc.

Therefore, the set of M_{Tq} is

$$M_{Tq} = \left\{ m(k,i,j) \in M(k) | m(k,i,j) = \sum_{t=n(q)}^{k} r_{i,j} p_{t-1}!^{p} + q \right\}$$

The set M of all Mo numbers admits the following partition:

$$M = M_{T2} = \{2\} \cup M_{T3} = \{2,3\} \cup M_{T5} \cup M_{T7}$$

with the disjointness property:

$$\{2,3\} \cap M_{T5} \cap M_{T7} = \emptyset$$

3.2. *Type-5 and Type-7 Modular Forms and Multiplication Rules*.

Properties of modulo 6:

$$\begin{cases} M_{T5} \equiv 5 \pmod 6 \\ M_{T7} \equiv 1 \pmod 6 \end{cases}$$

Theorem 3.2.1:

- $M_{T5} \times M_{T5} \to M_{T7}$:

$$\forall m_i, m_j \in M_{T5}, m_i \times m_j = m_k \in M \Rightarrow m_k \in M_{T7}$$

- $M_{T5} \times M_{T7} \to M_{T5}$:

$$\forall m_i \in M_{T5}, m_j \in M_{T7}, m_i \times m_j = m_k \in M \Rightarrow m_k \in M_{T5}$$

- $M_{T7} \times M_{T7} \to M_{T7}$:

$$\forall m_i, m_j \in M_{T7}, m_i \times m_j = m_k \in M \Rightarrow m_k \in M_{T7}$$

Proof:

(1) Proof of $M_{T5} \times M_{T5} \to M_{T7}$:
- Let $m_i = 6a + 5 \in M_{T5}$, $m_j = 6b + 5 \in M_{T5}$.
- Compute:

$$\begin{aligned} m_i \times m_j &= (6a+5) \times (6b+5) \\ &= 36ab + 30(a+b) + 25 \\ &= 6[6ab + 5(a+b) + 4] + 1 \\ &= m_k \end{aligned}$$

- If $m_k \in M$, then $m_k \in M_{T7}$, it follows that $M_{T5} \times M_{T5} \to M_{T7}$.

(2) Proof of $M_{T5} \times M_{T7} \to M_{T5}$:
- Let $m_i = 6a + 5 \in M_{T5}$, $m_j = 6b + 1 \in M_{T7}$.

- Compute:

$$\begin{aligned} m_i \times m_j &= (6a+5) \times (6b+1) \\ &= 6(6ab+a+5b)+5 \\ &= m_k \end{aligned}$$

- If $m_k \in M$, then $m_k \in M_{T5}$, it follows that $M_{T5} \times M_{T7} \to M_{T5}$.

(3) Proof of $M_{T7} \times M_{T7} \to M_{T7}$:

- Let $m_i = 6a+1 \in M_{T7}$, $m_j = 6b+1 \in M_{T7}$.
- Compute:

$$\begin{aligned} m_i \times m_j &= (6a+1) \times (6b+1) \\ &= 6(6ab+a+b)+1 \\ &= m_k \end{aligned}$$

- If $m_k \in M$, then $m_k \in M_{T7}$, it follows that $M_{T7} \times M_{T7} \to M_{T7}$.

Q.E.D.

Corollary 1:

For any prime number $p \geq 5$, we have $p^2 \in M_{T7}$, i.e., that is,

$$p^2 = \sum_{t=2}^{k} r_{i,j} p_{t-1}!^p + 7$$

More generally, if $p^n \in M$:

$$\begin{cases} p^n \in M_{T5} & \text{if } p \in M_{T5} \textit{ and } n \textit{ is odd} \\ p^n \in M_{T7} & \text{if } p \in M_{T7} \textit{ or } n \textit{ is even} \end{cases}$$

Corollary 2:

For any twin primes $p \geq 5$ and $p+2$, we have $p \in M_{T5}$, $p+2 \in M_{T7}$.

Therefore, if $p(p+2) \in M$, then $p(p+2) \in M_{T5}$, i.e.:

$$p(p+2) = \sum_{t=2}^{k} r_{i,j} p_{t-1}!^p + 5$$

3.3. ***Study of Sophie Germain Primes***.

Definition: A prime number p is called a Sophie Germain prime (SG prime) if $2p+1$ is also prime.

Mo Number Type Constraint:

For any Sophie Germain primes $p \geq 5$ must satisfy:

(1) $p \in M_{T5}$ (belongs to Type-5 Mo numbers);
(2) $p \not\equiv 7 \pmod{10}$ (the units digit cannot be 7);
(3) $2p+1 \in M_{T5}$ (belongs to Type-5 Mo numbers);
(4) $2p+1 \not\equiv 1 \pmod{10}$ ($p \geq 11$ the units digit cannot be 1).

Proof of Type Constraints:
Theorem 3.3.1: No Sophie Germain prime $p \geq 5$ belongs to M_{T7}.
Proof (by contradiction):
Assume there exists an SG prime $p \in M_{T7}$. Then it has the expression:

$$p = \sum_{t=2}^{k} r_t p_{t-1}!^p + 7$$

Compute $2p+1$:

$$\begin{aligned} 2p+1 &= 2(\sum_{t=2}^{k} r_t p_{t-1}!^p + 7) + 1 \\ &= 2\sum_{t=2}^{k} r_t p_{t-1}!^p + 15 \end{aligned}$$

Since $\sum_{t=2}^{k} r_t p_{t-1}!^p$ is a multiple of 3 and 15 is also a multiple of 3, $2p+1$ is a multiple of 3, which contradicting the definition of an SG prime.
Theorem 3.3.2: The units digit of a Sophie Germain prime cannot be 7.
Proof (by contradiction):
Assume there exists $p = 7 \pmod{10}$, i.e., $p = 10a + 7$.
Compute $2p+1$:

$$\begin{aligned} 2p+1 &= 2(10a+7) + 1 \\ &= 5(4a+3) \end{aligned}$$

Clearly, $2p+1$ is a multiple of 5, which contradicting the definition of an SG prime.
Theorem 3.3.3: For any Sophie Germain prime $p \in 5$, $2p+1 \in M_{T5}$.
Proof:
Since $p \in M_{T5}$, Then it has the expression:

$$p = \sum_{t=2}^{k} r_t p_{t-1}!^p + 5$$

Compute $2p+1$:

$$2p+1 = 2(\sum_{t=2}^{k} r_t p_{t-1}!^p + 5) + 1$$
$$= 2\sum_{t=2}^{k} r_t p_{t-1}!^p + 11$$

Since $2\sum_{t=2}^{k} r_t p_{t-1}!^p$ is a multiple of 6, $2p+1 \equiv 5 \pmod 6$, therefore $2p+1 \in M_{T5}$.

Theorem 3.3.4: The units digit of safe-primes cannot be 1 where Sophie Germain prime $p \geq 11$.

Proof (by contradiction):

Assume there exists an SG prime $p \geq 11$ and $2p+1 = 1 \pmod{10}$, i.e., $2p+1 = 10a+1$.

Thus $p = 5a$.

Clearly, p is a multiple of 5, which contradicting p is a prime.

3.4. ***Infinitude of Primes in*** M_{Tq}.

Theorem 3.4.1: For any prime q, the set M_{Tq} of Type-q Mo numbers contains infinitely many primes. That is:

$$\forall\ prime\ q, \quad |\{p \in M_{Tq} : p\ is\ prime\}| = \infty$$

Proof:

Let $s = n(q)$, then $b_{s+1} = p_s!^p$.

Since $q \geq p_{s+1} > p_s$, $gcd(q, b_{s+1}) = 1$.

By Dirichlet's theorem, the arithmetic progression

$$B_q = \{q, q+b_{s+1}, q+2b_{s+1}, q+3b_{s+1}, \dots\}$$

contains infinitely many primes.

Now we only need to show that $M_{Tq} \subseteq B_q$ and that the set difference $B_q \setminus M_{Tq}$ contains no primes.

For any $m(k,i,j) \in M_{Tq}$, we have

$$m(k,i,j) = \sum_{t=n(q)}^{k} r_{i,j} p_{t-1}!^p + q$$

For all $t \geq n(q) = s$ and if $t = s$ then $r_{i,j} = 0$, thus

$$b_{s+1} \mid \sum_{t=n(q)}^{k} r_{i,j} p_{t-1}!^p$$

Therefore,

$$m(k,i,j) = \sum_{t=n(q)}^{k} r_{i,j} p_{t-1}!^p + q \in B_q$$

which implies $M_{Tq} \subseteq B_q$.

Suppose, for contradiction, that there exists a prime $q+nb_{s+1}$ (with $n \in \mathbb{N}$) in the set difference $B_q \setminus M_{Tq}$.

Then $q + nb_{s+1} \notin M$.

But this contradicts the fact that $P \subseteq M$ (all primes belong to the Mo number set M).

Hence, the difference set $B_q \setminus M_{Tq}$ contains no primes.

Since B_q contains infinitely many primes (by Dirichlet's theorem), all these primes must lie in M_{Tq}, and therefore M_{Tq} contains infinitely many primes.

Q.E.D.

3.5. ***Classification and Conjectures on Mo number Pairs***.

For any consecutive Mo number pair $q, q+2h \in M(n(q))$ with $q \geq 5$, the set of Type-(q,q+2h) Mo number pairs is defined as:

$$M_{T(q,q+2h)} = \{(q, q+2h)\} \cup \{(m(k,i,j), m(k,i,j)+2h)\}$$

where:

- $m(k,i,j) = \sum_{t=n(q)}^{k} r_{i,j} p_{t-1}!^p + q$, $m(k,i,j), m(k,i,j)+2h \in M(k)$;
- p_{t-1} denotes the (t-1)-th prime;
- $r_{i,j} \in \mathbb{N}$, $r_{i,j} \leq p_t - 1$;
- $h \in \mathbb{N}_+$;
- If $q \in M(s)$, then $n(q) = s$;
- $t \geq n(q)$.

Conjecture 3.5.1:

When $h = 1$, for any twin prime pair $\{(q, q+2)\}$ with $q \geq 5$, the set of type-(q,q+2) twin Mo number pairs $M_{T(q,q+2)}$ contains infinitely many twin prime pairs.

Clearly, this is a refined subdivision of the Twin Prime Conjecture, which itself is equivalent to the statement that $M_{T(5,7)}$ contains infinitely many twin prime pairs.

Conjecture 3.5.2: Infinitude of Prime Pairs at Even Spacing in Mo Number Types.

For any even spacing $2h$ (where $h \in \mathbb{N}_+$), and for any initial prime pair $\{(q, q+2h)\}$ with $q \geq 5$, the set of type-(q,q+2h) Mo number pairs $M_{T(q,q+2h)}$ contains infinitely many prime pairs spaced exactly $2h$ apart.

Clearly, this is a refinement and subdivision of Polignac's conjecture.

For example, $M_{T(23,29)}$ contains an infinite number of prime pairs with an interval of 6.

4. **Study of the Matrix** $M(k)$

4.1. ***Properties of the Matrix*** $M(k)$.

- Basic Properties of the Matrix $M(k)$
  (1) When $k \geq 2$, the matrix $M(k)$ has $p_k - 1$ rows and $\prod_{t=1}^{k-1}(p_t - 1)$ columns (a proof will be given later). Therefore, the total number of elements in $M(k)$ is $\prod_{t=1}^{k}(p_t - 1)$.
  (2) Each column of the matrix $M(k)$ is an arithmetic progression with common difference $b_k = p_{k-1}!^p$.
  (3) The smallest element of the matrix $M(k)$ is $m(k, 1, 1) = p_{k-1}!^p + p_k$.
  (4) The largest element of the matrix $M(k)$ is $m(k, p_k - 1, \prod_{t=1}^{k-1}(p_t - 1)) = p_k!^p + 1$.
  (5) The smallest element in the r-th row of the matrix $M(k)$ is $m(k, r, 1) = r \times p_{k-1}!^p + p_k$.
  (6) The largest element in the r-th row of the matrix $M(k)$ is $m(k, r, \prod_{t=1}^{k-1}(p_t - 1)) = (r + 1) \times p_{k-1}!^p + 1$.
- Let $F^m_{min}(p_k)$ denote the set of Mo numbers in M whose smallest prime factor is p_k. Then it has the following properties:
  (1) $F^m_{min}(p_k) = \{m \in M \mid f^p_{min}(m) = p_k\}$.
  (2) The smallest element in the set $F^m_{min}(p_k)$ is p_k, and it is the only prime number in this set.
  (3) The second smallest element in $F^m_{min}(p_k)$ is p_k^2.
  (4) The largest element in $F^m_{min}(p_k)$ is $(p_{k-1}!^p - 1) \times p_k$, and for $k \geq 4$, it is located in row $p_k - 1$, column $\prod_{t=1}^{k-1}(p_t - 1) - 2$ of matrix $M(k)$.

Theorem 4.1.1: When $k \geq 2$, we have

$$C_k = \prod_{t=1}^{k-1}(p_t - 1) = |M(k-1)| = |F^m_{min}(p_k)|$$

where

- C_k = number of columns of the matrix $M(k)$;
- $|M(k-1)|$ = number of elements in the matrix $M(k-1)$;
- $|F^m_{min}(p_k)|$ = number of elements in the set $F^m_{min}(p_k)$.

Since the number of rows of the matrix $M(k-1)$ is $p_{k-1}-1$, the number of elements in $M(k-1)$ is:

$$|M(k-1)| = C_{k-1}(p_{k-1}-1) = \prod_{t=1}^{k-1}(p_t - 1)$$

Therefore, we only need to prove:

$$C_k = \prod_{t=1}^{k-1}(p_t - 1) = |F^m_{min}(p_k)|$$

Proof (by mathematical induction):
Base cases:

- For $k = 2$,

$$C_2 = 1 = \prod_{t=1}^{1}(p_t - 1) = |F^m_{min}(p_2)|$$

  holds.
- For $k = 3$,

$$C_3 = 2 = \prod_{t=1}^{2}(p_t - 1) = |F^m_{min}(p_3)|$$

  holds.
- For $k = 4$,

$$C_4 = 8 = \prod_{t=1}^{3}(p_t - 1) = |F^m_{min}(p_4)|$$

  holds.

Inductive hypothesis:
Assume for $k = n$, $n \geq 2$,

$$C_n = \prod_{t=1}^{n-1}(p_t - 1) = |F^m_{min}(p_n)|$$

holds.
Inductive step for $k = n + 1$:
Let $|F^m_{min}(n, p_n)|$ be the number of elements in the set $F^m_{min}(n, p_n)$—the subset of $M(n)$ whose elements have smallest prime factor p_n.

Let $|F^m_{min}([p_{n+1}, p_n!^p + 1], \geq p_{n+1})|$ be the number of elements in the set $F^m_{min}([p_{n+1}, p_n!^p+1], \geq p_{n+1})$—the subset of Mo numbers in the interval $[p_{n+1}, p_n!^p+1]$ with smallest prime factor $\geq p_{n+1}$.

According to the computation rule for matrix $M(n+1)$:

$$\begin{aligned} C_{n+1} &= |F^m_{min}([p_{n+1}, p_n!^p + 1], \geq p_{n+1})| \\ &= |M(n)| - |F^m_{min}(n, p_n)| + C_n - (|F^m_{min}(p_n)| - |F^m_{min}(n, p_n)|) \\ &= |M(n)| + C_n - |F^m_{min}(p_n)| \\ &= |M(n)| + C_n - C_n \\ &= (p_n - 1)\prod_{t=1}^{n-1}(p_t - 1) \\ &= \prod_{t=1}^{n}(p_t - 1) \end{aligned}$$

Also,

$$\begin{aligned} |F^m_{min}(p_{n+1})| &= C_{n+1} + |M(n+1)| - C_{n+2} \\ &= C_{n+1} + |M(n+1)| - |M(n+1)| \\ &= C_{n+1} \end{aligned}$$

Therefore, for $k = n+1$,

$$C_{n+1} = \prod_{t=1}^{n}(p_t - 1) = |F^m_{min}(p_{n+1})|$$

holds.

Conclusion:

By induction, the statement is true for all $k \geq 2$.

Q.E.D.

4.2. ***Distribution of Primes in the Matrix*** $M(k)$.

This chapter will introduce how to calculate the number of primes in the matrix $M(k)$ based on $F(k)$.

Define $\lfloor x \rfloor_p$ as the largest prime number not exceeding x, and $\lceil x \rceil_p$ as the smallest prime number not less than x.

We can easily see that the smallest prime factors of the elements in the set $M^{'}(k)$ lie within the interval $\left[p_k,\ \lfloor \sqrt{p_k!^p + 1} \rfloor_p \right]$.

Let $p_{k+s} = \lfloor \sqrt{p_k!^p + 1} \rfloor_p$. Then the set of smallest prime factors of the elements in $M^{'}(k)$ is:

$$\{p_k, p_{k+1}, \ldots, p_{k+s-1}, p_{k+s} \mid k, s \in \mathbb{N},\ k \geq 1\}$$

Let $N(p_k \mid M^{'}(k))$, where $k \geq 3$, denote the number of elements in the set $M^{'}(k)$ whose smallest prime factor is p_k.

For example:

(1) $M^{'}(3) = \{25\}$ contains only one element, whose smallest prime factor is 5, i.e., $N(5 \mid M^{'}(3)) = 1$.

(2) $M^{'}(4) = \{49, 77, 91, 119, 121, 133, 143, 161, 169, 187, 203, 209\}$

- $N(7 \mid M^{'}(4)) = 7$ (Elements: 49, 77, 91, 119, 133, 161, 203).
- $N(11 \mid M^{'}(4)) = 4$ (Elements: 121, 143, 187, 209).
- $N(13 \mid M^{'}(4)) = 1$ (Element: 169).

Let $N(p_k \mid [a, b])$ denote the number of elements in $M^{'}(k)$ within the interval $[a, b]$ whose smallest prime factor is p_k, and $N_F(\geq p_k \mid [c, d])$ denote the number of elements in $F(k)$ within the interval $[c, d]$ whose smallest prime factor is at least p_k.

Then

$$N(p_k \mid [a, b]) = N_F(\geq p_k \mid \left[max(\left\lceil \frac{a}{p_k} \right\rceil, p_k), \left\lfloor \frac{b}{p_k} \right\rfloor\right])$$

Therefore,

$$N(p_t \mid M^{'}(k)) = N(p_t \mid [p_{k-1}!^p + 2, p_k!^p + 1])$$
$$= N_F(\geq p_t \mid \left[max(\left\lceil \frac{p_{k-1}!^p + 2}{p_t} \right\rceil, p_t), \left\lfloor \frac{p_k!^p + 1}{p_t} \right\rfloor\right])$$

For example:

$$N(13 \mid M^{'}(6)) = N_F(\geq 13 \mid \left[max(\left\lceil \frac{11!^p + 2}{13} \right\rceil, 13), \left\lfloor \frac{13!^p + 1}{13} \right\rfloor\right]) = 443$$

$$N(41 \mid M^{'}(6)) = N_F(\geq 41 \mid \left[max(\left\lceil \frac{11!^p + 2}{41} \right\rceil, 41), \left\lfloor \frac{13!^p + 1}{41} \right\rfloor\right]) = 113$$

Furthermore, since:

$$|M^{'}(k)| = \sum_{n=0}^{s} N(p_{k+n} \mid M^{'}(k))$$

the number of primes in the matrix $M(k)$ is:

$$|P(k)| = |M(k)| - |M^{'}(k)|$$

Substituting the expressions for $|M(k)|$ and $|M^{'}(k)|$, we have:

$$|P(k)| = \prod_{t=1}^{k}(p_t - 1) - \sum_{n=0}^{s} N(p_{k+n} \mid M^{'}(k))$$

Continuing by substituting the expression for $N(p_{k+n} \mid M^{'}(k))$, we obtain:

$$|P(k)| = \prod_{t=1}^{k}(p_t - 1) - \sum_{n=0}^{s} N_F(\geq p_{k+n} \mid \left[max(\left\lceil \frac{p_{k-1}!^p + 2}{p_{k+n}} \right\rceil, p_{k+n}), \left\lfloor \frac{p_k!^p + 1}{p_{k+n}} \right\rfloor\right])$$

Certainly, when $k \geq 6$, it becomes necessary to use computer programs to facilitate the calculation. The corresponding Python code is provided in the appendix.

$M(k)$	$Interval$ $(p_{k-1}!^p+1, p_k!^p+1]$	$Number\ of$ $Elements$ $\lvert M(k)\rvert$	$Number\ of$ $Primes$ $\lvert P(k)\rvert$	$Prime$ $Density$ (%)
$M(0)$	(1,2]	1	1	100.00
$M(1)$	(2,3]	1	1	100.00
$M(2)$	(3,7]	2	2	100.00
$M(3)$	(7,31]	8	7	87.50
$M(4)$	(31,211]	48	36	75.00
$M(5)$	(211,2311]	480	297	61.88
$M(6)$	(2311,30031]	5760	2904	50.42
$M(7)$	(30031,510511]	92160	39083	42.41
$M(8)$	(510511,9699691]	1648880	603698	36.61
$M(9)$	(9699691,223092871]	36495360	11637502	31.89
$M(10)$	(223092871,6469693231]	1021870080	288086265	28.19

Table 1. Statistical of prime in matrix $M(k)$

From the table, it can be observed that as k increases, the number of prime numbers $|P(k)|$ in $M(k)$ also increase.

4.3. ***Prime Distribution Function*** $\pi(x)$.

Let $x \in (p_{k-1}!^p + 1,\ p_k!^p + 1]$, and let $p_{k+s} = \lfloor\sqrt{x}\rfloor_p$.

Let $N(p_k \mid M^{'}(k,x))$ denote the number of Mo numbers in the set $M^{'}(k)$ that do not exceed x and have smallest prime factor p_k. Then:

$$N(p_k \mid M^{'}(k,x)) = N_F(\geq p_k \mid \left[max(\left\lceil\frac{(p_{k-1}!^p+2)}{p_k}\right\rceil,\ p_k),\ \left\lfloor\frac{x}{p_k}\right\rfloor\right])$$

Let $|M^{'}(k,x)|$ denote the number of elements in the set $M^{'}(k)$ that do not exceed x. Then:

$$\begin{aligned}|M^{'}(k,x)| &= \sum_{n=0}^{s} N(p_{k+n} \mid M^{'}(k,x)) \\ &= \sum_{n=0}^{s}(N_F(\geq p_{k+n} \mid \left[max(\left\lceil\frac{p_{k-1}!^p+2}{p_{k+n}}\right\rceil,\ p_{k+n}), \left\lfloor\frac{x}{p_{k+n}}\right\rfloor\right])\end{aligned}$$

Let $|M(k,x)|$ denote the number of elements in the matrix $M(k)$ that do not exceed x. Then:

$$
\begin{aligned}
|M(k,x)| = (\left\lfloor \frac{x}{p_{k-1}!^p} \right\rfloor - 1)\prod_{i=1}^{k-1}(p_i - 1) \\
+ N_F(\geq p_k \mid \left[p_k,\ x - \left\lfloor \frac{x}{p_{k-1}!^p} \right\rfloor p_{k-1}!^p\right])
\end{aligned}
$$

Let $|P(k,x)|$ denote the number of primes in the set $P(k)$ that do not exceed x. Then:

$$|P(k,x)| = |M(k,x)| - |M^{'}(k,x)|$$

Therefore, the prime distribution function $\pi(x)$ can be expressed as:

$$
\begin{aligned}
\pi(x) &= \sum_{t=0}^{k-1} |P(t)| + |P(k,x)| \\
&= \sum_{t=0}^{k-1} |P(t)| + |M(k,x)| - |M^{'}(k,x)|
\end{aligned}
$$

For example:

Let $x = 139 \in (p_3!^p + 1,\ p_4!^p + 1]$. We know it lies in the matrix $M(4)$.

Let $p_{k+s} = \left\lfloor \sqrt{139} \right\rfloor_p = 11 = p_5$, so $s \in \{0,\ 1\}$, and the corresponding set of smallest prime factors is $\{7,\ 11\}$.

Given $F(4) = \{7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31\}$, we have:

$$
\begin{aligned}
|M(4,139)| &= (\left\lfloor \frac{139}{p_3!^p} \right\rfloor - 1)\prod_{i=1}^{3}(p_i - 1) + N_F(\geq p_4 \mid \left[p_4,\ 139 - \left\lfloor \frac{139}{p_3!^p} \right\rfloor p_3!^p\right]) \\
&= (\left\lfloor \frac{139}{30} \right\rfloor - 1) \times 8 + N_F(\geq 7 \mid \left[7,\ 139 - \left\lfloor \frac{139}{30} \right\rfloor \times 30\right]) \\
&= 24 + 5 \\
&= 29
\end{aligned}
$$

$$
\begin{aligned}
|M^{'}(4,139)| &= \sum_{n=0}^{1} N_F(\geq p_{4+n} \mid \left[max(\left\lceil \frac{p_3!^p + 2}{p_{4+n}} \right\rceil,\ p_{4+n}),\ \left\lfloor \frac{139}{p_{4+n}} \right\rfloor\right]) \\
&= 6
\end{aligned}
$$

Therefore:

$$\begin{aligned}\pi(139) &= \sum_{t=0}^{3} |P(t)| + |P(4,139)| \\ &= \sum_{t=0}^{3} |P(t)| + |M(4,139)| - |M^{'}(4,139)| \\ &= 11 + 29 - 6 \\ &= 34\end{aligned}$$

4.4. ***Obtaining the primes in the Matrix*** $M(k)$.

To facilitate computation, we construct the extended matrix $\overline{M(k)}$ of $M(k)$:

$$\overline{M(k)} = \begin{bmatrix} F(k) \\ M(k) \end{bmatrix}$$

and define $F(0) = \begin{bmatrix} 1 \end{bmatrix}$. Then we have:

$$\overline{M(0)} = \begin{bmatrix} F(0) \\ M(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Similarly, we can construct the sequence $\overline{M}$ with $\overline{M(k)}$ for $k \geq 0$ as its elements:

$$\overline{M} = \left\{ \overline{M(0)}, \overline{M(1)}, \overline{M(2)}, \ldots, \overline{M(k)}, \ldots \right\}$$

First, $F(k)$ is a row vector formed by removing the elements with smallest prime factor equal to p_{k-1} from the matrix $\overline{M(k-1)}$. Since $M(k)$ is a column arithmetic matrix with common difference $p_{k-1}!^p$, we can derive $M(k)$ from $\overline{M(k-1)}$, thereby obtaining $M(k)$.

Second, use $F(k)$ to generate the set $M^{'}(k)$:

Let $p_{k+s} = \left\lfloor \sqrt{p_k!^p + 1} \right\rfloor_p$. Then the set of smallest prime factors of the elements in $M^{'}(k)$ is:

$$\{p_k, p_{k+1}, \ldots, p_{k+s-1}, p_{k+s} \mid k, s \in \mathbb{N},\ k \geq 1\}$$

Step 1: Find the elements in $M^{'}(k)$ with smallest prime factor p_k:

- Identify elements in F(k) that belong to the interval
$$\left[max(\left\lceil \frac{(p_{k-1}!^p + 2)}{p_k} \right\rceil, p_k), \left\lfloor \frac{(p_k!^p + 1)}{p_k} \right\rfloor \right]$$
and have smallest prime factor $\geq p_k$.
- Multiply these elements by p_k to obtain the set $F^{m'}_{min}(k, p_k)$ of elements in $M^{'}(k)$ with smallest prime factor p_k.

Step 2: Repeat Step 1 to obtain the sets $F_{min}^{m'}(k, p_{k+1})$ to $F_{min}^{m'}(k, p_{k+s})$ for elements in $M^{'}(k)$ with smallest prime factors from p_{k+1} to p_{k+s}.

Step 3: Combine the results from Step 1 and Step 2 to form the set

$$M^{'}(k) = \bigcup_{n=0}^{s} F_{min}^{m'}(k, p_{k+n})$$

Finally, obtain P(k) by taking the set difference:

$$P(k) = M(k) \setminus M^{'}(k)$$

For example:

When $k = 3$

$$\overline{M(3)} = \begin{bmatrix} F(3) \\ M(3) \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 11 & 13 \\ 17 & 19 \\ 23 & 25 \\ 29 & 31 \end{bmatrix}$$

From this we obtain:

$$F(4) = \begin{bmatrix} 7 & 11 & 13 & 17 & 19 & 23 & 29 & 31 \end{bmatrix}$$

According to the common difference $p_3!^p = 30$, we can easily derive:

$$\overline{M(4)} = \begin{bmatrix} 7 & 11 & 13 & 17 & 19 & 23 & 29 & 31 \\ 37 & 41 & 43 & 47 & 49 & 53 & 59 & 61 \\ 67 & 71 & 73 & 77 & 79 & 83 & 89 & 91 \\ 97 & 101 & 103 & 107 & 109 & 113 & 119 & 121 \\ 127 & 131 & 133 & 137 & 139 & 143 & 149 & 151 \\ 157 & 161 & 163 & 167 & 169 & 173 & 179 & 181 \\ 187 & 191 & 193 & 197 & 199 & 203 & 209 & 211 \end{bmatrix}$$

Therefore:

$$M(4) = \begin{bmatrix} 37 & 41 & 43 & 47 & 49 & 53 & 59 & 61 \\ 67 & 71 & 73 & 77 & 79 & 83 & 89 & 91 \\ 97 & 101 & 103 & 107 & 109 & 113 & 119 & 121 \\ 127 & 131 & 133 & 137 & 139 & 143 & 149 & 151 \\ 157 & 161 & 163 & 167 & 169 & 173 & 179 & 181 \\ 187 & 191 & 193 & 197 & 199 & 203 & 209 & 211 \end{bmatrix}$$

Use $F(4)$ to generate the set $M^{'}(4)$:

Let $p_{4+s} = \left\lfloor \sqrt{p_4!^p + 1} \right\rfloor_p = 13 = p_6$. Then the set of smallest prime factors of the elements in $M^{'}(4)$ is:

$$\{p_4, p_5, p_6\} = \{7, 11, 13\}$$

(1) Case with smallest prime factor $p_4 = 7$:

- Interval:

$$[max(\frac{p_3!^p+1}{p_4}, p_4), \frac{p_4!^p+1}{p_4}] = [7, 30]$$

- Elements of $F(4)$ in $[7, 30]$:

$$\{7, 11, 13, 17, 19, 23, 29\}$$

- After multiplying by 7:

$$\{49, 77, 91, 119, 133, 161, 203\}$$

(2) Case with smallest prime factor $p_5 = 11$:

- Interval:

$$[max(\frac{p_3!^p+1}{p_5}, p_5), \frac{p_4!^p+1}{p_5}] = [11, 19]$$

- Elements of $F(4)$ in $[11, 19]$:

$$\{11, 13, 17, 19\}$$

- After multiplying by 11:

$$\{121, 143, 187, 209\}$$

(3) Case with smallest prime factor $p_6 = 13$:

- Interval:

$$[max(\frac{p_3!^p+1}{p_6}, p_6), \frac{p_4!^p+1}{p_6}] = [13, 16]$$

- Elements of $F(4)$ in $[13, 16]$:

$$\{13\}$$

- After multiplying by 13:

$$\{169\}$$

Combining the results above:

$$\begin{aligned} M^{'}(4) &= \bigcup_{n=0}^{2} F_{min}^{m'}(4, p_{4+n}) \\ &= \{49, 77, 91, 119, 133, 161, 203\} \cup \{121, 143, 187, 209\} \cup \{169\} \\ &= \{49, 77, 91, 119, 121, 133, 143, 161, 169, 187, 203, 209\} \end{aligned}$$

Deriving $P(4)$:
Compute the set difference:

$$P(4) = M(4) \setminus M^{'}(4)$$

Result:

$$P(4) = \{37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211\}$$

This is an **efficient method for generating prime tables**.

5. Study of Twin Primes

5.1. *The Origin of Twin Numbers*.

Twin primes originate from the computation of $M(2)$:

$$M(2) = \begin{bmatrix} 1 \times b_2 + p_2 \\ 2 \times b_2 + p_2 \end{bmatrix} = \begin{bmatrix} 1 \times p_1!^p + p_2 \\ 2 \times p_1!^p + p_2 \end{bmatrix} = \begin{bmatrix} 1 \times 2 + 3 \\ 2 \times 2 + 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

- $\{3,\ 5\}$ is the only pair of twin primes that spans across matrices.
- All subsequent twin primes are directly or indirectly generated from $M(2) = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ and belong to the type $M_{T(5,7)}$.
- $M(2)$ is also the only pair of twin primes within a matrix that spans across rows.
- The value $b_2 = p_1!^p = 2$ is the fundamental reason for the generation of twin numbers.

5.2. *Properties of the Twin Number Matrix* $M_2(k)$.

Let $M_2(k)$ denote the set of all twin number pairs in the matrix $M(k)$, $P_2(k)$ denote the set of all twin prime pairs in $M(k)$, and $M_2^{'}(k)$ denote the set of all twin number pairs in $M(k)$ that are not twin primes. Then:

$$M_2(k) = P_2(k) \cup M_2^{'}(k)$$

Let $F_2(k)$ be the set of twin number pairs in $F(k)$, and T_k be the number of columns in the matrix $M_2(k)$. Then:

$$T_k = \prod_{t=2}^{k-1} (p_t - 2) \ ,\ k \geq 3$$

The number of twin number pairs in $M(k)$ is:

$$|M_2(k)| = (p_k - 1) \prod_{t=2}^{k-1} (p_t - 2) \ ,\ k \geq 3$$

Let $F^m_{2min}(p_k)$ denote the set of twin number pairs in M whose smallest prime factor is p_k. Then the number of twin number pairs in this set is:

$$|F^m_{2min}(p_k)| = 2\prod_{t=2}^{k-1}(p_t - 2)\ ,\ k \geq 3$$

Theorem 5.2.1: When $k \geq 3$,

$$T_k = \prod_{t=2}^{k-1}(p_t - 2) = \frac{1}{2}|F^m_{2min}(p_k)|$$

Proof:

(1) Base Cases:
- For $k = 3$:

$$T_3 = 1 = \prod_{t=2}^{2}(p_t - 2) = \frac{1}{2}|F^m_{2min}(p_3)|$$

- For $k = 4$:

$$T_4 = 3 = \prod_{t=2}^{3}(p_t - 2) = \frac{1}{2}|F^m_{2min}(p_4)|$$

- For $k = 5$:

$$T_5 = 15 = \prod_{t=2}^{4}(p_t - 2) = \frac{1}{2}|F^m_{2min}(p_5)|$$

(2) Inductive Hypothesis:
Assume that for $k = n$ (where $n \geq 3$) the following holds:

$$T_n = \prod_{t=2}^{n-1}(p_t - 2) = \frac{1}{2}|F^m_{2min}(p_n)|$$

(3) Inductive Step for $k = n + 1$:
Let $|F^m_{2min}(n, p_n)|$ be the number of twin number pairs in matrix $M_2(n)$ with smallest prime factor p_n.
Let $|F^m_{2min}([p_{n+1},\ p_n!^p + 1]\,, \geq p_{n+1})|$ be the number of twin number pairs in $F_2(n + 1)$ within the interval $[p_{n+1}, p_n!^p + 1]$ with smallest prime factor at least p_{n+1}.

According to the computation rule for matrix $M_2(n+1)$:

$$\begin{aligned} T_{n+1} &= |F^m_{2min}([p_{n+1},\ p_n!^p + 1]\ ,\ \geq p_{n+1})| \\ &= |M_2(n)| + |F^m_{2min}([p_n,\ p_{n-1}!^p + 1]\ ,\ \geq p_n)| - |F^m_{2min}(p_n)| \\ &= (p_n - 1)T_n + T_n - 2T_n \\ &= (p_n - 2)\prod_{t=2}^{n-1}(p_t - 2) \\ &= \prod_{t=2}^{n}(p_t - 2) \end{aligned}$$

The number of elements in the set of twin number pairs in M with smallest prime factor p_{n+1}:

$$\begin{aligned} |F^m_{2min}(p_{n+1})| &= T_{n+1} + |M_2(n+1)| - T_{n+2} \\ &= T_{n+1} + (p_{n+1} - 1)T_{n+1} - (p_{n+1} - 2)T_{n+1} \\ &= T_{n+1}(1 + p_{n+1} - 1 - p_{n+1} + 2) \\ &= 2T_{n+1} \\ &= 2\prod_{t=2}^{n}(p_t - 2) \end{aligned}$$

Therefore, for $k = n + 1$, the following holds:

$$T_{n+1} = \prod_{t=2}^{n}(p_t - 2) = \frac{1}{2}|F^m_{2min}(p_{n+1})|$$

(4) Conclusion:

For all $k \geq 3$,

$$T_k = \prod_{t=2}^{k-1}(p_t - 2) = \frac{1}{2}|F^m_{2min}(p_k)|$$

holds.

Q.E.D.

If $(p_k, p_k + 2)$ is a twin prime pair, then it is the only twin prime pair in the set $F^m_{2min}(p_k)$; otherwise, there will be no twin prime pairs in the set $F^m_{2min}(p_k)$.

The sequence formed by the sets $M_2(k)$ for $k \geq 0$ as elements is denoted as M_2. Thus,

$$M_2 = \{M_2(0), M_2(1), M_2(2), \ldots, M_2(k), \ldots\}$$

The sequence formed by the sets $P_2(k)$ for $k \geq 0$ as elements is denoted as P_2. Thus,

$$P_2 = \{P_2(0), P_2(1), P_2(2), \ldots, P_2(k), \ldots\}$$

The table below provides statistics on the number of twin prime pairs and twin number pairs in the matrix $M_2(k)$.

$M(k)$	*Interval* $(p_{k-1}!^p+1, p_k!^p+1]$	*The number of twinprime pairs*	*The number of twin number pairs*	*proportion of twin prime pairs*
$M(0)$	(1,2]	0	0	-
$M(1)$	(2,3]	0	0	-
$M(2)$	(3,7]	1	1	100.00%
$M(3)$	(7,31]	3	4	75.00%
$M(4)$	(31,211]	10	18	55.56%
$M(5)$	(211,2311]	55	150	36.67%
$M(6)$	(2311,30031]	398	1620	24.57%
$M(7)$	(30031,510511]	4168	23760	17.54%
$M(8)$	(510511,9699691]	52817	400950	13.17%
$M(9)$	(9699691,223092871]	838609	8330850	10.07%
$M(10)$	(223092871,6469693231]	17567651	222660900	7.89%

Table 2. Statistical of twin prime pairs in matrix $M(k)$

The table reveals that for $k \geq 2$, as k increases, both the number of twin prime pairs $|P_2(k)|$ and the number of twin pairs $|M_2(k)|$ grow exponentially. However, the proportion of twin prime pairs exhibits a declining trend. This indicates that the growth rate of twin pairs $|M_2(k)|$ surpasses that of twin prime pairs $|P_2(k)|$ as k increases. Consequently, we propose the following conjecture:

$$|P_2(k+1)| > |P_2(k)| > 0, and \lim_{k\to\infty} \frac{|P_2(k)|}{|M_2(k)|} = 0, k \geq 2$$

Since P_2 is an infinite sequence, the validity of the above conclusion would imply the truth of the Twin Prime Conjecture.

5.3. ***Obtaining the twin primes in the matrix*** $M_2(k)$.

In Section 4.4, we introduced how to obtain the set $P(k)$. Since $P_2(k) \subseteq P(k)$, we can obtain $P_2(k)$ from $P(k)$.

However, here I would like to introduce another method for obtaining $P_2(k)$ using $F(k)$ and $b_k = p_{k-1}!^p$.

Let $F_2(k)$ be the set of twin number pairs in $F(k)$, and let $p_{k+s} = \lfloor \sqrt{p_k!^p+1} \rfloor_p$. Then we obtain the set of smallest prime factors of the twin number pairs in $M_2'(k)$:

$$\{p_k,\ p_{k+1},\ \dots\ ,\ p_{k+s-1}, p_{k+s} \mid s \in N\}$$

Step 1: Construct the Column Vector

Construct the column vector:

$$\begin{bmatrix} 1 \times p_{k-1}!^p \\ 2 \times p_{k-1}!^p \\ \vdots \\ (p_k - 1) \times p_{k-1}!^p \end{bmatrix}$$

Take the modulus of each element in the column vector with respect to the set $\{p_k, p_{k+1}, \dots, p_{k+s-1}, p_{k+s}\}$, resulting in the remainder matrix R_b:

$$R_b = \begin{array}{c} \begin{array}{cccc} p_k & p_{k+1} & \cdots & p_{k+s} \end{array} \\ \begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,s+1} \\ r_{2,1} & r_{2,2} & \cdots & r_{2,s+1} \\ \vdots & \vdots & \ddots & \vdots \\ r_{p_k-1,1} & r_{p_k-1,2} & \cdots & r_{p_k-1,s+1} \end{bmatrix} \end{array}$$

Step 2: Construct the Row Vector $F_2(k)$

Take the modulus of each element in the row vector $F_2(k)$ with respect to the set $\{p_k, p_{k+1}, \dots, p_{k+s-1}, p_{k+s}\}$, resulting in the remainder matrix R_f:

$$\begin{bmatrix} (f_{1,1,1}, f_{1,1,2}) & (f_{1,2,1}, f_{1,2,2}) & \cdots & (f_{1,T_k,1}, f_{1,T_k,2}) \\ (f_{2,1,1}, f_{2,1,2}) & (f_{2,2,1}, f_{2,2,2}) & \cdots & (f_{2,T_k,1}, f_{2,T_k,2}) \\ \vdots & \vdots & \ddots & \vdots \\ (f_{s+1,1,1}, f_{s+1,1,2}) & (f_{s+1,2,1}, f_{s+1,2,2}) & \cdots & (f_{s+1,T_k,1}, f_{s+1,T_k,2}) \end{bmatrix} \begin{array}{c} p_k \\ p_{k+1} \\ \vdots \\ p_{k+s} \end{array}$$

Here, in $f_{s,i,j}$:

- s represents the index in the set $\{p_k, p_{k+1}, \dots, p_{k+s-1}, p_{k+s}\}$.
- i represents the i-th twin number pair in the row vector $F_2(k)$, $T_k = \prod_{t=2}^{k-1}(p_t - 2)$.
- j represents the index of the number in the i-th twin number pair.

Step 3: Combine R_b and R_f to Form $R_{bf}(n)$

$$\begin{bmatrix} r_{1,n} + (f_{n,1,1}, f_{n,1,2}) & r_{1,n} + (f_{n,2,1}, f_{n,2,2}) & \cdots & r_{1,n} + (f_{n,T_k,1}, f_{n,T_k,2}) \\ r_{2,n} + (f_{n,1,1}, f_{n,1,2}) & r_{2,n} + (f_{n,2,1}, f_{n,2,2}) & \cdots & r_{2,n} + (f_{n,T_k,1}, f_{n,T_k,2}) \\ \vdots & \vdots & \ddots & \vdots \\ r_{p_k-1,n} + (f_{n,1,1}, f_{n,1,2}) & r_{p_k-1,n} + (f_{n,2,1}, f_{n,2,2}) & \cdots & r_{p_k-1,n} + (f_{n,T_k,1}, f_{n,T_k,2}) \end{bmatrix}$$

where $n \in [1, s+1], n \in Z$.

Step 4: Sieve and Obtain $P_2(k)$

For $n = 1$ to $s + 1$, sieve out elements in $R_{bf}(n)$ that contain p_{k-n+1}. The remaining elements correspond to the positions of twin prime pairs in the matrix $M_2(k)$. Based on the computational rules of $M_2(k)$, we can then obtain $P_2(k)$.

Example: Obtaining $P_2(4)$

(1) Determine p_{4+s}:
- $p_{4+s} = \left\lfloor \sqrt{p_4!^p + 1} \right\rfloor_p = 13 = p_6$
- Thus, $s = 2$, and the set of smallest prime factors in $M_2^{'}(k)$ is $\{p_4, p_5, p_6\} = \{7, 11, 13\}$.

(2) Construct the Column Vector:

$$\begin{bmatrix} 1 \times p_3!^p \\ 2 \times p_3!^p \\ 3 \times p_3!^p \\ 4 \times p_3!^p \\ 5 \times p_3!^p \\ 6 \times p_3!^p \end{bmatrix} = \begin{bmatrix} 30 \\ 60 \\ 90 \\ 120 \\ 150 \\ 180 \end{bmatrix}$$

(3) Compute Remainder Matrix R_b:
- Take the modulus of each element in the column vector with respect to $\{7, 11, 13\}$:

$$R_b = \begin{bmatrix} 2 & 8 & 4 \\ 4 & 5 & 8 \\ 6 & 2 & 12 \\ 1 & 10 & 3 \\ 3 & 7 & 7 \\ 5 & 4 & 11 \end{bmatrix}$$

(4) Construct the Row Vector $F_2(4)$:

$$F_2(4) = [(11, 13), (17, 19), (29, 31)]$$

(5) Compute Remainder Matrix R_f:
- Take the modulus of each element in $F_2(4)$ with respect to $\{7, 11, 13\}$:

$$R_f = \begin{bmatrix} (4, 6) & (3, 5) & (1, 3) \\ (0, 2) & (6, 8) & (7, 9) \\ (11, 0) & (4, 6) & (3, 5) \end{bmatrix}$$

(6) Combine R_b and R_f to Form $R_{bf}(1)$:

- Add the first column of R_b and the first row of R_f:

$$R_{bf}(1) = \begin{bmatrix} 2+(4,6) & 2+(3,5) & 2+(1,3) \\ 4+(4,6) & 4+(3,5) & 4+(1,3) \\ 6+(4,6) & 6+(3,5) & 6+(1,3) \\ 1+(4,6) & 1+(3,5) & 1+(1,3) \\ 3+(4,6) & 3+(3,5) & 3+(1,3) \\ 5+(4,6) & 5+(3,5) & 5+(1,3) \end{bmatrix}$$
$$= \begin{bmatrix} (6,8) & (5,7) & (3,5) \\ (8,10) & (7,9) & (5,7) \\ (10,12) & (9,11) & (7,9) \\ (5,7) & (4,6) & (2,4) \\ (7,9) & (6,8) & (4,6) \\ (9,11) & (8,10) & (6,8) \end{bmatrix}$$

- Sieve out elements containing 7:

$$\begin{bmatrix} (6,8) & & (3,5) \\ (8,10) & & \\ (10,12) & (9,11) & \\ & (4,6) & (2,4) \\ & (6,8) & (4,6) \\ (9,11) & (8,10) & (6,8) \end{bmatrix}$$

(7) Combine R_b and R_f to Form $R_{bf}(2)$:

- Add the second column of R_b and the second row of R_f:

$$R_{bf}(2) = \begin{bmatrix} 8+(0,2) & & 8+(7,9) \\ 5+(0,2) & & \\ 2+(0,2) & 2+(6,8) & \\ & 10+(6,8) & 10+(7,9) \\ & 7+(6,8) & 7+(7,9) \\ 4+(0,2) & 4+(6,8) & 4+(7,9) \end{bmatrix}$$
$$= \begin{bmatrix} (8,10) & & (15,17) \\ (5,7) & & \\ (2,4) & (8,10) & \\ & (16,18) & (17,19) \\ & (13,15) & (14,16) \\ (4,6) & (10,12) & (11,13) \end{bmatrix}$$

- Sieve out elements containing 11:

$$\begin{bmatrix} (8,10) & & (15,17) \\ (5,7) & & \\ (2,4) & (8,10) & \\ & (16,18) & (17,19) \\ & (13,15) & (14,16) \\ (4,6) & (10,12) & \end{bmatrix}$$

(8) Combine R_b and R_f to Form $R_{bf}(3)$:
- Add the third column of R_b and the third row of R_f:

$$R_{bf}(3) = \begin{bmatrix} 4+(11,0) & & 4+(3,5) \\ 8+(11,0) & & \\ 12+(11,0) & 12+(4,6) & \\ & 3+(4,6) & 3+(3,5) \\ & 7+(4,6) & 7+(3,5) \\ 11+(11,0) & 11+(4,6) & \end{bmatrix}$$
$$= \begin{bmatrix} (15,4) & & (7,9) \\ (19,8) & & \\ (23,12) & (16,18) & \\ & (7,9) & (4,6) \\ & (11,13) & (10,12) \\ (22,11) & (15,17) & \end{bmatrix}$$

- Sieve out elements containing 13:

$$\begin{bmatrix} (15,4) & & (7,9) \\ (19,8) & & \\ (23,12) & (16,18) & \\ & (7,9) & (4,6) \\ & & (10,12) \\ (22,11) & (15,17) & \end{bmatrix}$$

(9) Obtain $P_2(4)$:

- Based on $F_2(4) = \begin{bmatrix}(11,13) & (17,19) & (29,31)\end{bmatrix}$, we compute:

$$P_2(4) = \begin{bmatrix} 1\times 30 + (11,13) & & 1\times 30 + (29,31) \\ 2\times 30 + (11,13) & & \\ 3\times 30 + (11,13) & 3\times 30 + (17,19) & \\ & 4\times 30 + (17,19) & 4\times 30 + (29,31) \\ & & 5\times 30 + (29,31) \\ 6\times 30 + (11,13) & 6\times 30 + (17,19) & \end{bmatrix}$$

$$= \begin{bmatrix} (41,43) & & (59,61) \\ (71,73) & & \\ (101,103) & (107,109) & \\ & (137,139) & (149,151) \\ & & (179,181) \\ (191,193) & (197,199) & \end{bmatrix}$$

References

This article is entirely original and has not referenced any literature or materials!