Resolution of the Abraham–Minkowski Controversy

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The Abraham–Minkowski controversy has arisen because the Minkowski stress-energy tensor for the electromagnetic field in polarized and magnetized matter is not symmetric. This note derives a closely related variant that is both symmetric and traceless.

I. INTRODUCTION

A. Conventions Adopted

The Minkowski metric is $(-,+,+,+)$.

$\nabla\cdot\overleftrightarrow{\sigma}\equiv\nabla_i\sigma_{ij}$ so that a stress-energy (or stress-energy-momentum) tensor takes the form

$$\begin{pmatrix} u & c\mathbf{g} \\ \mathbf{S}/c & -\overleftrightarrow{\sigma} \end{pmatrix} \tag{1}$$

B. Background

The Abraham–Minkowski controversy is a century-long dispute as to the true form of the electromagnetic stress-energy tensor in and, hence, the electromagnetic force on matter that is polarized and magnetized.

It has its origin in 1908, when Minkowski derived a stress-energy tensor for the electromagnetic field in polarized and magnetized matter [1, eqn 75],

$$\Theta^{\mu\nu}+\eta^{\mu\beta}P_{\alpha\beta}F^{\alpha\nu}-\tfrac{1}{4}\eta^{\mu\nu}P_{\alpha\beta}F^{\alpha\beta} \tag{2}$$

where $\Theta^{\mu\nu}$ is the (symmetric) electromagnetic stress-energy tensor, $\eta^{\mu\nu}$ is the (Minkowski) metric tensor, $P^{\mu\nu}$ is the polarization-magnetization tensor [2, eqn 21], and $F^{\mu\nu}$ is the electromagnetic field tensor.

The controversy started the next year, when Abraham [3] derived relations with '... *eine merkwürdige Symmetrieeigenschaft jenes Gleichungssystemes, die sich in Minkowski's Ansätzen nicht findet.*' The underlying reason for this is the source of the controversy: Minkowski's stress-energy tensor is not symmetric.

Despite having been 'resolved' numerous times [4, 5], the controversy persists to this day. This resolution will not be the last.

II. DERIVATION

A. Electromagnetic Force Density

The Lorentz force density on *free* charges and currents is

$$(\nabla\cdot\mathbf{D})\mathbf{E}+\left(\nabla\times\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t}\right)\times\mathbf{B} \tag{3}$$

which is the same as

$$-\partial_\mu\Theta^{\mu j}-(\partial^\beta P_{\alpha\beta})F^{\alpha j} \tag{4}$$

where the Latin index j denotes spatial components.

The Lorentz force density on *bound* charges and currents is [6, eqn 64]

$$(\mathbf{P}\cdot\nabla)\mathbf{E}+\frac{\partial\mathbf{P}}{\partial t}\times\mathbf{B}+\mathbf{M}\times(\nabla\times\mathbf{B})+(\mathbf{M}\cdot\nabla)\mathbf{B} \tag{5}$$

which is *almost* the same as

$$-P_{\alpha\beta}(\partial^\beta F^{\alpha j}) \tag{6}$$

the difference being that the latter expression includes an additional term

$$-\frac{\partial}{\partial t}(\mathbf{P}\times\mathbf{B}) \tag{7}$$

Omitting this term would break the Lorentz covariance of Expression 6, which constitutes compelling evidence that it is real. However, it does not contribute to the Lorentz force density, so how to account for it?

B. Intrinsic Momentum of Electric Dipoles

That $\mathbf{P}\times\mathbf{B}$ has the form of a momentum density suggests treating it as if it were a component of the momentum density of matter.

According to Newton's second law of motion, the force density on matter is

$$\partial_\mu\Upsilon^{\mu j} \tag{8}$$

where $\Upsilon^{\mu\nu}$ is the stress-energy tensor for matter. This is the same as

$$\frac{\partial\mathbf{g}_\Upsilon}{\partial t}-\nabla\cdot\overleftrightarrow{\sigma}_\Upsilon \tag{9}$$

where $\mathbf{g}_\Upsilon$ is the momentum density of the matter and $\overleftrightarrow{\sigma}_\Upsilon$ its stress tensor. If Expression 7 is a real force density, then the electromagnetic force density on matter is equal to the sum of Expressions 4 and 6,

$$\begin{aligned}\partial_\mu\Upsilon^{\mu j}&=-\partial_\mu\Theta^{\mu j}-(\partial^\beta P_{\alpha\beta})F^{\alpha j}-P_{\alpha\beta}(\partial^\beta F^{\alpha j})\\&=-\partial_\mu\Pi^{\mu j}\end{aligned} \tag{10}$$

where

$$\Pi^{\mu\nu}=\Theta^{\mu\nu}+\eta^{\mu\beta}P_{\alpha\beta}F^{\alpha\nu} \tag{11}$$

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is Medina and Stephany's stress-energy tensor for the electromagnetic field [7, eqn 14]. The similarity to Minkowski's stress-energy tensor (Expression 2) is evident.

Consequently, the Lorentz force density (the sum of Expressions 3 and 5) is equal to

$$\frac{\partial}{\partial t}(\mathbf{g}_\Upsilon + \mathbf{P} \times \mathbf{B}) - \nabla \cdot \overleftrightarrow{\sigma}_\Upsilon \tag{12}$$

and therefore, the effective momentum of a body of finite extent that is subject to the Lorentz force is

$$\int_{\mathcal{V}} (\mathbf{g}_\Upsilon + \mathbf{P} \times \mathbf{B})\, d\tau \tag{13}$$

which implies that an electric dipole $\mathbf{p}$ in a magnetic field has linear momentum $\mathbf{p} \times \mathbf{B}$ independent of its motion.

Magnetic dipoles have a corresponding attribute in the form of '"hidden momentum" $G_l = -\epsilon_0 E \times \mu_0 m$ associated with energy flow in a current loop of magnetic dipole strength m situated in an electric field E.' [8] However, unlike a magnetic dipole, an electric dipole has no moving parts in which momentum can hide: it consists of separated electric charges, not a current loop of magnetic charges.

So, where does its momentum reside? Griffiths [9, eqn 3.16] locates it in the sources of the magnetic field. Minkowski [1], however, hides it in plain sight by including it in the momentum density of the electromagnetic field. The following resolution favours Minkowski.

C. Resolution

$\Pi^{\mu\nu}$, like Minkowski's stress-energy tensor, is problematic in that it:

1. is not symmetric
2. does not account for the hidden momentum of magnetic dipoles.

The remedy is to introduce a new stress-energy tensor

$$\Delta^{\mu\nu} = \eta^{\mu\beta} M_{\alpha\beta} G^{\alpha\nu} \tag{14}$$

where $M^{\mu\nu}$ and $G^{\mu\nu}$ are the duals of $P^{\mu\nu}$ and $F^{\mu\nu}$ respectfully. Defining $T^{\mu\nu}$ such that

$$\Upsilon^{\mu\nu} = T^{\mu\nu} + \eta^{\mu\beta} M_{\alpha\beta} G^{\alpha\nu} \tag{15}$$

which mirrors Equation 11, implies that Equation 10 is equivalent to

$$\partial_\mu T^{\mu j} = -\partial_\mu V^{\mu j} \tag{16}$$

where

$$V^{\mu\nu} = \Theta^{\mu\nu} + \eta^{\mu\beta}(P_{\alpha\beta} F^{\alpha\nu} + M_{\alpha\beta} G^{\alpha\nu}) \tag{17}$$

is symmetric and traceless. The momentum density component of $\Delta^{\mu\nu}$ is

$$c\mathbf{g}_\Delta = \frac{1}{c}\mathbf{M} \times \mathbf{E} \tag{18}$$

so that the effective momentum of a body of finite extent that is subject to the Lorentz force becomes

$$\begin{aligned} &\int_{\mathcal{V}} (\mathbf{g}_T + \mathbf{g}_\Delta + \mathbf{P} \times \mathbf{B})\, d\tau \\ &= \int_{\mathcal{V}} \left(\mathbf{g}_T + \frac{1}{c^2}\mathbf{M} \times \mathbf{E} + \mathbf{P} \times \mathbf{B}\right) d\tau \end{aligned} \tag{19}$$

where $c\mathbf{g}_T$ is the momentum density component of $T^{\mu\nu}$.

This invites the following interpretations.

1. $\Pi^{\mu\nu}$ is the stress-energy tensor for the field, which follows from $\Upsilon^{\mu\nu}$ being the stress-energy tensor for matter.
2. $T^{\mu\nu}$ is the stress-energy tensor for matter prior to its being polarized or magnetized, so $\Delta^{\mu\nu}$ is the change in the stress-energy tensor for matter arising from its polarization and/or magnetization.
3. Equation 16 is the (kinetic) electromagnetic force density.

Returning to the issue of the location of the momentum of electric dipoles, the first interpretation implies that it is accounted for by the (Minkowski) momentum density of the field, $\mathbf{D} \times \mathbf{B}$, which Barnett and Loudon [5, 10] demonstrate to be the canonical momentum density.

III. CONCLUSION

Barnett [5] concludes that the '… resolution of the Abraham–Minkowski dilemma requires us to recognize that there are two distinct electromagnetic momenta, the kinetic momentum and the canonical momentum.'

In this he is correct. There are two distinct electromagnetic stress-energy tensors:

1. $V^{\mu\nu}$ is the kinetic stress-energy tensor, which is symmetric and traceless
2. $\Pi^{\mu\nu}$ is the stress-energy tensor for the electromagnetic field, the momentum density component of which corresponds to the canonical momentum.

In the absence of polarization and magnetization both are equal to the familiar electromagnetic stress-energy tensor, $\Theta^{\mu\nu}$.

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