

# The method of dividing the $60^\circ$ angle into three equal parts

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The problem of dividing a  $60^\circ$  angle into three equal parts in modern mathematics has not yet been solved. This involves the infinite extension of this trigonometric function in a generalized perspective. After research, it was found that the solution is located between  $0r$  and one-thirds of  $r$ . The former represents a curve, while the latter represents a horizontal line. This article aims to utilize the relationships between various shapes to divide the  $60^\circ$  angle into three equal parts, which can then be extended to any angle less than  $180^\circ$  between  $r$  and two-thirds of  $r$ .

**Keywords:**  $60^\circ$  angle trisection; Bold attempt; Between  $0r$  and one-thirds of  $r$ ; trigonometric function

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## Introduction

In the long history of geometry, the problem of dividing angles into three equal parts has always been a fascinating topic. Since ancient Greece, mathematicians have been exploring how to accurately divide an arbitrary angle into three equal parts using only a ruler and compass. Although the problem of trisecting angles has been proven to be unsolvable in general, this challenge has inspired countless mathematicians and propelled the development of mathematical theory. This article will use the strings of a circle and isosceles trapezoids to solve this problem, hoping that readers can patiently watch.

## Literature review

Mathematician Wantzel once proved that it is impossible to directly make a trisecting angle. The reason is that  $\cos 20^\circ$  is a root of the equation  $(8x^3 - 6x - 1 = 0)$  (derived from the triple angle formula). This equation is a cubic equation, and its solution cannot be expressed by finite square root operations. Therefore, it is not possible to determine the length of the line segment at  $\cos 20^\circ$ , but he ignored the chord of the circle, which also represents the circle (Pierre Wantzel, 1837).

Lindemann proved the transcendence of  $\pi$ , indirectly supporting the unsolvable trisecting angle. His work shows that many problems related to ruler and gauge drawing, such as squaring circles, involve transcendental numbers and cannot be solved through ruler and gauge drawing. (Lindemann, 1882).

$\cos 20^\circ$  is a root of the equation  $(8x^3 - 6x - 1 = 0)$  (derived from the triple angle formula). This equation is a cubic equation, and its solution cannot be expressed by finite square root operations. But my answer is not contradictory to the views of two famous scholars, because my starting point is the properties of circular chords and isosceles trapezoids.

## Result

### 60° divide into three equal parts

Due to the slow processing of compass shapes on computers, paper images are used. This article will use the simplest method to draw to ensure that it is not too complicated.

(1) As shown in Figure 1, draw a circle with radius  $r$  and dot  $O$ , and then draw a  $60^\circ$  equilateral triangle inside the circle, which is  $\triangle ABC$ . Divide the  $BC$  edge into two parts, with the dividing point being  $N$ , which is connected to vertex  $A$  and extended to  $M$  to form the dividing line  $NAM$ ;

(2) Divide  $\angle B$  and  $\angle C$  into four parts, each at a  $45^\circ$  angle, with three parts forming one angle at  $45^\circ$  and the other angle also at  $45^\circ$ . The three-quarters of the two angles intersect at point  $A'$ , connecting three points to form an isosceles right triangle ( $180 - 45 - 45 = 90$ ). Connect  $A'A$  and  $O'B$  to obtain angle  $AA'B$ , then divide  $90^\circ$  into three equal parts to obtain an angle of  $30^\circ$ , and obtain points  $E$  and  $F$  of the three equal parts;

(3) Draw a circle with A'B and A'C as the centers, and the intersection point of the arc and the bisector NAM is A". Connect A" C and A"B to obtain the angle BA" C. Because the central angle is twice the circumference angle of the circle,  $\angle BA''C$  is  $45^\circ$ . Then divide the  $45^\circ$  angle into three equal parts to obtain a  $15^\circ$  angle, and obtain the points E' and F' of the three equal parts. The same applies to the  $180^\circ$  divide, where the three are in a straight line;

(4) The center of the  $60^\circ$  angle is between the center of the  $45^\circ$  angle and the center of the  $90^\circ$  angle; connect point E, point F, point E', and point F' to form an isosceles trapezoid, which expands from the trisecting segment EF to the remaining trisecting segment E'F'. As the angle decreases, the chord length of its three equal angles also decreases. The chord lengths of the three equal angles are the EF lines of the isosceles trapezoid. The straight line of the middle string continuously shrinks as the angle decreases. You can imagine having a semicircle with a constant chord length. The angle of the semicircle decreases continuously and becomes a straight line, while the chord length of the three equal angle changes from r to two-thirds r.

(5) Draw a circle with A as the center and AB as the radius. The intersection points with the isosceles trapezoid are E" and F", which are also the  $60^\circ$  bisector points. Obtain the line segment E"F" that divides into three equal parts. It is an isosceles trapezoid that divides  $45^\circ$  and  $90^\circ$  equally, and this trapezoid naturally has equal division. You can draw a semicircle and divide it into three equal parts, with the  $45^\circ$ ,  $90^\circ$ , and  $180^\circ$  points on a straight line.

(6) After obtaining an angle of  $20^\circ$ , we can obtain 18 equal circular parts, and then obtain 9, 36, and 72 equal circular parts;

(7) The same applies to Y equal parts. After dividing into 36 equal parts, each angle is 10 degrees. Find the middle angle or half of the angle between  $140^\circ$  degrees and  $70^\circ$  degrees, and then construct an isosceles trapezoid. Any angle can be divided into 7 equal parts; Similarly,  $50^\circ$  and  $100^\circ$  can also be found and divided into 5 equal parts. For example,  $50^\circ$ - $100^\circ$ , when divided into five equal parts, each angle is  $20^\circ$ . Observe the chord lengths of the third  $10^\circ$  and  $20^\circ$  and construct an isosceles trapezoid. After 72 equal divisions, with each angle being 5 degrees, find  $85^\circ$  and  $170^\circ$ ;

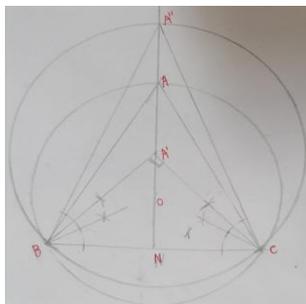


Figure 1

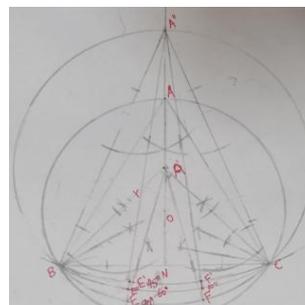


Figure 2

### Any angle divided into any equal parts

To summarize, divide the points into three equal parts at  $90^\circ$  and  $45^\circ$ , and connect the four points to construct a trapezoid. From Figure 3, it can be seen that there are two sides with an extension angle of  $\angle X$  less than  $180^\circ$ , resulting in an isosceles triangle  $A'OB'$ . Make a vertical bisector so that  $AB$  equals  $A'B'$ . Then draw a circle with  $OA'$  as the radius, where the arc intersects with the trapezoid. The connecting intersection point is the three equal parts of any angle. Then we need to start dividing it into any portion at any angle.

(1) If there are  $Y$  equal divisions (odd division and even division), you can first draw  $Y$  small angles similar to  $X$  degrees. Combine  $Y$  small angles into a large angle (large angle less than  $180^\circ$ ). You can combine  $Y$  equal angles into a large angle, find the chord length of that angle, and draw a circle with a radius of 0.5 chord length. Then divide the chord length equally by  $Y$  line segments, connect four points, and construct an isosceles trapezoid. (Must be converted to odd numbers, even numbers cannot be in the middle). You can imagine having a  $180^\circ$  semicircle with a fixed chord length ( $AB$ ). As the angle decreases, the equal division chord length becomes shorter and shorter. Finally, it becomes one- $Y$ th of the chord length at a large angle. For example, figure 4. If it is divided into 10 equal parts, then divide by 2 and place the double angle in the middle. The two points of waist length must pass through an isosceles trapezoid of  $0^\circ - 180^\circ$ . By connecting two points, a trapezoid shape of  $0^\circ - 180^\circ$  can also be drawn.

(2) If the degree of  $X$  is greater than  $180^\circ$ , using  $360^\circ - x^\circ$  for processing can achieve  $Y$  equal parts.

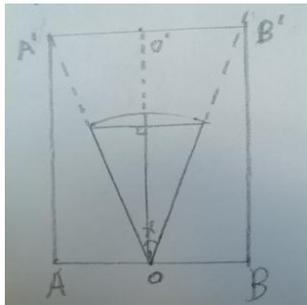


Figure3

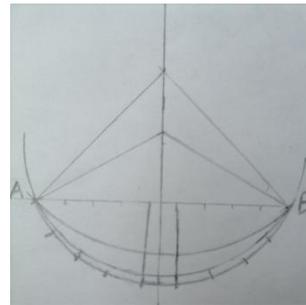


Figure4

### The chord length of a three equal circle

The vertex angles of the three isosceles triangles are  $45^\circ$ ,  $60^\circ$ , and  $90^\circ$ , respectively. The base is the same, both are  $\sqrt{3}$ . The top corner corresponds to the bottom edge. The ratio of the waist lengths of three isosceles triangles (expressed as trigonometric functions) is  $(\sqrt{3}\sin 67.5^\circ / \sin 45^\circ) : \sqrt{3} : (\sqrt{3}\sin 45^\circ / \sin 90^\circ)$ . Then draw a circle with the waist circumference of the three triangles as the radius, and calculate the chord lengths corresponding to  $15^\circ$ ,  $20^\circ$ , and  $30^\circ$ :  $[(\sqrt{3}\sin 67.5^\circ / \sin 45^\circ) \times (\sin 82.5^\circ / \sin 15^\circ)] : [\sqrt{3} \times (\sin 80^\circ / \sin 20^\circ)] : [(\sqrt{3}\sin 45^\circ / \sin 90^\circ) \times (\sin 75^\circ / \sin 30^\circ)]$ .

## References

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