SURFACE AREA OF THE MÖBIUS STRIP

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ABSTRACT. The (half) area of the surface of the Möbius strip is the expected product of the length of the circular spine times the width of the sweep line times a positive correction factor. The manuscript writes down this factor as a Taylor series of the ratio of width over circle radius; it approaches one if that ratio approaches zero. [vixra:2503.0103]

1. Incentive

The Guldin rule (Pappus' theorem) provide a formula for the surface generated by revolving a planar curve with known center of mass around a circle [1, (8.72)]. The naïve expectation is that the Möbius strip has an area equal to the product of length of a circular center line by the width. This manuscript corrects this hypothesis and evaluates a correction factor for this product.

2. Mathematical Model, Coordinates

We look at a Möbius strip of guide line radius R located in the $x-y-$plane with a paddle of width w staying with its middle at the guide line. A point on the guide line has the Cartesian coordinates

$$\begin{pmatrix} R\cos\lambda \\ R\sin\lambda \\ 0 \end{pmatrix} \tag{1}$$

parameterized by an azimuthal angle $0 \le \lambda \le 2\pi$. The tangent line to the circle points into the orthogonal direction

$$\begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix}. \tag{2}$$

A point on the strip at a distance t to the guide line has a torsion angle θ relative to the $x-y$-plane, such that its z-coordinate is $t\sin\theta$ in the range $-w/2 \le t \le w/2$. This leaves the factor $t\cos\theta$ for the x and y coordinates. Since the paddle is obtained by rotation around the tangent (2), its direction must be orthogonal to that, so dispersion of the $t\cos\theta$ factor gives a paddle vector of

$$\begin{pmatrix} t\cos\theta\cos\lambda \\ t\cos\theta\sin\lambda \\ t\sin\theta \end{pmatrix}. \tag{3}$$

Date: May 19, 2025.
2020 *Mathematics Subject Classification.* Primary 28A75; Secondary 51M04.
Key words and phrases. Mobius strip, area, integration.

Attaching it to the circle (1) gives the Cartesian coordinates of a point on the strip parameterized by λ and t:

$$\vec{r}(\lambda,t) = \begin{pmatrix} R\cos\lambda \\ R\sin\lambda \\ 0 \end{pmatrix} + \begin{pmatrix} t\cos\theta\cos\lambda \\ t\cos\theta\sin\lambda \\ t\sin\theta \end{pmatrix} = \begin{pmatrix} (R+t\cos\theta)\cos\lambda \\ (R+t\cos\theta)\sin\lambda \\ t\sin\theta \end{pmatrix}. \tag{4}$$

The principle of the definition now lets the torsion angle θ increase linearly with λ such that a point of constant t initially at

$$\vec{r}(0,w/2) = \begin{pmatrix} R+w/2 \\ 0 \\ 0 \end{pmatrix} \tag{5}$$

ends up at

$$\vec{r}(2\pi,w/2) = \begin{pmatrix} R-w/2 \\ 0 \\ 0 \end{pmatrix} \tag{6}$$

after one λ-rotation through the circle. This is achieved by setting

$$\theta = \lambda/2. \tag{7}$$

Continuous surfaces with larger numbers of twists as in Figure 1 can be constructed by selecting other positive integers k:

$$\theta = k\lambda/2. \tag{8}$$

Insertion into (4) defines a family of Möbius strips [3, 6]:

$$\vec{r} = \begin{pmatrix} (R+t\cos\frac{k\lambda}{2})\cos\lambda \\ (R+t\cos\frac{k\lambda}{2})\sin\lambda \\ t\sin\frac{k\lambda}{2} \end{pmatrix}. \tag{9}$$

3. Gaussian Parameters

Two tangential directions on the surface are constructed as the partial derivatives:

$$\frac{\partial\vec{r}}{\partial t} \equiv \vec{r}_t = \begin{pmatrix} \cos\frac{k\lambda}{2}\cos\lambda \\ \cos\frac{k\lambda}{2}\sin\lambda \\ \sin\frac{k\lambda}{2} \end{pmatrix}; \quad E = |\vec{r}_t| = 1; \tag{10}$$

$$\frac{\partial\vec{r}}{\partial\lambda} \equiv \vec{r}_\lambda = \begin{pmatrix} -\frac{tk}{2}\sin\frac{k\lambda}{2}\cos\lambda - R\sin\lambda - t\sin\lambda\cos\frac{k\lambda}{2} \\ -\frac{tk}{2}\sin\frac{k\lambda}{2}\sin\lambda + R\cos\lambda + t\cos\lambda\cos\frac{k\lambda}{2} \\ \frac{tk}{2}\cos\frac{k\lambda}{2} \end{pmatrix}. \tag{11}$$

These are orthogonal:

$$F = \vec{r}_\lambda\cdot\vec{r}_t = 0. \tag{12}$$

The cross product (direction of the surface normal, not of unit length) is

$$\vec{r}_t\times\vec{r}_\lambda = \begin{pmatrix} \frac{tk}{2}\sin\lambda - R\sin\frac{k\lambda}{2}\cos\lambda - t\cos\lambda\sin\frac{k\lambda}{2}\cos\frac{k\lambda}{2} \\ -\frac{tk}{2}\cos\lambda - R\sin\frac{k\lambda}{2}\sin\lambda - t\sin\lambda\sin\frac{k\lambda}{2}\cos\frac{k\lambda}{2} \\ (R+t\cos\frac{k\lambda}{2})\cos\frac{k\lambda}{2} \end{pmatrix}. \tag{13}$$

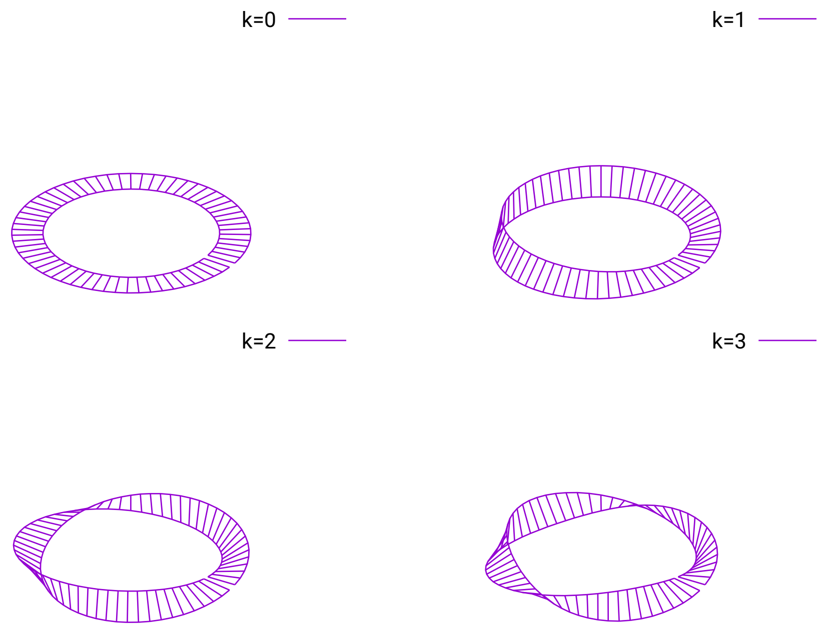

FIGURE 1. Möbius ribbons for twist numbers 0 to 3.

The length of the cross product is

$$|\vec{r}_t \times \vec{r}_\lambda| = |\vec{r}_\lambda| = \sqrt{G} = \sqrt{(R + t\cos\frac{k\lambda}{2})^2 + (\frac{tk}{2})^2}. \tag{14}$$

4. Edge length

The derivatives of the position as a function of the λ parameter in (11) define the line segment

$$\sqrt{(\partial r_x/d\lambda)^2 + (\partial r_y/d\lambda)^2 + (\partial r_z/d\lambda)^2} \\ = \sqrt{(R + t\cos\frac{k\lambda}{2})^2 + (\frac{tk}{2})^2} = R\sqrt{(1 + \frac{t}{R}\cos\frac{k\lambda}{2})^2 + (\frac{tk}{2R})^2} \tag{15}$$

for curves that run at constant distance t to the circular backbone.

The length S_k of such a line along the ribbon (up to the usual debatable factor of 2 if k is odd) is

$$S_k(t) = \int_{\lambda=0}^{2\pi} R\sqrt{(1 + \frac{t}{R}\cos\frac{k\lambda}{2})^2 + (\frac{tk}{2R})^2}\, d\lambda. \tag{16}$$

The λ-integral leads to Elliptic integrals which we shall avoid here (App. B).

The Taylor expansion of the kernel in powers of small t/R is

$$\sqrt{(1+\frac{t}{R}\cos\frac{k\lambda}{2})^2+(\frac{tk}{2R})^2} = 1+\cos(\frac{k\lambda}{2})\frac{t}{R}+\frac{k^2}{8}(\frac{t}{R})^2-\frac{k^2}{8}\cos(\frac{k\lambda}{2})(\frac{t}{R})^3 + \frac{k^2}{128}[4\cos(\frac{k\lambda}{2})-k][4\cos(\frac{k\lambda}{2})+k](\frac{t}{R})^4-\frac{k^2}{128}\cos(\frac{k\lambda}{2})[16\cos^2(\frac{k\lambda}{2})-3k^2](\frac{t}{R})^5 + \frac{k^2}{1024}[128\cos^4\frac{k\lambda}{2}-48k^2\cos^2\frac{k\lambda}{2}+k^4](\frac{t}{R})^6 - \frac{k^2}{1024}\cos\frac{k\lambda}{2}[128\cos^4\frac{k\lambda}{2}-80k^2\cos^2\frac{k\lambda}{2}+5k^4](\frac{t}{R})^7+\cdots \tag{17}$$

Term-by-term integration of the power series over $\lambda = 0\cdots 2\pi$ yields

$$S_0 = 2\pi R\big[1+\frac{t}{R}\big]; \tag{18}$$

$$S_1 = 2\pi R\big[1+\frac{1}{8}(\frac{t}{R})^2+\frac{7}{128}(\frac{t}{R})^4+\frac{25}{1024}(\frac{t}{R})^6+\frac{75}{32768}(\frac{t}{R})^8-\frac{2793}{262144}(\frac{t}{R})^{10}+\cdots\big]; \tag{19}$$

$$S_2 = 2\pi R\big[1+\frac{1}{2}(\frac{t}{R})^2+\frac{1}{8}(\frac{t}{R})^4-\frac{1}{8}(\frac{t}{R})^6-\frac{15}{128}(\frac{t}{R})^8+\frac{21}{128}(\frac{t}{R})^{10}+\cdots\big]; \tag{20}$$

$$S_3 = 2\pi R\big[1+\frac{9}{8}(\frac{t}{R})^2-\frac{9}{128}(\frac{t}{R})^4-\frac{783}{1024}(\frac{t}{R})^6+\frac{37035}{32768}(\frac{t}{R})^8+\frac{267183}{262144}(\frac{t}{R})^{10}+\cdots\big]; \tag{21}$$

$$S_4 = 2\pi R\big[1+2(\frac{t}{R})^2-(\frac{t}{R})^4-\frac{5}{4}(\frac{t}{R})^6+\frac{75}{8}(\frac{t}{R})^8-\frac{1533}{64}(\frac{t}{R})^{10}+\cdots\big]; \tag{22}$$

$$S_5 = 2\pi R\big[1+\frac{25}{8}(\frac{t}{R})^2-\frac{425}{128}(\frac{t}{R})^4+\frac{1825}{1024}(\frac{t}{R})^6+\frac{928875}{32768}(\frac{t}{R})^8-\frac{56366625}{262144}(\frac{t}{R})^{10}+\cdots\big]. \tag{23}$$

The fact that these $S_k(t)$ are larger than $2\pi R$ for $k > 0$ is no surprise, because these are basically lengths measured along the cutting edges of screws for screws that do not have straight but circular axes of length $2\pi R$.

The length of the rim of the stripe is obtained by inserting $t = \pm w/2$. (The sign obviously matters only for the planar case S_0.)

5. Area

The area is [7, (8.19)][1, (3.498b)]

$$A_k = \iint \sqrt{EG - F^2} d\lambda dt = \iint |\vec{r}_t \times \vec{r}_\lambda| d\lambda dt \tag{24}$$
$$= \int_0^{2\pi} d\lambda \int_{-w/2}^{w/2} dt \sqrt{(R + t\cos\frac{k\lambda}{2})^2 + (\frac{tk}{2})^2}$$
$$= R\int_0^{2\pi} d\lambda \int_{-w/2}^{w/2} dt \sqrt{(1 + \frac{t}{R}\cos\frac{k\lambda}{2})^2 + (\frac{tk}{2R})^2}$$
$$= \int_{-w/2}^{w/2} dt S_k(t)$$
$$= \frac{wR}{2}\int_0^{2\pi} d\lambda \int_{-1}^{1} dx \sqrt{(1 + \frac{xw}{2R}\cos\frac{k\lambda}{2})^2 + (\frac{xwk}{4R})^2}.$$

Remark 1. *Optionally one could multiply this by 2 to cover the 'back-side' area, i.e., to sweep this in the range $0 \le \lambda \le 4\pi$.*

Remark 2. *The t-integral may be executed* [4, 2.262.1,2.262.2]

$$\int_{-w/2}^{w/2} dt \sqrt{R^2 + 2Rt\cos\frac{k\lambda}{2} + t^2\cos^2\frac{k\lambda}{2} + \frac{t^2k^2}{4}} \tag{25}$$
$$= \frac{(\cos^2\frac{k\lambda}{2} + k^2/4)t + R\cos\frac{k\lambda}{2}}{2(\cos^2\frac{k\lambda}{2} + k^2/4)} \sqrt{(R + t\cos\frac{k\lambda}{2})^2 + \frac{t^2k^2}{4}}$$
$$+ \frac{R^2k^2}{8(\cos^2\frac{k\lambda}{2} + 2k^2)^{3/2}} \operatorname{arsinh}\frac{(\cos^2\frac{k\lambda}{2} + k^2/4)t + R\cos\frac{k\lambda}{2}}{kR/2} \Big|_{t=-w/2}^{w/2}$$

but since this still leaves a pending λ-integration, this analysis is not continued from there.

The case $k = 0$ is the trivial planar hollow circle, difference of areas of circles with radii $R \pm w/2$, with $A_0 = \pi[(R + w/2)^2 - (R - w/2)^2] = 2\pi wR$.

The further strategy is to utilize the power series expansion of $S_k(t)$ assuming w is small, where the integration over the powers of t is elementary.

Definition 1.

$$\hat{w} = w/R \tag{26}$$

is the unitless ratio of the strip width by the radius of the backbone circle.

The terms with odd powers of t disappear while integrating because the $t-$limits are symmetric. A_k is $2\pi Rw$ multiplied by an even function of $\hat{w}$.

Insertion of the S_k-series into (24) and term-by-term integration of (19) over $-w/2 \le t \le w/w$ yields

$$A_1 = 2\pi wR[1 + \frac{1}{96}\hat{w}^2 + \frac{7}{10240}\hat{w}^4 + \frac{25}{458752}\hat{w}^6 + \frac{25}{25165824}\hat{w}^8 - \frac{2793}{2952790016}\hat{w}^{10} - \frac{53277}{223338299392}\hat{w}^{12} + \cdots] \tag{27}$$

There is an apparent discrepancy between this formula and the usual manual construction of a Möbius model which attaches two ends of a rectangular stripe of dimension $2\pi R \times w$ after bending/twisting. In fact the paper model does not keep the center line of the rectangular stripe on a planar circle; its 2-dimensional surface is even more complex than the mathematical model (4) [8, 5, 9].

No new aspect arises in the analysis if twist numbers $k \ge 2$ are computed—besides the fact that for even k the computed area is indeed the area of only one of two sides.

$$A_2 = 2\pi w R\left[1+\frac{1}{24}\hat{w}^2+\frac{1}{640}\hat{w}^4-\frac{1}{3584}\hat{w}^6-\frac{5}{98304}\hat{w}^8+\frac{21}{1441792}\hat{w}^{10}+\frac{105}{27262976}\hat{w}^{12}+\cdots\right]; \tag{28}$$

$$A_3 = 2\pi w R[1+\frac{3}{32}\hat{w}^2-\frac{9}{10240}\hat{w}^4-\frac{783}{458752}\hat{w}^6+\frac{4115}{8388608}\hat{w}^8 +\frac{267183}{2952790016}\hat{w}^{10}-\frac{28573965}{223338299392}\hat{w}^{12}+\cdots]; \tag{29}$$

$$A_4 = 2\pi w R\left[1+\frac{1}{6}\hat{w}^2-\frac{1}{80}\hat{w}^4-\frac{5}{1792}\hat{w}^6+\frac{25}{6144}\hat{w}^8-\frac{1533}{720896}\hat{w}^{10}-\frac{399}{6815744}\hat{w}^{12}+\cdots\right]; \tag{30}$$

$$A_5 = 2\pi w R[1+\frac{25}{96}\hat{w}^2-\frac{85}{2048}\hat{w}^4+\frac{1825}{458752}\hat{w}^6+\frac{309625}{25165824}\hat{w}^8 -\frac{56366625}{2952790016}\hat{w}^{10}+\frac{3746147475}{223338299392}\hat{w}^{12}+\cdots]. \tag{31}$$

6. Summary

The (quasi one-sided) surface area of the Möbius strip of width w with a planar guide line of radius R is given by (27), where (26) denotes the unitless ratio of the two main parameters.

Appendix A. Embedding

The parameters of the second quadratic fundamental normal form are listed here [1, (3.503c)][7, (8.26)]. The normal vector of the plane is

$$\vec{n} = \frac{1}{\sqrt{G}}\vec{r}_t \times \vec{r}_\lambda. \tag{32}$$

The products of partial derivatives are

$$L = -\vec{n}_\lambda \cdot \vec{r}_\lambda = \frac{1}{\sqrt{G}}\sin\frac{k\lambda}{2}\left[(R+t\cos\frac{k\lambda}{2})^2+\frac{t^2k^2}{2}\right]; \tag{33}$$

(the term in the square brackets is not the same as the discriminant of the root in (14))

$$N = -\vec{n}_t \cdot \vec{r}_t = 0; \tag{34}$$

$$M = -(\vec{n}_\lambda \cdot \vec{r}_t + \vec{n}_t \cdot \vec{r}_\lambda)/2 = \frac{kR}{2\sqrt{G}}. \tag{35}$$

Appendix B. Elliptic Integrals

The integrals along the spine of the strip have the shape

$$\int_0^{2\pi} d\lambda \sqrt{(1+\frac{xw}{2R}\cos\frac{k\lambda}{2})^2+(\frac{xwk}{4R})^2} = 2\int_0^{\pi} d\phi \sqrt{(1+\frac{xw}{2R}\cos\phi)^2+(\frac{xwk}{4R})^2}. \tag{36}$$

The substitution $\cos\phi = \xi$ rephrases this as an elliptic integral

$$\begin{aligned} \ldots &= 2\int_{-1}^{1} d\xi \sqrt{(1+\frac{xw}{2R}\xi)^2+(\frac{xwk}{4R})^2}\frac{1}{\sqrt{1-\xi^2}} \\ &= \frac{2xw}{2R}\int_{-1}^{1} d\xi \sqrt{(\frac{2R}{xw}+\xi)^2+(\frac{k}{2})^2}\frac{1}{\sqrt{1-\xi^2}} \\ &= \frac{xw}{R}\int_{-1}^{1} d\xi \sqrt{(i\frac{k}{2}+\frac{2R}{xw}+\xi)(-i\frac{k}{2}+\frac{2R}{xw}+\xi)}\frac{1}{\sqrt{(1-\xi)(1+\xi)}} \\ &= x\hat{w}\sum_{m=0}^{2}\beta_m \int_{-1}^{1} d\xi \frac{\xi^m}{\sqrt{(i\frac{k}{2}+\frac{2R}{xw}+\xi)(-i\frac{k}{2}+\frac{2R}{xw}+\xi)(1-\xi)(\xi-(-1))}} \end{aligned} \tag{37}$$

with expansion coefficients

$$\beta_0 = (\frac{k}{2})^2 + (\frac{2}{x\hat{w}})^2; \quad \beta_1 = \frac{4}{x\hat{w}}; \quad \beta_2 = 1. \tag{38}$$

In particular for $m = 0$ [2, 259.00] at Byrd-Friedmann parameters $b = -1$, $a = 1$, $c = -\frac{2R}{xw} + i\frac{k}{2}$, $b_1 = R/(xw)$,

$$\int_{-1}^{1} d\xi \frac{1}{\sqrt{(i\frac{k}{2}+\frac{2R}{xw}+\xi)(-i\frac{k}{2}+\frac{2R}{xw}+\xi)(1-\xi)(\xi-(-1))}} = gK(\hat{k})$$

is a complete elliptic integral of the first kind where

$$\begin{aligned} g &= \frac{2}{x\hat{w}\tau^{1/4}}; \\ \tau &\equiv 16x^2\hat{w}^2(x\hat{w}-4+x\hat{w}\beta_0)(x\hat{w}+4+x\hat{w}\beta_0); \\ \hat{k}^2 &= \frac{-4x^2\hat{w}^2+4x^2\hat{w}^2\beta_0-\sqrt{\tau}}{2\sqrt{\tau}}. \end{aligned}$$

The other two cases $m = 1, 2$ are linear combinations of elliptic integrals of the first, second and third kind [2, 259.03].

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