Proof of Goldbach conjecture

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Abstract. This paper is a trial to prove Goldbach conjecture according to the following process.
1. We find that {the total number of ways to divide an even number n into 2 prime numbers} : $l(n)$ diverges to ∞ with $n \to \infty$.
2. We find that $1 \leq l(n)$ holds true in $4 * 10^{18} < n$ from the probability of $l(n) = 0$.
3. Goldbach conjecture is already confirmed to be true up to $n = 4 * 10^{18}$.
4. Goldbach conjecture is true from the above item 2 and 3.

1. Introduction

1.1 When an even number n is divided into 2 odd numbers x and y, we can express the situation as pair (x, y) like the following (1).

$$n = x + y = (x, y) \qquad (n = 6, 8, 10, 12, \cdots\cdots \quad x, y : \text{odd number}) \tag{1}$$

n has $n/2$ pairs like the following (2).

$$(1, n-1), (3, n-3), (5, n-5), \cdots\cdots, (n-5, 5), (n-3, 3), (n-1, 1) \tag{2}$$

We define as follows.

Prime pair : the pair where both x and y in (x, y) are prime numbers
Composite pair : the pair other than the above prime pair
$l(n)$: the total number of the prime pairs which exist in $n/2$ pairs shown by the above (2). (p, q) is regarded as the different pair from (q, p). (p, q : prime number) Then when $n/2$ is a composite number $l(n)$ is an even number and when $n/2$ is a prime number $l(n)$ is an odd number.

1.2 Goldbach conjecture can be expressed as the following (3) i.e. any even number $n(\geq 6)$ can be divided into 2 prime numbers.

$$1 \leq l(n) \qquad (n = 6, 8, 10, 12, \cdots\cdots) \tag{3}$$

Since Goldbach conjecture is already confirmed to be true up to $n = 4 * 10^{18}$, we can try to prove Goldbach conjecture in the following condition.

$$4 * 10^{18} < n \tag{4}$$

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2. Investigation of $l(n)$

2.1 When an even number n is divided into 2 odd numbers x and y, we can find the pair of $\pi(n), l(n), m_{xx}, m_x, m_y$ and m_{xy} in $n/2$ pairs of (x, y) as shown in the following (Figure 1).

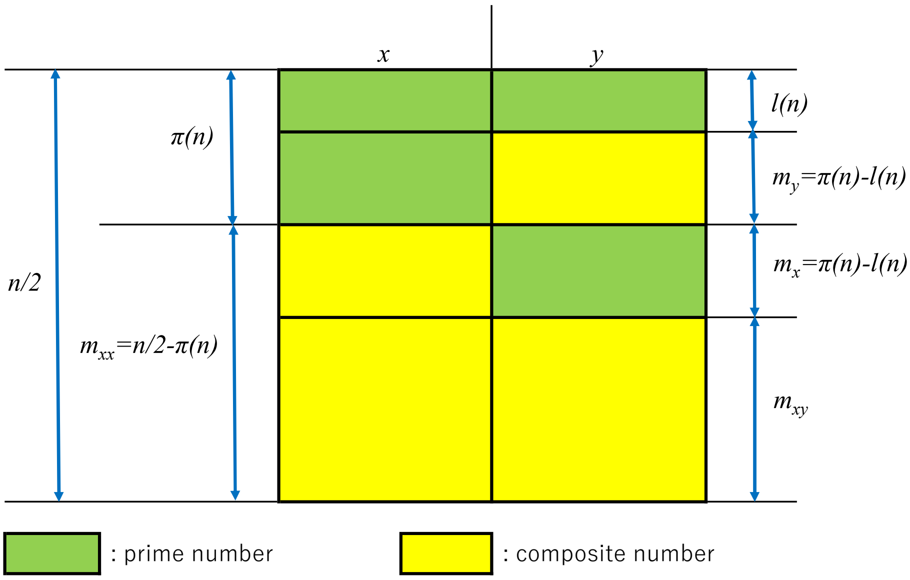


Figure 1 : Various pairs in $n/2$ pairs of (x, y)

We define as follows.

$\pi(n)$: $\pi(n)$ shows the total number of prime numbers which exist between 1 and n. But we use $\pi(n)$ in the above (Figure 1) for the total number of prime numbers which exist in $n/2$ odd numbers of $(1, 3, 5, \cdots\cdots,$ $n-5, n-3, n-1)$. Strictly speaking, this value must be $\pi(n-1)-1$. But we can say $\pi(n-1)-1 = \pi(n)-1 \fallingdotseq \pi(n)$ because n is an even number and a large number as shown in (4).

m_{xx} : the total number of pairs where x is a composite number. 1 is regarded as a composite number.

m_x : the total number of pairs where x and y are composite number and prime number respectively

2.2 We have the following (5) from Prime number theorem.

$$\frac{\pi(n)}{n} \sim \frac{n/\log n}{n} = \frac{1}{\log n} \qquad (n \to \infty) \tag{5}$$

We have $\lim_{n\to\infty} \dfrac{\pi(n)}{n} = 0$ from the above (5). Then we have the following (6) from (Figure 1) and $\lim_{n\to\infty} \dfrac{\pi(n)}{n} = 0$

$$m_{xx} = n/2 - \pi(n) = (n/2)\{1 - 2\pi(n)/n\} \sim n/2 \qquad (n \to \infty) \tag{6}$$

When m_{xx} approaches $n/2$ with $n \to \infty$ as shown in the above (6), m_x approaches $\pi(n)$ with $n \to \infty$ due to the following reasons.

2.2.1 m_x shows the total number of prime numbers which exist in y of m_{xx} as shown in (Figure 1).

2.2.2 y of m_{xx} approaches $n/2$ odd numbers of $(1,\ 3,\ 5,\ \cdots\cdots, n-5, n-3, n-1)$ with $n \to \infty$ as shown in the above (6).

2.2.3 $(1,\ 3,\ 5,\ \cdots\cdots, n-5, n-3, n-1)$ has $\pi(n)$ prime numbers.

Then we can have the following (7) from (Figure 1).

$$m_x = \pi(n) - l(n) = \pi(n)\{1 - l(n)/\pi(n)\} \quad \sim \quad \pi(n) \qquad (n \to \infty) \tag{7}$$

We have $\lim_{n\to\infty} \dfrac{l(n)}{\pi(n)} = 0$ from the above (7). We have the following (8) from the above (6) and (7).

$$\frac{\pi(n) - l(n)}{n/2 - \pi(n)} \quad \sim \quad \frac{\pi(n)}{n/2} \qquad (n \to \infty) \tag{8}$$

We have the following (9) from the above (8) and Prime number theorem.

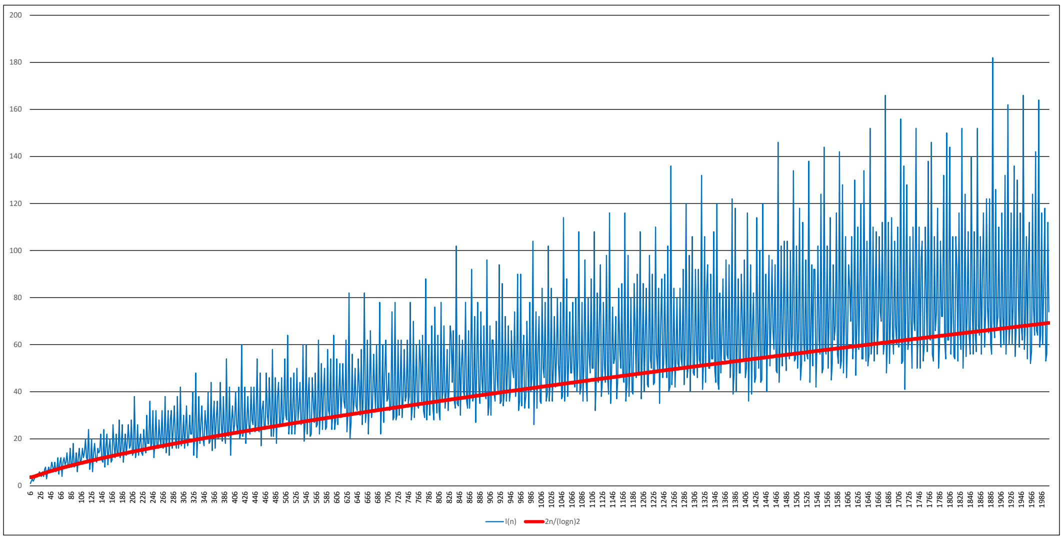
$$l(n) \quad \sim \quad \frac{\{\pi(n)\}^2}{n/2} \quad \sim \quad \frac{\{n/\log n\}^2}{n/2} = \frac{2n}{(\log n)^2} \qquad (n \to \infty) \tag{9}$$

We can find that $l(n)$ has the following properties from the above (9).

2.2.4 $l(n)$ repeats increases and decreases with increase of n as shown in the following (Graph 1). But overall $l(n)$ is an increasing function regarding n because $\dfrac{2n}{(\log n)^2}$ is an increasing function regarding n.

2.2.5 $l(n)$ diverges to ∞ with $n \to \infty$ because $\dfrac{2n}{(\log n)^2}$ diverges to ∞ with $n \to \infty$.

2.3 $\dfrac{2n}{(\log n)^2}$ seems to approximate $l(n)$ sufficiently well as shown in the following (Graph 1).



Graph 1 : $l(n)$(blue line)[1] and $\dfrac{2n}{(\log n)^2}$(red line) from $n = 6$ to $n = 2,000$

3. Investigation of zero point of $l(n)$

3.1 Since both x and $(n-x)$ in pair $(x, y) = (x, n-x)$ are always an odd number, we must consider the probability that x or $(n-x)$ is a prime number in the world where only odd numbers exist.
Prime number theorem shows that {the probability that randomly selected integer m is a prime number} approaches to $1/\log m$ with $m \to \infty$. Then we can have {the probability that randomly selected odd number N is a prime number} : $P(N)$ like the following (10) because an even mumber cannot be a prime number.

$$P(N) \quad \sim \quad \frac{2}{\log N} \qquad (N \to \infty \quad N : \text{odd number}) \tag{10}$$

3.2 Since the probability that $(x, n-x)$ or $(n-x, x)$ is a prime pair is $P(x)*P(n-x)$, the probability that $(x, n-x)$ or $(n-x, x)$ is a composite pair is $\{1 - P(x) * P(n-x)\}$. Therefore the probability that all of $n/2$ pairs are a composite pair i.e. {the probability of $l(n) = 0$} : $A(n)$ can be expressed like the following (11).
Since $(1, n-1)$ and $(n-1, 1)$ are always a composite pair, we don't include these pairs in (11). Then x does not include 1 in the following (11-1) and (11-2). (11) has $(n/2 - 2)$ terms of $\{1 - P(x) * P(n-x)\}$ altogether.

$$\begin{aligned}
A(n) &= \{1 - P(3) * P(n-3)\}^2\{1 - P(5) * P(n-5)\}^2\{1 - P(7) * P(n-7)\}^2 \ldots\ldots \\
&\quad \{1 - P(x) * P(n-x)\}^2 \ldots\ldots \{1 - P(n/2-4) * P(n/2+4)\}^2 \\
&\quad \{1 - P(n/2-2) * P(n/2+2)\}^2\{1 - P(n/2)^2\} \qquad (n/2 : \text{odd number}) \\
&= \{1 - P(3) * P(n-3)\}^2\{1 - P(5) * P(n-5)\}^2\{1 - P(7) * P(n-7)\}^2 \ldots\ldots \\
&\quad \{1 - P(x) * P(n-x)\}^2 \ldots\ldots \{1 - P(n/2-5) * P(n/2+5)\}^2 \\
&\quad \{1 - P(n/2-3) * P(n/2+3)\}^2\{1 - P(n/2-1) * P(n/2+1)\}^2
\end{aligned}$$

$$(n/2 : \text{even number}) \tag{11}$$

$$(x = 3, 5, 7, 9, \ldots\ldots, n/2-4, n/2-2, n/2 \qquad n/2 : \text{odd number}) \tag{11-1}$$

$$(x = 3, 5, 7, 9, \ldots\ldots, n/2-5, n/2-3, n/2-1 \qquad n/2 : \text{even number}) \tag{11-2}$$

3.3 We have the following (12) from the above (11-1) and (11-2).

$$3 \leq x \leq n/2 \leq n-x < n+1 \ll 10^{18} * n + 1 \tag{12}$$

Since $P(N)$ decreases with increase of N as shown in [Appendix 1 : Investigation of $P(N)$], if n is large enough, we have the following (13) from (10) and (12).

$$\begin{aligned} 1 \geq P(x) \geq P(n-x) \geq P(n+1) &\fallingdotseq \frac{2}{\log n} \\ > P(10^{18} * n + 1) &\fallingdotseq \frac{2}{\log(10^{18} * n)} = \frac{2}{\log n + 41.4} \end{aligned} \tag{13}$$

We have the following (14) from (13).

$$0 < 1 - P(x) * P(n-x) < 1 - \{P(10^{18} * n + 1)\}^2 \tag{14}$$

We have the following (15) from (11), (13) and (14).

$$\begin{aligned} 0 < A(n) < B(n) &= [1 - \{P(10^{18} * n + 1)\}^2]^{n/2-2} \\ &\sim \quad \{1 - \frac{4}{(\log n + 41.4)^2}\}^{n/2} \\ &= [\{1 - \frac{1}{\{(\log n + 41.4)/2\}^2}\}^{\{(\log n + 41.4)/2\}^2}]^{(n/2)/\{(\log n + 41.4)/2\}^2} \\ &\sim \quad (\frac{1}{e})^{(n/2)/\{(\log n + 41.4)/2\}^2} = \frac{1}{e^{(n/2)/\{(\log n + 41.4)/2\}^2}} \qquad (n \to \infty) \end{aligned} \tag{15}$$

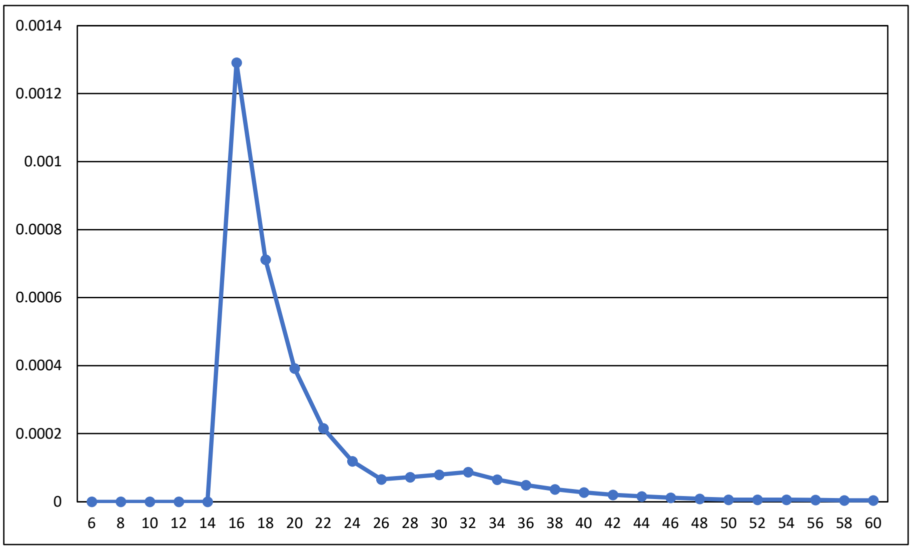
We have the following (16) from the above (15).

$$\lim_{n \to \infty} A(n) = 0 \tag{16}$$

If n is large enough, i.e. if $4 * 10^{18} \leq n$ is satisfied, $B(n)$ can be approximated to $\frac{1}{e^{(n/2)/\{(\log n + 41.4)/2\}^2}}$ from the above (15) and $\frac{1}{e^{(n/2)/\{(\log n + 41.4)/2\}^2}}$ decreases with increase of n in $4 * 10^{18} \leq n$. Therefore we have the following (17).

$$0 < A(n) < B(n) < B(4 * 10^{18}) \qquad (4 * 10^{18} < n) \tag{17}$$

3.4 Since we can calculate {the probability that N is a composite number} : $Q(N)$ as shown in item 1.1 and 1.2 of [Appendix 1], we can have $P(N) = 1 - Q(N)$ from $Q(N)$. Then we can calculate $A(n)$ from $P(N)$ and (11) as shown in the following (Graph 2).



Graph 2 : $A(n)$ from $n = 6$ to $n = 60$

$A(n)$ has the following properties.

3.4.1 $A(n) = 0$ holds true in $6 \leq n \leq 14$ as shown in item 1.5 of [Appendix 1]. $A(16)$ has the value of 0.0013 and $A(n)$ almost decreases with increase of n in $16 \leq n \leq 60$ as shown in the above (Graph 2).

3.4.2 The above (17) holds true.

3.4.3 $A(n)$ converges to zero with $n \to \infty$.

3.5 When $l(n_0) = 0$ holds true we define n_0 as {zero point of $l(n)$}. Possible zero point distribution of $l(n)$ is limited to 4 cases which are classified according to location of zero point as shown in the following (Table 1).

	Location of zero point		Contradiction with	Can this case exist as real $l(n)$?
	$n \leqq 4*10^{18}$	$4*10^{18} < n$		
Case 1	●	●	item 3.5.2	NO
Case 2	●	X	item 3.5.2	NO
Case 3	X	●	item 3.5.1	NO
Case 4	X	X	nothing	YES

● : zero points exist. X : no zero points exist.

Table 1 : 4 cases of zero point distribution of $l(n)$

Distribution of zero point of $l(n)$ is affected by the following facts.

3.5.1 $A(n)$ has the properties shown in item 3.4.

3.5.2 Goldbach conjecture is already confirmed to be true up to $n = 4*10^{18}$ as shown in item 1.2. Therefore any zero points of $l(n)$ do not exist in $n \leq 4 * 10^{18}$.

Case 1 and Case 2 cannot exist because they contradict item 3.5.2.
Case 3 cannot exist because it contradicts item 3.5.1 as shown in the following item 3.6.

3.6 From (17) we have the following (18) which shows that $A(n)$ is extremely small in $4*10^{18} < n$. $B(n)$ is defined in (15).

$$A(n) < B(4*10^{18}) \fallingdotseq \frac{1}{e^{(2*10^{18})/[\{\log(4*10^{18})+41.4\}/2]^2}} = \frac{1}{e^{(2*10^{18})/1774}} = e^{-1.1*10^{15}}$$
$$= (e^{1.1})^{-10^{15}} = (10^{0.47})^{-10^{15}} = 10^{-4.7*10^{14}} \qquad (4*10^{18} < n) \qquad (18)$$

We can have $A(16) = 0.0013$ as shown in item 3.4.1.
Since Case 3 has zero points only in $4*10^{18} < n$, Case 3 contradicts $A(n)$ as follows.

3.6.1 The larger $A(n)$ is, the more likely a zero point will appear. Then the situation where a zero point can exist in $A(n) < 10^{-4.7*10^{14}}$ as (18) shows contradicts the situation where a zero point cannot exist in $A(16) = 0.0013$. In other words, Case 3 shows the situation that is completely opposite to the situation expected from $A(n)$ as shown in the following item 3.6.2 and 3.6.3.

3.6.2 0.0013 is extremely larger than $10^{-4.7*10^{14}}$ and zero points already exist in $A(n) < 10^{-4.7*10^{14}}$. Therefore a new zero point must exist near $n = 16$. But Case 3 does not have any zero point in $n \leq 4*10^{18}$.

3.6.3 $10^{-4.7*10^{14}}$ is extremely smaller than 0.0013 and zero points do not exist near $n = 16$. Therefore zero points must not exist in $4*10^{18} < n$. But Case 3 has zero points in $4*10^{18} < n$.

The following (Figure 2) shows the contradiction between Case 3 and $A(n)$.

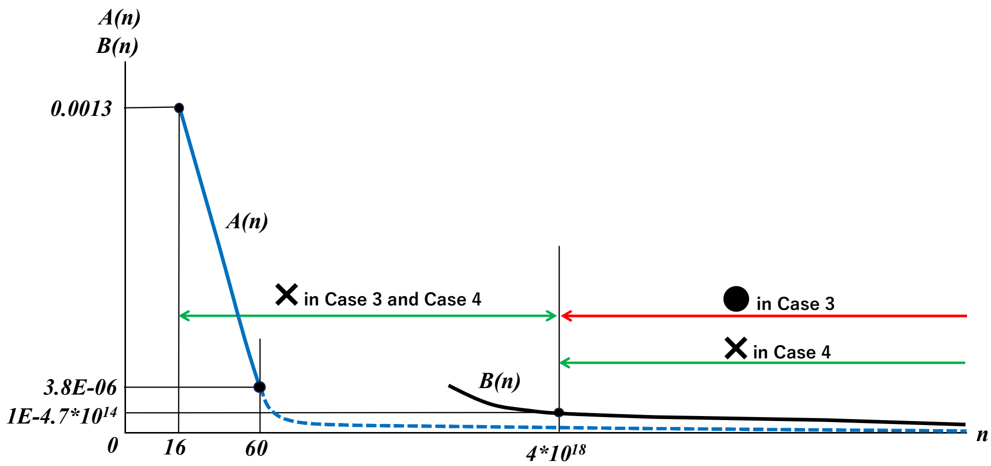


Figure 2 : the contradiction between Case 3 and $A(n)$

3.7 If no zero points exist in $4*10^{18} < n$, the contradiction shown in item 3.6.1 does not occur as shown in the above (Figure 2). In other words Case 4 is consistent with $A(n)$. Among 4 cases of zero point distribution of $l(n)$ shown in (Table 1), only Case 4 meets both item 3.5.1 and 3.5.2. Therefore Case 4 shows the real $l(n)$. We have the following (19) from Case 4 because Case 4 does not have any zero point in $4 * 10^{18} < n$.

$$1 \leq l(n) \qquad\qquad (4 * 10^{18} < n) \tag{19}$$

4. Conclusion

Goldbach conjecture is true from the following item 4.1 and 4.2.

4.1 Goldbach conjecture is already confirmed to be true up to $n = 4 * 10^{18}$.

4.2 Goldbach conjecture is true in $4 * 10^{18} < n$ from the above (19).

Appendix 1. : Investigation of $P(N)$

We can find that $P(N)$ decreases with increase of N in this appendix.

1.1 When odd number N is a composite number, N is divisible by any of $\{\pi(\lfloor\sqrt{N}\rfloor)-1\}$ prime numbers of $\{p_2 = 3, p_3 = 5, p_4 = 7, \cdots\cdots, p_k, \cdots\cdots, p_{\pi(\lfloor\sqrt{N}\rfloor)-1}, p_{\pi(\lfloor\sqrt{N}\rfloor)}\}$. The above $\{\pi(\lfloor\sqrt{N}\rfloor)-1\}$ prime numbers satisfy $3 \leq p \leq \sqrt{N}$. (p : prime number) Then {the probability that N is a composite number} i.e. {the probability that N is divisible by any of the above $\{\pi(\lfloor\sqrt{N}\rfloor)-1\}$ prime numbers} : $Q(N)$ can be expressed like the following (20). Because $(Q_2, Q_3, Q_4, Q_5, \cdots\cdots, Q_k, \cdots\cdots, Q_{\pi(\lfloor\sqrt{N}\rfloor)-1}, Q_{\pi(\lfloor\sqrt{N}\rfloor)})$ in (20) are the probabilities of mutually exclusive events.

$$Q(N) = Q_2 + Q_3 + Q_4 + Q_5 + \cdots\cdots + Q_k + \cdots\cdots + Q_{\pi(\lfloor\sqrt{N}\rfloor)-1} + Q_{\pi(\lfloor\sqrt{N}\rfloor)}$$
$$(2 \leq k \leq \pi(\lfloor\sqrt{N}\rfloor)) \tag{20}$$

Q_2 : the probability that N is divisible by $p_2 = 3$
Q_3 : the probability that N is divisible by $p_3 = 5$ but not by $p_2 = 3$
Q_4 : the probability that N is divisible by $p_4 = 7$ but not by $p_3 = 5$ or $p_2 = 3$
Q_5 : the probability that N is divisible by $p_5 = 11$ but not by $p_4 = 7$, $p_3 = 5$ or $p_2 = 3$
Q_k : the probability that N is divisible by p_k but not by any of $(p_{k-1}, p_{k-2}, \cdots\cdots, p_4 = 7, p_3 = 5, p_2 = 3)$

1.2 We have the values of Q_2, Q_3, Q_4, Q_5 and Q_6 as follows.

1.2.1 We have $\qquad Q_2 = 1/3 = 0.333$
because the probability that randomly selected odd number N is divisible by $p_2 = 3$ is $1/3$.

1.2.2 We have $Q_3 = 1/5 - 1/(5*3) = 0.133$
because the probability that randomly selected odd number N is divisible by $p_3 = 5$ is $1/5$ and the probability that randomly selected odd number N is divisible by both $p_3 = 5$ and $p_2 = 3$ is $1/(5*3)$.

1.2.3 Similarly we have
$Q_4 = 1/7 - \{1/(7*5) + 1/(7*3) - 1/(7*5*3)\} = 0.0762.$
Here $1/(7*5*3)$ is necessary because both {the probability that N is divisible by both $p_4 = 7$ and $p_3 = 5$} and {the probability that N is divisible by both $p_4 = 7$ and $p_2 = 3$} contain {the probability that N is divisible by all of $p_4 = 7$, $p_3 = 5$ and $p_2 = 3$}.

1.2.4 In the same way as Q_4 we have

$$\begin{aligned}Q_5 = 1/11 - \{&1/(11*7) + 1/(11*5) + 1/(11*3)\\ &-1/(11*7*5)-1/(11*7*3)-1/(11*5*3)+2/(11*7*5*3)\} = 0.0410\end{aligned}$$

$$\begin{aligned}Q_6 = 1/13 - \{&1/(13*11) + 1/(13*7) + 1/(13*5) + 1/(13*3)\\ &- 1/(13*11*7) - 1/(13*11*5) - 1/(13*11*3) - 1/(13*7*5)\\ &- 1/(13*7*3) - 1/(13*5*3)\\ &+1/(13*11*7*5)+1/(13*11*7*3)+1/(13*11*5*3)+2/(13*7*5*3)\\ &- 4/(13*11*7*5*3)\} = 0.0322\end{aligned}$$

1.3 We can find the properties of Q_k as follows.

1.3.1 We define as follows.

1.3.1.1 N_k : odd composite number which are divisible by prime number p_k

1.3.1.2 N_{k+} : N_k that is divisible by any of $(p_{k-1}, p_{k-2}, \cdots\cdots, p_4 = 7, p_3 = 5, p_2 = 3)$

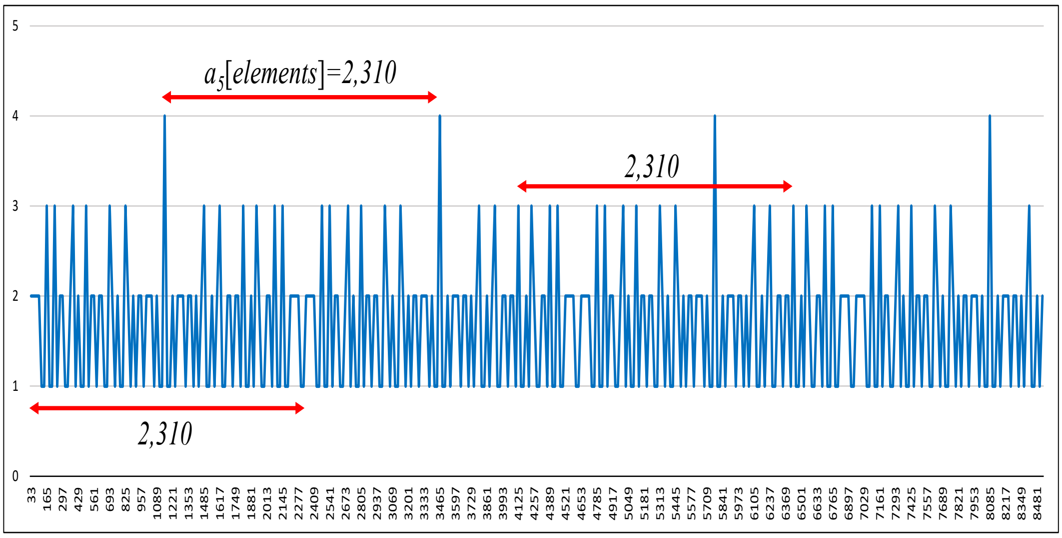
1.3.1.3 N_{k-} : N_k that is not divisible by any of $(p_{k-1}, p_{k-2}, \cdots\cdots, p_4 = 7, p_3 = 5, p_2 = 3)$

The following S_k is the set of N_k and N_k is arranged in ascending order.

$$S_k =\{p_k * 3, p_k * 5, p_k * 7, p_k * 9, \cdots\cdots, p_k * (p_k - 2), p_k^2, p_k * (p_k + 2), \cdots\cdots\} \tag{21}$$

The element of N_{k+} which is divisible by $p_2 = 3, (p_k * 3, p_k * 9, p_k * 15, p_k * 21, \cdots\cdots)$ appears every 3 elements in the above S_k. Similarly the element of N_{k+} which is divisible by prime number $(p_{k-1}, p_{k-2}, \cdots\cdots, p_4 = 7, p_3 = 5)$ appears every $(p_{k-1}, p_{k-2}, \cdots\cdots, p_4 = 7, p_3 = 5)$ elements respectively in S_k. Then the distribution of element of N_{k+} repeats the same pattern every $a_k (= p_{k-1} * p_{k-2} * \cdots\cdots * 7 * 5 * 3)$ elements in S_k.

1.3.2 The distribution of element of N_{k-} also repeats the same pattern every a_k elements in S_k because N_k consists only of N_{k+} and N_{k-}.
The following (Graph 3) shows {the total number of distinct prime factors $p(\le p_5 = 11)$ of N_5} : $\omega(N_5)$ from $N = 11 * 3 = 33$ to $N = 11 * 775 = 8,525$.



Graph 3 : $\omega(N_5)$ from $N = 33$ to $N = 8,525$

The value of $\omega(N_5)$ shows the properties of N_5 as follows.
$\omega(N_5) = 4$: N_5 is divisible by all of 3, 5, 7 and 11.
$\omega(N_5) = 3$: N_5 is divisible by 11 and (2 prime numbers out of 3, 5, and 7).
$\omega(N_5) = 2$: N_5 is divisible by 11 and (one prime number out of 3, 5, and 7).
$\omega(N_5) = 2$, 3, or 4 : N_5 is divisible by any of 3, 5 and 7. Then N_5 is N_{5+}.
$\omega(N_5) = 1$: N_5 is divisible by only 11. Then N_5 is N_{5-}.
From (Graph 3) we can find that the distributions of both N_{5+} and N_{5-} repeat the same pattern every a_5 elements ($= 2*p_5*a_5 = 2*11*7*5*3 = 2,310$) in S_5. Because the distance between adjacent elements of S_5 is $2*p_5$ as shown in (21).

1.3.3 Q_k is the probability that N is N_{k-}. Since the distance between adjacent elements of S_k is $2*p_k$, we can calculate Q_k in $3*p_k \le N$ by the following (22). No matter where we calculate Q_k in $3*p_k \le N$, we can always have a same value. Because the distribution of N_{k-} repeats the same pattern every $2*p_k*a_k$ in $3*p_k \le N$ as shown in item 1.3.2. Therefore the value of Q_k is a constant which depends on k but not on N.

$$
\begin{aligned}
Q_k =& \frac{\text{the total number of } N_{k-} \text{ in } N_0 \le N \le (N_0 + 2*p_k*a_k)}{\text{the total number of odd number in } N_0 \le N \le (N_0 + 2*p_k*a_k)} \\
=& \frac{\text{the total number of } N_{k-} \text{ in } N_0 \le N \le (N_0 + 2*p_k*a_k)}{p_k*a_k}
\end{aligned} \tag{22}
$$

(N_0 : any N which satisfies $3*p_k \le N$)
($p_k*a_k = p_k*p_{k-1}*p_{k-2}*\cdots\cdots*7*5*3$)

1.3.4 The above N_{k-} has the following properties.

1.3.4.1 All prime factors of N_{k-} satisfy $p_k \le p$ from item 1.3.1.3.

1.3.4.2 N_{k-} has at least 2 prime factors from item 1.3.1.1.

1.3.4.3 N_{k-} has at least one prime factor p_k from item 1.3.1.1.

1.3.4.4 N_{k-} does not exist in $N < p_k^2$. There exist an infinite number of N_{k-} in $p_k^2 \le N$ from item 1.3.4.1—1.3.4.3.

We can have the following (23) from item 1.2, 1.3.3 and 1.3.4.4.

$$\begin{aligned} Q_k =& 0 && (N < p_k^2) \\ Q_k =& 1/p_k - C_k > 0 && (p_k^2 \le N) \end{aligned} \tag{23}$$

C_k : the correction value for the fact that N is not divisible by any of $(p_{k-1}, p_{k-2}, \cdots\cdots, p_4 = 7, p_3 = 5, p_2 = 3)$
$(0 \le C_k < 1/p_k \quad 0 = C_k$ only at $k = 2)$

We have the following (24) from (23).

$$0 < Q_k \le 1/p_k \qquad (p_k^2 \le N \quad Q_k = 1/p_k \text{ only at } k = 2) \tag{24}$$

1.4 $Q(N)$ increases with increase of N due to the following reasons.

1.4.1 $\pi(\lfloor\sqrt{N}\rfloor)$ increases with increase of N.

1.4.2 Since $Q(N)$ has $\{\pi(\lfloor\sqrt{N}\rfloor) - 1\}$ terms as shown in (20), the total number of term of $Q(N)$ increases with increase of N.

1.4.3 Q_k has positive value as shown in (24).

1.4.4 Since the value of Q_k is a constant which depends on k but not on N as shown in item 1.3.3, even if the total number of term of $Q(N)$ increases by 1 after increase of N, the value of each Q_k which already existed before increase of N does not change.

1.4.5 When N increases from $N = N_1 - 2$ to $N = N_1$, if a prime number does not exist in the range of $\sqrt{N_1 - 2} < r \le \sqrt{N_1}$ (r : real number), $Q(N)$ does not change. But if a prime number $p_{\pi(\lfloor\sqrt{N_1}\rfloor)}$ exists in the range of $\sqrt{N_1 - 2} < r \le \sqrt{N_1}$, $Q(N)$ increases by $Q_{\pi(\lfloor\sqrt{N_1}\rfloor)}(> 0)$.

Since $Q(N)$ increases with increase of N, $P(N) = 1 - Q(N)$ decreases with increase of N.

We have the following (25) from (10) and we have the following (26) from (25).

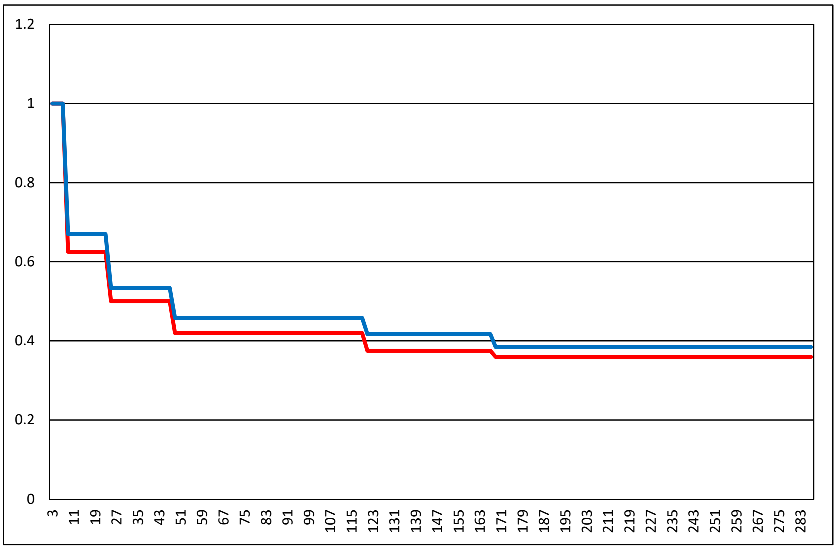
$$\lim_{N\to\infty}\{1 - Q(N)\} = \lim_{N\to\infty} P(N) = 0 \tag{25}$$

$$\lim_{N\to\infty} Q(N) = 1 \tag{26}$$

1.5 In order for $Q(N)$ to exist, $2 \le \pi(\lfloor\sqrt{N}\rfloor)$ must hold true from item 1.4.2. Then $Q(N)$ is defined in $9 \le N$. Since $\pi(\lfloor\sqrt{N}\rfloor) \le 1$ holds true in $3 \le N \le 7$, we should think that $Q(N) = 0$ i.e. $P(N) = 1$ holds true in $3 \le N \le 7$. Because the real probability that N is a prime number in $3 \le N \le 7$ is 100%. Therefore $A(n) = 0$ holds true in $6 \le n \le 14$ from (11).

The following (Graph 4) shows $P(N)$ calculated from $Q(N)$ and $p(N)$: the real probability calculated from the following (27).

$$p(N) = \frac{\text{the total number of prime number in } p_j^2 \le N < p_{j+1}^2}{\text{the total number of odd number in } p_j^2 \le N < p_{j+1}^2}$$
$$(j = 1, 2, 3, \cdots\cdots \qquad p_1 = \sqrt{3}, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11 \cdots\cdots) \qquad (27)$$



Graph 4 : $P(N)$(blue line) and $p(N)$(red line) from $N = 3$ to $N = 287$

References

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