The House of St. Nicholas

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Abstract

We define for each finite graph nine constants. They can serve as a 'characteristic number' or 'dimension'. Further, we define ten sequences for such graph. The elements are natural numbers or zero.
We will not provide examples.

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The defined numbers tradim, orddim, compdim, vertdim, vertnodim, edgedim, edgenodim, graphdim, and nodim, respectively, can be used as a 'characteristic number' or 'dimension' for a finite graph.

The following well-known graph is called 'The House of St. Nicholas', and it can be drawn without lifting the pencil from the paper.

This idea can be extended.

In the following graphs, '$\times$' means a vertex.

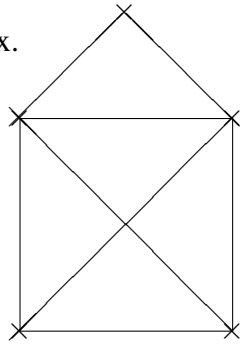


Figure 1:
We see the 'House of St. Nicholas'.
It consists of 5 vertices and 8 edges.
The edges are straight lines.

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We call a *traverse line* a graph which can be drawn without lifting the pencil from the paper, and every edge of the graph is drawn a single time.

Let n be a natural number, i.e. $\mathsf{n} \in \mathbb{N} = \{1, 2, 3, \ldots\}$.

The graph 'The House of St. Nicholas' can be drawn without lifting the pencil from the paper. It is a traverse line. You can speak one syllable of the German sentence 'Das ist das Haus vom Nikolaus' when you draw each edge.

Definition 1. We define an n-*Nicholas* as n Houses of St. Nicholas, which are put in a row. They touch each other at one edge. See Figure 2.
We define an n-*Rectangle* as n rectangles which are put in a row, and they touch each other at one edge. The shape of an n-Rectangle is a narrow rectangle. See Figure 3.

Remark 1. An 1-Nicholas is just the House of St. Nicholas. An 1-Rectangle is just a square.

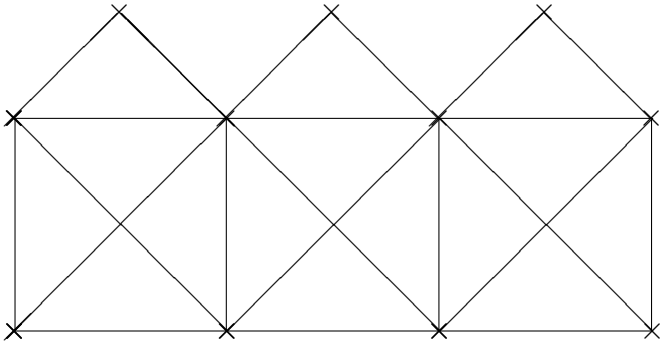


Figure 2:

The graph 3-Nicholas.

It consists of 11 vertices and 22 edges.

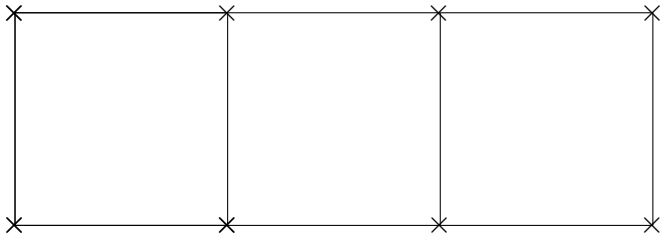


Figure 3:

The graph 3-Rectangle.

It consists of 8 vertices and 10 edges.

Remark 2. n-Nicholas is a traverse line if and only if $\mathsf{n} = 1$. n-Rectangle is a traverse line if and only if n is less than 3, i.e. either it is a 2-Rectangle or a square. This is shown by the old theorem of Leonard Euler and the problem of the bridges of Königsberg since only in a 2-Rectangle the number of vertices which have an odd number of edges is 2. This number is also larger than 2 in an n-Nicholas for $\mathsf{n} > 1$.

An edge e and a vertex $\vec{v}$ are *incident* if and only if $\vec{v}$ is a vertex of e.
Two different vertices are *adjacent* if and only if they belong to the same edge.
We say that two different edges are *neighboring* if and only if they have a common vertex.

'To color a graph' means for a concrete graph (i.e. a graph which is a subset of $\mathbb{R}^p$ for any p), that we color the vertices and the inner points of the edges (i.e. all points of an edge except the two incident vertices). The interior of an edge gets a single color.
If two or more edges intersect, the intersection point gets one of the colors of the edges, besides that the intersection point is a vertex.
We assume that we have as many colors as the number of the sum of the number of the vertices and the number of the edges.
For an abstract graph, its only a defintion to color it. We assign colors to the vertices and the edges.

'To color an edge' means for a concrete graph that we color all inner points with one color. For an abstract graph, we assign it to a color.

We say that two colorings C_1 and C_2 are different if and only if there is a vertex or an edge such that the color it gets by C_1 differs from the color that it gets from C_2.

Let in the following H be an arbitrary finite graph.
We assume that we have s colors, where s is the number of the sum of the number of the vertices and edges of H. This ensures that the following constants all are from $\mathbb{N}$.

Definition 2. We define for the graph H the *traverse line dimension* (in short *tradim*) as the least natural number k such that it is composed of k traverse lines.
We define for the graph H the *component dimension* (in short *compdim*) as the number of components of H.
We define for the graph H the *order dimension* (in short *orddim*) as the number of possibilities that it can be drawn. We draw exclusively a complete edge or multiple edges. We have s colors.
We define for the graph H the *vertex dimension* (in short *vertdim*) as the number of possibilities that the vertices can be colored.
We define for the graph H the *vertex dimension* (in short *vertnodim*) as the number of possibilities to color the vertices such that adjacent vertices get different colors.
We define for the graph H the *edge dimension* (in short *edgedim*) as the number of possibilities that the edges can be colored.
We define for the graph H the *edge dimension* (in short *edgenodim*) as the number of possibilities to color the edges such that neighboring edges get different colors.
We define for the graph H the *graph dimension* (in short *graphdim*) as the number of possibilities that it can be colored.
We define for the graph H the *nothing dimension* (in short *nodim*) as the number of possibilities that it can be colored such that adjacent vertices, neighboring edges, and also pairs of an incident vertex and edge get different colors.

We will not provide examples.

Now we generate for every finite graph H for each natural number m ten constants. They are from $\mathbb{N}$ or zero. Therefore, we get for each finite graph ten sequences.

Definition 3. In the following, we assume that we have m colors, $\mathsf{m} \in \mathbb{N}$.

Let *vertices* (H, m) be defined as the number of possibilities to color the vertices of H.

Let *adjacent* (H, m) be defined as the number of possibilities to color the vertices of H, where adjacent vertices have different colors.

Let *edges* (H, m) be defined as the number of possibilities to color the edges of H.

Let *neighboring* (H, m) be defined as the number of possibilities to color the edges of H, where neighboring edges get different colors.

Let *incident* (H, m) be defined as the number of possibilities to color the graph H, where incident pairs of an edge and a vertex get different colors.

Let *graph* (H, m) be defined as the number of possibilities to color the graph H.

Let *graph, adjacent* (H, m) be defined as the number of possibilities to color the graph H, where adjacent vertices get different colors.

Let *graph, neighboring* (H, m) be defined as the number of possibilities to color the graph H, where neighboring edges get different colors.

Let *graph, incident* (H, m) be defined as the number of possibilities to color the graph H, where pairs of an incident vertex and edge get different colors.

Let *graph, none* (H, m) be defined as the number of possibilities to color the graph H, where adjacent vertices, neighboring edges, and also pairs of an incident vertex and edge get different colors.

References

[1] Joan M. Aldous and Robin J. Wilson: *Graphs and Applications*, Springer (2000)