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"A new formula for the Riemann hypothesis "

KEYWORDS,Non standard analysis,Non standard rearrangement,Riemann Dini's theorem,Zeta function of Riemann

MSC classification: 11M16, 26E35, 30B40, 65B15

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INTRODUCTION

Since the monumental Riemann's article of 1859 [9] the demonstration of what will become the Riemann hypothesis,it was and still is the central problem of number theory and all mathematics,having profound implications in various fields of it,and even in quantum mechanics [2] .From great German mathematician Riemann onwards, a great number of famous mathematicians have tried in vain to prove it.Just to nome a few Hardy,Littlewood,Hilbert,and in more recent times,Weil and others[2]. Furthermore the Riemann hypothesis is one of seven "Millennium Problems" [1].In this article the author tackles the problem of proving the Riemann hypothesis in a completely new different way from previous attempts,in fact in this article we will use some fundamental concept of non standard analysis founded by mathematical logician Abraham Robinson in the sixties of twenty century [10].Subsequently it was greatly simplified,without losing in mathematical rigor,by Jerome Keisler [6]which is the approach used in this article.

DEFINITION AND PRELIMINARIES

We give here some concept and definition that we will use for the continuations of this article.Foundamentally Keisler approach is based on the following two principles [6][7].

THE EXTENSION PRINCIPLE

a) The real numbers form a subset of the hyperreal numbers, and the order relation x< y for the real numbers is a subset of the order relation for the hyperreal numbers
b)There is a hyperreal number that is greater than zero but less than every positive real number
c) For every real function f of one or more variables we are given a corresponding hyperreal function f^* of the same number of variables , f^* is called the natural extension of f (in this article f^* is called the extension of f at non standard model of analysis) .
Furthermore with each relation X on R there is corresponding relation X* on R^* called the natural extension of X [7]

THE TRANSFER PRINCIPLE

Every real statement that hold for one or more particular real function holds for the hyperreal natural extensions of these functions, the transfer principle is equivalent to Leibniz' principle ,which is the property that for each real bounded sentence $\phi \in L$,is true if and only if ϕ^* is true . L is the language of the first order predicate [7]we still give the following definitions always of Keisler [6]:

DEFINITIONS

A hyperreal number b is said to be:
positive infinitesimal if b is positive but less than every positive real number,
negative infinitesimal if b is negative but greater than every negative real number.
A hyperreal number b is said to be:
finite if b is between two real numbers,
positive infinite if b is greater than every real number,
negative infinite if b is less than every real number.
Finally in the text we will assume that zeta function is an analytic function throughout $\mathbb{C} - 1$,and monodromous as demonstrated by Riemann in 1859 [9]

ABSTRACT

By a simple extension of rearrangement definition of a simply converging series, at non standard model of analysis, the author finds a new formula for $\zeta^*(s)$ with $s \in C^*$ $complex\ hyperreal\ numbers\ set$ with a *> 0 and with $s \neq 1$.
However this new formula is very easily extendable at the whole complex plane with $s \neq 1$.
Notable result is that with the definition of " non standard rearrangement " the commutative property of addition continues to hold even for simply convergent series (such as harmonic series with alternate signs).
Moreover the author, by means of the new formula of $\zeta^*(s)$ and the corresponding functional equation, gives a proof of the Riemann hypothesis.

TEXT

The new formula (for the zeta function of Riemann) is the follower:

$$\zeta^*(s)=\sum_{1}^{\omega} n^{-s} - \frac{\omega^{-s+1}}{-s+1} \qquad \text{(I)}$$

$s \in C^*$ $with\ the\ form\ a + ib\ (a\ and\ b\ hyperreal\ numbers)$ $s \neq 1$ and a *>0 (*> hyperreal order relation [7]). (I) has a simple pole at s = 1 where: ω=infinite number positive (see definitions and preliminaries) *=asterisk indicates the extension of $\zeta(s)$ at non standard model of analysis (see definitions and preliminaries) for the demonstration of Riemann's hypothesis we also use the functional equation of $\zeta(s)$ extended at non standard model ,it is in the standard model is the following:

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\sin\left(\frac{1-s}{2}\pi\right)(s-1)!\,\zeta(s) \qquad \text{(II)} \qquad [2].$$

Applying to (I) the Euler-Maclaurin formula (E.M.F) we have:

$$\zeta^*(s)= \sum_{1}^{R} n^{-s} - \frac{(R+1)^{-s+1}}{-s+1} + \frac{1}{2}(R+1)^{-s} - \frac{1}{12}\left[-s(R+1)^{-s-1}\right]+$$

$+\frac{1}{720}\left[-s(-s-1)(-s-2)R+1)^{-s-3}\right]+\ldots..$ (III)

R ∈ N =1,2,3…

In this article by convention we will indicate this formula (E.M.F) without distinction with its extension at non standard model of analysis (see definitions and preliminares), therefore also when this formula is applied in R* (hyperreal numbers set) or in C* complex hyperreal numbers set or with form $s = a^* + ib^*$ (with a^* and b* hyperreal numbers).It is easy to see that (I) with a *>1 (*> hyperreal order relation [7]) becomes:

$$\zeta^*(s)=\sum_{1}^{\omega} n^{-s}$$

that is the extension at non standard model of analysis of the classical definition:

$$\zeta(s)=\sum_{1}^{\infty} n^{-s}$$ with $a>1$ [1][2][3][9] .

Therefore by principle of identity of analytic functions (and for the transfer principle,see definitions and preliminaries) (I) is the analytic continuation of

$$\zeta^*(s) = \sum_{1}^{\omega} n^{-s}$$ (with a *>1)

In fact it is not difficult to prove with the (E.M.F) that (I) converges with a *> 0.We can formalize and generalize this in the following way:Let F(s) and G(s) be two analytic functions in a region $of\ C$ R_1 and R_2 respectively with $R_1 \cap R_2 \neq \emptyset$ (a small arc in common is sufficient)

$(\forall s \in (R_1 \wedge R_2)\ (F(s) = G(s)) \rightarrow (\forall s \in (R_1 \vee R_2)\ (F(s) = G(s))$.
estendit it at non standard model we have:
$(\forall s^* \in (R^*_1 \wedge R^*_2)\ (F^*(s) = G^*(s)) \rightarrow (\forall s^* \in (R^*_1 \vee R^*_2)\ (F^*(s) = G^*(s))$.
since the first formula written is true in C (set of complex numbers), in fact it is represents the principle of identity of analytic functions which also guarantees that the zeta function admits a unique analytic continuation, the last formula is true too in C * (complex hyperreal numbers set) according to the transfer principle or Leibniz' principle , in fact standard and non standard models are elementary equivalents ,it is not possible to find a formula (of the

first order language) that is true in one model that is not true in the other model (see the transfer principle).
Moreover (III) permits to find all values of $\zeta^*(s)$ (and therefore of $\zeta(s)$)for example here we give some values of $\zeta^*(s)$:

for $\zeta^*(0)$ using (III) and R=1 we have :

$$\zeta^*(0) = 1-2+\frac{1}{2}\left(\frac{1}{2}\right)^0 = -\frac{1}{2}$$

for $\zeta^*(-1)$ we have (with R=1)

$$\zeta^*(-1) = 1-\frac{2^2}{2}+\frac{1}{2}\cdot 2-\frac{1}{12}\left(1\cdot 2^0\right) = \frac{-1}{12}$$

for $\zeta^*(-2)$ (with R=1) we have: $\zeta^*(-2) = 1-\frac{2^3}{3}+\frac{1}{2}2^2-\frac{1}{12}(2\cdot 2) = 0$ (the first trivial zero of $\zeta(s)$)

for $\zeta^*(\frac{1}{2})$ (with R=3) we have:

$$\zeta^*\left(\frac{1}{2}\right) = 1+2^{-\frac{1}{2}}+3^{-\frac{1}{2}}-\frac{4^{\frac{1}{2}}}{\frac{1}{2}}+\frac{1}{2}4^{-\frac{1}{2}}-\frac{1}{12}\left[-\frac{1}{2}\bullet 4^{-1,5}\right]+$$
$$\frac{1}{720}\left[-\frac{1}{2}\cdot -1,5\cdot -2,5\cdot 4^{-3,5}\right]+\ldots=$$
$$= -1,460354\ldots$$

To deduce (III) it was taken into account that $\frac{\omega^{-s+1}}{-s+1}$ it is the primitive of the (non standard) function n^{-s}calculates at the ω point.If you want to find the same formula (III),but obtained with methods of classical analysis,and in ways very different from those of this article see [3].Moreover for the error term to be attributed to (III),where the derivatives of odd order do not vanish,as in the last example seen above, following Edwards [3] “it’s a general rule of thumb in applying the (E.M.F.) that as long as terms are decreasing rapidly in size,the bulk of the errors is in the term omitted".Now we prove (I).Stated harmonic series with alternate signs :

$$S = \sum_{1}^{\infty}\frac{(-1)^{n+1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ \ldots = In2 \qquad (1)$$

extending it at “non standard model” we have :

$$\sum_{1}^{\omega}\frac{(-1)^{n+1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ \ldots = In2 \qquad (1)^*$$

with ω = infinite number positive .We call “non standard rearrangement” series of (1)* a series with both the conditions following ,the first condition is: "the rearranged series of (1)* must consist in a permutation of all standard terms (i.e with form $\frac{1}{N}$) contained in (1)* " the second condition is : "the rearranged series must consist in a permutation of all non standard terms (i.e with form $\frac{1}{N^*}$) contained in (1)*and no another terms "(N is the Natural numbers set , N^* is the infinite hypernatural numbers set i.e the extension of N at non standard model of analysis).Since real numbers are a subset of hyperreals (see extension principle) the non standard rearrangement could be defined by a single condition ,but to Better highlight the difference between two rearrangements two were written.We give an example of standard rearrangement (using only the first condition) and correspondent “non standard rearrangement” (using the first and the second condition) It knows that:

$$(1-\frac{1}{2}-\frac{1}{4})+(\frac{1}{3}-\frac{1}{6}-\frac{1}{8})+(\frac{1}{5}-\frac{1}{10}-\frac{1}{12})+\ldots\ldots=$$

$$=\sum_{n=1}^{\infty}(\frac{1}{2n-1}-\frac{1}{4n-2}-\frac{1}{4n}) = \underline{S} = \frac{1}{2}S = \frac{In2}{2} \qquad (2).$$

(2) is an example of standard rearrangement of (1).According to Riemann-Dini ‘s theorem the result of (2) is different from result of (1),(the value of (2) i.e $\frac{In2}{2}$ can be easily obtained by Ohm' s rearrangement theorem as we will see shortly).On the contrary using the first and second condition we rewrite (2)* as :

$$\sum_{1}^{\frac{\omega}{2}}\frac{1}{2n-1}-\sum_{1}^{\frac{\omega}{4}}\frac{1}{4n-2}-\sum_{1}^{\frac{\omega}{4}}\frac{1}{4n}$$

considering that: $\sum_{1}^{\frac{\omega}{4}}\frac{1}{4n-2}=\frac{1}{2}\sum_{1}^{\frac{\omega}{4}}\frac{1}{2n-1}$

we have:

$$\frac{1}{2}\left(\sum_{1}^{\frac{\omega}{4}}\frac{1}{2n-1}-\sum_{1}^{\frac{\omega}{4}}\frac{1}{2n}\right)+\sum_{\frac{\omega}{4}+1}^{\frac{\omega}{2}}\frac{1}{2n-1} \qquad (2)^{*} \quad \text{with :}$$

$$\left(\sum_{1}^{\frac{\omega}{4}}\frac{1}{2n-1}-\frac{1}{2n}\right) = S = In2$$

the value of this convergent series is $In2$ (it is the harmonic series with alternating signs calculated until to ω/4, by the Cauchy convergent test in non standard analysis [3] we have that the sum of harmonic series with alternating signs is invariant ,to less than infinitesimals ,from the particular infinite number that we use for to calculate it ,for example as in this case ω/4,or others infinite numbers as ω/2,or ω,or ω^2,or....etc) and using E.M.F.(Euler-Maclaurin formula):

$$\sum_{\frac{\omega}{4}+1}^{\frac{\omega}{2}}\frac{1}{2n-1}\sim\int_{\frac{\omega}{4}}^{\frac{\omega}{2}}\frac{1}{2n-1}dn=\frac{1}{2}In2$$ (~ sign of equivalence that is the two members are equal to less than infinitesimals) therefore we have:

$$\frac{1}{2}In2+\frac{1}{2}In2=S=In2 \qquad (2)^{*}.$$

This result confirms validity of commutative property of addition in opposition to Riemann-Dini’s theorem (or Riemann's rearrangement theorem) in fact:

$$\sum_{\frac{\omega}{2}+1}^{\omega}\frac{1}{2n-1}-\sum_{\frac{\omega}{4}+1}^{\omega}\frac{1}{4n-2}-\sum_{\frac{\omega}{4}+1}^{\omega}\frac{1}{4n}=-\frac{1}{2}In2 \qquad (3)^{*}$$

Since $(3)^{*}$ does not include in $(1)^{*}$ we must remove it (according to the second conditions before seen) from the simple extension at non standard model of (2) that is :

$$\sum_{1}^{\omega}\left(\frac{1}{2n-1}-\frac{1}{4n-2}-\frac{1}{4n}\right) \qquad (2^{bis})\ *$$

therefore we have :

$$(2^{bis})\ *-(3)\ *=(2)\ *=S=In2$$

Now we generalize and formalize the two previous condition which together constitute the new concept of "non standard rearrangement"of a series,for the first condition that characterize the classical standard rearrangement of a series we have:give the series$\sum_k a_k$ with real or complex terms and one bijective function $\pi : N \to N$ it's called rearranged series of $\sum_k a_k$ according π , the series $\sum_k a_{\pi(k)}$.

The second condition is the following:given the series $\sum_k a_k{}^*$ with non standard and complex terms (infinitesimal complex and infinitesimal real numbers) and one bijective non standard function (see definition and preliminaries)

$\pi^* : N^* \to N^*$ (N^* are also called infinite hypernatural numbers [6]).It is called rearranged series according to π^* the series:

$$\sum_k a_{\pi^*(k)}$$

The biunivocity of π and π^* ensure in particular that the number of terms of the rearranged series have the same number of terms as the originals series respectively in$N$ and in N^*.Note that Riemann - Dini theorem concerns simply convergent series in R, while the concept of "non standard rearrangement" is easily extendable at C and C*.It is shown in standard analysis (Ohm's rearrangement theorem) that in (1) taking p positive and q negative terms we have the sum S = ln(2)+(1/2)ln(p/q).In the previous standard example it was p = 1 and q = 2 instead with our definition of "non standard rearrangement" it is always $p^* = q^*$ both infinite numbers in the previous non standard example it was p^* = ω/2 and q^* = ω/4+ω/4 = ω/2 .The validity of the commutative property in (1)* should not be surprising as this property is obviously valid for real numbers,according to the transfer principle it is also valid for hyperreal numbers in fact we have:

$(\forall x\ \forall y) \in R\ (x+y=y+x)$

it is true in R,extending it at non standard model we have :

$(\forall x^* \ \forall y^*) \in R^* (x^*+y^*=y^*+x^*)$
(R^*= set of hyperreal numbers)
(Following common usage we omit the asterisk for the sum between hyperreal numbers).Why is Riemann -Dini theorem valid with " standard rearrangement" despide the commutativity of addition?The answer is simple

in fact formalizing this theorem we have: let $\sum_k a_k$ a simply convergent series

$(\forall x \in R \cup \{-\infty, +\infty\} \ \exists \pi : N \leftrightarrow N)$:

$$\lim\ N \rightarrow \infty \sum_{k=1}^{N} a_{\pi(k)} = x$$

Extending at non standard model we have:

$(\forall x^* \in R^* \ \exists \pi^* : N^* \leftrightarrow N^*)$:

$$\sum_{k=1}^{\omega} a_{\pi^*(k)} = x^*$$

It is precisely the limit with $N \rightarrow \infty \ \sum_{k=1}^{N} a_{\pi(k)}$ or in non standard model $\sum_{k=1}^{\omega} a_{\pi^*(k)}$ that introduces an infinite quantity of infinitesimals whose sum is different from zero (and from being infinitesimal) not existing in the original series (1)* these quantities are excluded only with the" non standard rearrangement" as seen in (3)*.Now we study the following series with alternate signs:

$$\eta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \ ... \qquad (s \in C)$$

it is convergent (with $a > 0$) [2] and we have

$$\eta(s) = (1 + \frac{1}{2^s} + \frac{1}{3^s} + \ ...) - 2(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + ...) =$$

$$= \zeta(s) - \frac{2\zeta(s)}{2^s} = (1 - \frac{2}{2^s})\ \zeta(s) \qquad (4)$$

but extending (4) at non standard model we have:

$$\eta^*(s)= \sum_{1}^{\omega} n^{-s} - 2(\sum_{1}^{\omega} (2n)^{-s}) \qquad (4)^*$$

but the last series on the right following the second condition of the " non standard rearrangement" seen before is not correct (in fact the number of positive terms is p* = ω while the number of negative terms is q* = 2ω) since all terms of:

$\sum_{\frac{\omega}{2}+1}^{\omega} (2n)^{-s}$ are not in $\zeta^*(s) = \sum_{1}^{\omega} n^{-s}$

therefore the correct formula is:

$$\eta^*_{correct}(s) = \sum_{1}^{\omega} n^{-s} - \frac{2}{2^s}(\sum_{1}^{\omega} n^{-s} - \sum_{\frac{\omega}{2}+1}^{\omega} n^{-s}) \qquad (5)^*$$

(in (5)* we have now p* = q* = ω.Furthermore since $(5)^*$ is convergent with $a >^* 0$, we can apply the Cauchy convergent test which extended at non standard model [6] tells us (as seen before) that the value of (5)* i.e.the standard part [6][7] of (5)* (or the finite part of (5)*) is invariant for the particular infinite number that we use for p* e q* for example 2ω, ω/2, ω,... ω^2 etc , the thing important for the " non standard rearrangement" of (5)* is that always is p* = q*) therefore we have:

$\zeta^*(s)= \frac{\eta^*(s)}{1-\frac{2}{2^s}}$ with our "correction" is $\zeta^*(s) = \frac{\eta^*(s)_{correct}}{1-\frac{2}{2^s}} =$

$$= \sum_{1}^{\omega} n^{-s} + \frac{1}{2^{s-1}-1} \bullet \sum_{\frac{\omega}{2}+1}^{\omega} n^{-s}$$

using (E.M.F.) we have

$$\sum_{\frac{\omega}{2}+1}^{\omega} n^{-s} \sim \left\| \frac{n^{-s+1}}{-s+1} \right\|_{\frac{\omega}{2}}^{\omega} = \frac{\omega^{-s+1}}{-s+1} - \frac{(\frac{\omega}{2})^{-s+1}}{-s+1}$$

with $s \neq 1$

at last we have (after simple calculations):

$$\zeta^{*}_{correct}(s) = \zeta^{*}(s) = \sum_{n=1}^{\omega} n^{-s} - \frac{\omega^{-s+1}}{-s+1} \qquad (I)$$

with $s \in C^{*}$ and $\neq 1$ (with $a^{*} > 0$).Now we demonstrate using (I) and (II) Riemann hypothesis (RH).However, let us first prove some facts concerning the zeros of $\zeta(s)$ (such facts can be extended to $\zeta^{*}(s)$ in a trivial way).In (II) replacing s with 2n+1 we have :

$$\zeta(-2n) = 2^{-2n}\pi^{-2n-1}\sin(-n\pi)(-2n)!\zeta(2n+1) \qquad (6).$$

The only factor that can be canceled at the second member of (6) is

$$\sin(-n\pi) = 0 \quad \forall n \in N$$

therefore (6) admits zeros with $s = -2n$ ($-2, -4, -6, \ldots -2n$),they are called trivial zeros of $\zeta(s)$,and they do not concern the (RH) [2].It is easy to prove using the Euler's product formula [1][2][3][8]

$$\zeta(s) = \prod_{p=primes} \frac{p^{s}}{p^{s}-1} \text{ with } a > 1, \quad (s \in C)$$

(p = primes = 2,3,5,7,….) that doesn't exist $s \in C$ for which p^{s}=0 (since the numerator never vanishes).Therefore $\zeta(s)$ with $a > 1$ does not admit zeros,and using (II) we have that $\zeta(s)$ does not admit zeros with $a < 0$ (except for the trivial zeros).Therefore all non trivial zeros of $\zeta(s)$ are in the range $0 \leq a \leq 1$.In 1896 La Vallée' Poussin[8] and Jacques Hadamard [4] independently proved that $\zeta(s)$ has not zeros with a = 1,and therefore using (II) it has not zeros even with a = 0, so all non trivial zeros are in the range $0 < a < 1$ (this range is called "the critical strip" [2][3]) and we can use in this range the formula (I) and the corresponding equation below:

$$\zeta^{*}(1-s) = \sum_{1}^{\omega} n^{s-1} - \frac{\omega^{s}}{s} \qquad \text{with } s \in C^{*} \text{ and } s \neq 0 \qquad (IV)$$

(IV) has a simple pole at $s = 0$,obtained from (I) replacing s with $1-s$ in fact (IV) is valid with $a^{*} < 1$.Now let s_k be the (non trivial) zeros of $\zeta^{*}(s)$ (with $k \in$ N=1,2,3,...). In (II)* since in the critical strip the only factor than can be canceled is precisely $\zeta^{*}(1 - s_k)$ [2]

therefore we have: $\zeta^{*}(s_k) = \zeta^{*}(1 - s_k) = 0 \qquad (V)$.

(V) is valid $\forall s_k : \zeta^*(s_k) = 0$ with $0 *< a_k *< 1$.

Obviously by (II)* we mean the extension of (II) at non- standard model i.e.

$\zeta^*(1\text{-}s) = 2^{1-s}\ \pi^{-s}\ \sin^*(\frac{1-s}{2}\pi)(s - 1)!^*\ \zeta^*(s) \quad s \in C^*$ (II)*

For (II)* as for all complex functions extended to the non- standard model,the obvious property applies $f^*(s) = f(s)\ \forall s \in C$ (if f(x) is real function $f^*(x) = f(x)\ \forall x \in R$). By means of (V) we have that if $s_k = \frac{1}{2} + ib_k$ is a zero of $\zeta^*(s)$ then also $s_k = \frac{1}{2} - ib_k$ will be a zero of $\zeta^*(s)$, but this is true also for all possible non trivial zeros by reflection principle [3] (of schwartz) , i.e the non trivial zeros arrange themselves in pairs of conjugate complex numbers [2] .Using (I) and (IV) into (V) we obtain (with $s = s_k$):

$$\sum_{1}^{\omega} n^{-s_k} - \frac{\omega^{-s_k+1}}{-s_k+1} = \sum_{1}^{\omega} n^{s_k-1} - \frac{\omega^{s_k}}{s_k} = 0 \qquad \text{(VI).}$$

Since the zeta function is a monodromous function [9] that is one value in the domain corresponds to one and only one value in the codomain,by the means of (V) we have that the four elements of (VI) are also monodromous.It is easy to prove that (VI) is solved with

$- s_k = s_k - 1$ or $- a_k \pm ib_k = a_k - 1 \pm ib_k$

Separating the (hyper)real and the (hyper)imaginary part we have:

$- a_k = a_k - 1 \to a_k = \frac{1}{2}$; $\pm b_k i = \pm b_k i \quad (it\ is\ always\ walid\,)$

Therefore (VI) is solved precisely by value admitted by the RH ($a_k = \frac{1}{2}$) for the non trivial zeros of $\zeta^*(s)$ (naturally also the real number $\frac{1}{2} \in R^*$ it is a finite hyperreal).But of course the (RH) absolutely demands that all non-trivial zeros must lie on the critical line $a_k = \frac{1}{2}$ in this regard it is good to recall a work of Hardy [5] in which he demonstrates that $\zeta(s)$ with $a_k = \frac{1}{2}$ admits infinitely many zeros.But infinitely many zeros do not mean all!Alternatively (and equivalently) the (RH) can thus be stated “no zero of $\zeta^*(s)$ with $a_k{}^* \neq \frac{1}{2}$ can exist in the critical stripe” and this will be what we will prove.Note that (V) is invariant under substitution : $s_k = 1 - s_k$ this substitution admits $s_k = \frac{1}{2} \pm b_k i$ as the only possible value for (V) as we will now demonstrate (that is we demonstrate the Riemann hypothesis).Now let $s_k = \frac{1}{2} + \varepsilon \pm b_k i$ with ε real number $0 \leq \varepsilon < \frac{1}{2}$ (the case with ε positive infinitesimal we will see later) if (VI) admits $s_k = \frac{1}{2} + \varepsilon \pm b_k i$ with $\varepsilon = 0$ as the only one solution

then (RH) is valid,if instead (VI) admits also only one solution with $\varepsilon \neq 0$ *then* (RH) *is* certainly false.Let therefore be fixed

$$s_k = \frac{1}{2} + \varepsilon \pm b_k i \quad \text{with } (0 \leq \varepsilon < \tfrac{1}{2}) \qquad \text{(VII)}$$

(VI) in addition to trivial solution given equalizing (I) and (IV) to zero, admits another simple solution,it is easy to see that the other simple solution of (VI) is given by following chain of equalities :

$$\sum_{1}^{\omega} n^{-s_k} = \frac{\omega^{-s_k+1}}{-s_k+1} = \sum_{1}^{\omega} n^{s_k-1} = \frac{\omega^{s_k}}{s_k} \qquad \text{(VIII).}$$

Since into (VIII) no element can be zero, because we are in the critical strip,the (VIII) is the last possible solution of (VI) as it is easy verify, in fact for graphic simplicity we denote the four elements of (VI) in an orderly manner with A,B,C,D we can write (VI) as:

A−B=C−D=0 ⇒

A=B & C=D in particular from chain of equalities (VI) we have:

A−B=C−D putting B and C to the other member :

A−C =B−D that admits an ulterior solution (equalizing both members to zero) A=C & B=D together the two solutions can be written as: A=B=C=D that is (VIII), and considering that by (VIII) in particular is : $\frac{\omega^{-s_k+1}}{-s_k+1} = \frac{\omega^{s_k}}{s_k}$

(IX) dividing both the members of (IX) by the first member and inserting (VII) into (IX) we have : $\frac{\left(\frac{1}{2} - \varepsilon \pm ib_k\right)}{\left(\frac{1}{2} + \varepsilon \pm ib_k\right)} \omega^{2\varepsilon} = 1.$ (IX)

To the first member of (IX) the term $\omega^{2\varepsilon} = e^{2\varepsilon ln\omega}$ with $\varepsilon > 0$ is always infinite number (this result for the extension principle is valid also for every positive hyperreal numbers excluding the case with ε positive infinitesimal number that for definition is smaller than every positive real numbers even if $>^* 0$ this case will be seen later) while the other term is finite term non -zero, therefore (IX) (and therefore (VI) too) is not satisfied with $\varepsilon\ number\ real > 0$. For this the only admissible value for (IX) (and therefore for (VI) too) is $\varepsilon = 0$, in fact $\omega^{2\varepsilon}$ with $\varepsilon = 0$ is $e^{0ln\omega} = e^{0} = 1$.But this condition is necessary but not yet sufficient.In fact for (IX) to be satisfied it,must have for solution $\varepsilon = 0$ also the following finite term no-zero before seen:

$$\frac{\left(\frac{1}{2} - \varepsilon \pm ib_k\right)}{\left(\frac{1}{2} + \varepsilon \pm ib_k\right)} = 1$$ which admits

effectively exactly the solution with $\varepsilon = 0$ (just substitute the value $\varepsilon = 0$ in the above equation).Since as just demonstrated there are no zeros of $\zeta^*(s)$ with $s_k = \frac{1}{2} + \varepsilon \pm ib_k$ with $\varepsilon \neq 0$ then for (II)* there can be no zeros of $\zeta^*(s)$ with $s_k = \frac{1}{2} - \varepsilon \pm i\ b_k$ $with\ \varepsilon \neq 0$, $therefore\ now$ $\forall\ s_k$ with $0 <^* a_k^* < 1$ (excluding the case with ε positive infinitesimal in (VII) and therefore in (IX)).The hypothesis remains that (IX) is also resolved with ε positive infinitesimal ,but this is not possible since the ratio of the two quantities in brackets in the first member of (IX) can never be a non- complex hyperreal number (i.e. without imaginary unit i) as instead is the second member of (IX) (i.e. the number 1),except in the case before seen with ε =0, or the case with $b_k = 0$ ($but\ there\ is\ not$ non trivial zero for zeta function with $b = 0$), while $\omega^{2\varepsilon} = e^{2\varepsilon ln\omega}$ using Taylor series expansion for the extension principle we have : $e^{2\varepsilon ln\omega} = 1 + 2\varepsilon ln\omega + 2\varepsilon^2 ln^2\omega$ + ... is not a complex number (it is a pure hyperreal number or ∈ R*). For this reason finally the Riemann hypothesis is proven to be true, more precisely we have demonstrate that an extension at non standard model of $\zeta(s)$ admits all its non trivial zeros only with $a_k^* = \frac{1}{2}$ this result can be transferred to the standard function $\zeta(s)$ we in fact can write : be $s_k \in C^*$ and $0 <^* a_k^* < 1$ $(\forall\ s_k :\ \zeta^*(s_k) = 0) \rightarrow\ a_k^* = \frac{1}{2}$

or in standard model : be $s_k \in C$ and $0 < a_k < 1$

$(\forall s_k : \zeta(s_k)=0) \rightarrow a_k = \frac{1}{2}$. Since the first proposition is demonstrate true in C* the second proposition is true in C therefore Riemann hypothesis is demonstrate true also in C.Furthermore according to (VIII) the demonstration of the Riemann hypothesis is also equivalent to demonstration that the following equalities :

$$\sum_{1}^{\omega} n^{-s_k} = \sum_{1}^{\omega} n^{s_k - 1} \qquad \text{(X)}$$

is valid only with the value given by (VII) with $\varepsilon = 0$, or $s_k = \frac{1}{2} \pm b_k i \ \forall s_k$ with $0 \ {}^{*}\!< a_k {}^{*}\!< 1$. In fact (X) is equivalently by means (VIII) to (IX) where we already have demonstrated that the only admissible value is precisely that required for the validity of the Riemann hypothesis .It 's immediate to see by (VIII) that the two series of (X) are divergent since they are evaluated in the critical stripe . However (X) can be written equivalently if you prefer as :

$$\sum_{1}^{\omega} (n^{-s_k} - n^{s_k - 1}) = 0. \qquad \text{(X).}$$

So this one series has a finite value. (X) finally in standard model becomes:

$$\sum_{1}^{\infty} (n^{-s_k} - n^{s_k - 1}) = 0. \quad \text{with} \quad s_k \in C \ \text{and} \quad 0 < a_k < 1$$

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