The fundamental structure of quantum algorithms for finding the roots of a polynomial function

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Abstract

The Abel–Ruffini theorem (also known as Abel's impossibility theorem) states that there is no solution in radicals to general polynomial equations of degree five or higher with arbitrary coefficients. Here we solve partly the mathematical problem by finding the roots of the following example polynomial function $f(x) = (x+2)(x-3)(x-4)(x-6)(x-8)$ without knowing the five roots. We first propose necessary and sufficient conditions for root-finding problem. These are the essences of quantum algorithms for finding the roots of a polynomial function $f(x) = x^m + a_{m-1}x^{m-1} + ... + a_1 x + a_0$. As a result, we find a simple relation between the root-finding problem and the oracle. Let r be a root. Then $f(r) = 0$, thus the oracle becomes the identity operator I. That is, $U_f = U_0 = I$ if f is zero. In more detail, the phase kickback does not occur if f is zero. The phase kickback occurs if f is not zero. Here all the roots are in the real numbers $\mathbf{R}$. All the roots are different numbers and the number of the roots is m. The essence of our deep desire to describe our honest view is a new insight into root-finding problem for future studies over the academic world as we discover straightly the importance of the phase kickback in the root-finding problems.

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I. INTRODUCTION

The great success of quantum mechanics (cf. [1–8]) is recognized by the scientific community for physical theories. Between the articles of research for constructing theoretical quantum algorithms [9] it may be mentioned as follows: In 1985, the Deutsch algorithm was introduced and constructed for the function property problem [10–12]. In 1993, the Bernstein–Vazirani algorithm was proposed for identifying linear functions [13, 14]. Generalization of the Bernstein–Vazirani algorithm beyond qubit systems is reported [15]. In 1994, Simon's algorithm [16] and Shor's algorithm [17] were discussed for period finding of periodic functions. In 1996, Grover [18] provided an algorithm for unordered object finding and the motivation for exploring the computational possibilities offered by quantum mechanics. In 2020, a parallel computation for all of the combinations of values in variables of a logical function was proposed by Nagata and Nakamura [19]. In 2021, concrete quantum circuits for addition of two numbers of arbitrary length were discussed by Nakamura and Nagata [20].

Continuous-variable quantum information is the area of quantum information science that makes use of physical observables, such as the strength of an electromagnetic field, whose numerical values belong to continuous intervals. In 1998, Braunstein studied error correction for continuous quantum variables [21] and quantum error correction for communication with linear optics [22]. In 1999, Lloyd and Braunstein proposed quantum computation over continuous variables [23]. The same year, Ralph considered continuous-variable quantum cryptography [24]. In 2000, Hillery discussed quantum cryptography with squeezed states [25], while Reid described quantum cryptography with a predetermined key using continuous-variable Einstein-Podolsky-Rosen correlations [26].

In 2001, secure quantum key distribution using squeezed states was studied by Gottesman and Preskill [27]. A year later, continuous-variable quantum cryptography using coherent states was first proposed by Grosshans and Grangier [28]. Efficient classical simulation of continuous-variable quantum information processes is studied by Bartlett, Sanders, Braunstein, and Nemoto [29]. Continuous-variable quantum computing and its applications to cryptography are discussed by Diep, Nagata, and Wong [30].

Recently, Nagata and Nakamura discuss a quantum algorithm for storing the roots of a polynomial function by using the generalized Bernstein–Vazirani algorithm [31]. They restrict themselves to an assumption that all the roots are in the integers $\mathbf{Z}$. Here, all the roots considered here are extending to the real numbers $\mathbf{R}$. All the roots are different numbers. Is there a quantum algorithm for finding such roots of the polynomial function? The essence of our deep desire to describe our honest view is a new insight into root-finding problem for future studies over the academic world.

In this paper, we first propose necessary and sufficient conditions for the root-finding problem. A quantum algorithm for finding the roots of a polynomial function $f(x) = x^m + a_{m-1}x^{m-1} + ... + a_1x + a_0$ is studied in term of the phase kickback as an application of the necessary and sufficient condition. As a result, we find a simple relation to the root-finding problem. Here all the roots are in the real numbers $\mathbf{R}$. All the roots are different numbers and the number of the roots is m. The Abel–Ruffini theorem states that there is no solution in radicals to general polynomial equations of degree five or higher with arbitrary coefficients. Here we solve partly the mathematical problem by finding the roots of the following example polynomial function $f(x) = (x + 2)(x - 3)(x - 4)(x - 6)(x - 8)$ without knowing the five roots. The essence of our deep desire to describe our honest view is a new insight into root-finding problem for future studies over the academic world as we discover straightly the importance of the phase kickback in the root-finding problems.

II. NECESSARY AND SUFFICIENT CONDITIONS FOR THE ROOT-FINDING PROBLEM

Let us consider necessary and sufficient conditions for finding the roots of a polynomial function $f(x) = x^m + a_{m-1}x^{m-1} + ... + a_1x + a_0$. Here all the roots are in the real numbers $\mathbf{R}$. All the roots are different numbers and the number of the roots is m. That is, $r_1 < r_2 < ... < r_m, r_j \in \mathbf{R}$, $f(x) \in \mathbf{R}$, $x \in \mathbf{R}$, and $a_j \in \mathbf{R}$.

Here the problem is of searching necessary and sufficient conditions for finding the roots of the polynomial function in the interval $[-p, +p] (0 \leq p < +\infty, p \in \mathbf{R})$. We suppose $-p < r_1 < r_2 < ... < r_m < p$. We introduce a natural number d and suppose the following relation:

$$d \geq 2. \tag{1}$$

Let us discuss the structure of quantum computing. To this end, we introduce the transformation U_f (using the polynomial function f) defined by the mapping

$$U_f|x\rangle|j\rangle = |x\rangle|(|f(x)| + j) \bmod d\rangle, \tag{2}$$

where $|f(x)|$ is the absolute value of the function $f(x)$. We define a quantum state $|\phi_d\rangle$ as:

$$|\phi_d\rangle = \frac{1}{\sqrt{d}} \int_0^d dj \omega(d)^{d-j} |j\rangle, \tag{3}$$

where $\omega(d) = e^{2\pi i/d}$. By the phase kickback [32] (See Appendix A) we have the following relation:

$$U_f|x\rangle|\phi_d\rangle = \omega(d)^{|f(x)|}|x\rangle|\phi_d\rangle. \tag{4}$$

Notice that

$$(U_f)^d|x\rangle|j\rangle = |x\rangle|(d|f(x)| + j) \bmod d\rangle = |x\rangle|j\rangle. \tag{5}$$

Therefore, the mapping U_f is a cyclic transformation. Here, we define the input state as:

$$|\psi\rangle_d = \int_{-p}^{+p} dx|x\rangle|\phi_d\rangle. \tag{6}$$

By applying U_f, to $|\psi\rangle_d$, we obtain the following output state by the phase kickback:

$$U_f|\psi\rangle_d = \int_{-p}^{+p} dx\omega(d)^{|f(x)|}|x\rangle|\phi_d\rangle. \tag{7}$$

Thus, by looking at the state $U_f|\psi\rangle_d$, we see the phase factor $\omega(d)^{|f(x)|}$. Again, we define the input state as (d and e are relatively prime and $d < e$):

$$|\psi\rangle_e = \int_{-p}^{+p} dx|x\rangle|\phi_e\rangle. \tag{8}$$

By applying U_f, to $|\psi_0\rangle$, we obtain the following output state by the phase kickback:

$$U_f|\psi\rangle_e = \int_{-p}^{+p} dx\omega(e)^{|f(x)|}|x\rangle|\phi_e\rangle. \tag{9}$$

Thus, by looking at the state $U_f|\psi\rangle_e$, we see the phase factor $\omega(e)^{|f(x)|}$. We have several necessary and sufficient conditions for finding all the roots of a polynomial function.

Theorem

$$\begin{aligned} &|f(r)| = 0 \\ &\Leftrightarrow \omega(d)^{|f(r)|} = 1 \wedge \omega(e)^{|f(r)|} = 1 \\ &\Leftrightarrow U_f = I \\ &\Leftrightarrow U_f|\psi\rangle_d = |\psi\rangle_d \wedge U_f|\psi\rangle_e = |\psi\rangle_e, \end{aligned} \tag{10}$$

where d and e are relatively prime and $d < e$, I is an identity operator, and r is a root of $f(x)$. First, we have

$$|f(r)| = 0 \Rightarrow \omega(d)^{|f(r)|} = 1 \wedge \omega(e)^{|f(r)|} = 1. \tag{11}$$

If $|f(r)| = 0$, then $\omega(d)^0 = 1$ and $\omega(e)^0 = 1$. And

$$|f(r)| = 0 \Leftarrow \omega(d)^{|f(r)|} = 1 \wedge \omega(e)^{|f(r)|} = 1. \tag{12}$$

If $\omega(d)^{|f(r)|} = 1$, then $|f(r)| = 0$ or $|f(r)| = dp, (p = 1, 2, 3, ...)$. If $\omega(e)^{|f(r)|} = 1$, then $|f(r)| = 0$ or $|f(r)| = eq, (q = 1, 2, 3, ...)$. d and e are relatively prime and $d < e$. Thus $|f(r)| = dp$ and $|f(r)| = eq$ are not realized simultaneously. Therefore, $\omega(d)^{|f(r)|} = 1 \wedge \omega(e)^{|f(r)|} = 1$ implies $|f(r)| = 0$.

Moreover, we have

$$\omega(d)^{|f(r)|} = 1 \wedge \omega(e)^{|f(r)|} = 1 \Leftarrow U_f|\psi\rangle_d = |\psi\rangle_d \wedge U_f|\psi\rangle_e = |\psi\rangle_e. \tag{13}$$

We define the input state as follows:

$$|\psi\rangle_d = \int_{-p}^{+p} dx|x\rangle|\phi_d\rangle. \tag{14}$$

By applying U_f, to $|\psi\rangle_d$, we obtain the following output state by the phase kickback:

$$U_f|\psi\rangle_d = \int_{-p}^{+p} dx\omega(d)^{|f(x)|}|x\rangle|\phi_d\rangle. \tag{15}$$

Thus, by looking at the state $U_f|\psi\rangle_d$, we see the phase factor $\omega(d)^{|f(x)|}$. Thus, we have

$$U_f|\psi\rangle_d = |\psi\rangle_d \Rightarrow \int_{-p}^{+p} dx\omega(d)^{|f(x)|}|x\rangle|\phi_d\rangle = \int_{-p}^{+p} dx|x\rangle|\phi_d\rangle \Rightarrow \omega(d)^{|f(x)|} = 1. \tag{16}$$

Similarly, we have, using e,

$$U_f|\psi\rangle_e = |\psi\rangle_e \Rightarrow \omega(e)^{|f(x)|} = 1. \tag{17}$$

Therefore, $\omega(d)^{|f(r)|} = 1 \wedge \omega(e)^{|f(r)|} = 1 \Leftarrow U_f|\psi\rangle_d = |\psi\rangle_d \wedge U_f|\psi\rangle_e = |\psi\rangle_e$.

Further, we see

$$U_f = I \Rightarrow U_f|\psi\rangle_d = |\psi\rangle_d \wedge U_f|\psi\rangle_e = |\psi\rangle_e. \tag{18}$$

If $U_f = I$, then $U_f|\psi\rangle_d = |\psi\rangle_d$ and $U_f|\psi\rangle_e = |\psi\rangle_e$. Finally, we have

$$|f(r)| = 0 \Rightarrow U_f = I. \tag{19}$$

If $|f(r)| = 0$, then $U_f|r\rangle|j\rangle = |r\rangle|(|f(r)| + j) \bmod d\rangle = |r\rangle|j\rangle$.

Thus, we prove the theorem (10).

III. APPLICATION OF THE NECESSARY AND SUFFICIENT CONDITION

Let us consider a quantum algorithm for finding the roots of a polynomial function $f(x) = x^m + a_{m-1}x^{m-1} + ... + a_1x + a_0$. We use necessary and sufficient conditions for finding all the m roots of the polynomial function. See the theorem (10). Here all the roots are in the real numbers $\mathbf{R}$. All the roots are different numbers and the number of the roots is m. That is, $-p < r_1 < r_2 < ... < r_m < p$. Here the problem is of searching quantum algorithm for finding the roots of the polynomial function.

We define a quantum state $|\phi_d\rangle$ as:

$$|\phi_d\rangle = \frac{1}{\sqrt{d}} \int_0^d dj\omega(d)^{d-j}|j\rangle, \tag{20}$$

where $\omega(d) = e^{2\pi i/d}$. By the phase kickback [32] (See Appendix A) we have the following relation:

$$U_f|x\rangle|\phi_d\rangle = \omega(d)^{|f(x)|}|x\rangle|\phi_d\rangle. \tag{21}$$

Here, we define the input state as:

$$|\psi\rangle_d = \int_{-p}^{+p} dx|x\rangle|\phi_d\rangle. \tag{22}$$

By applying U_f, to $|\psi\rangle_d$, we obtain the following output state by the phase kickback:

$$U_f|\psi\rangle_d = \int_{-p}^{+p} dx\omega(d)^{|f(x)|}|x\rangle|\phi_d\rangle. \tag{23}$$

Thus, by looking at the state $U_f|\psi\rangle_d$, we see the phase factor $\omega(d)^{|f(x)|}$. If r is a root of $f(x)$, then $f(r) = 0$. Thus, by looking at the state $U_f|\psi\rangle_d$, we do not see the phase factor $\omega(d)^{|f(r)|}$. There are m points, in the interval $[-p, +p]$, such that

$$U_f|\psi\rangle_d = |\psi\rangle_d. \tag{24}$$

Similarly, we have, using e,

$$U_f|\psi\rangle_e = |\psi\rangle_e. \tag{25}$$

Hence we have

$$(U_f|\psi\rangle_d - |\psi\rangle_d) + (U_f|\psi\rangle_e - |\psi\rangle_e) = \begin{cases} \int_{-p}^{+p} dx\{\omega(d)^{|f(x)|} - 1\}|x\rangle|\phi_d\rangle \\ + \int_{-p}^{+p} dx\{\omega(e)^{|f(x)|} - 1\}|x\rangle|\phi_e\rangle & \text{if } f(x) \neq 0, \\ \mathbf{0} & \text{if } f(x) = 0. \end{cases} \tag{26}$$

Therefore, we determine all the m roots by evaluating the relation (26). As a result, we find a simple relation (26) to the root-finding problem. Let r be a root. Then $f(r) = 0$, thus the oracle becomes the identity operator I. That is, $U_f = U_0 = I$ if f is zero. In more detail, the phase kickback does not occur if f is zero. The phase kickback occurs if f is not zero. When $f(x) = 0$ we have

$$(U_f|\psi\rangle_d - |\psi\rangle_d) + (U_f|\psi\rangle_e - |\psi\rangle_e) = \mathbf{0}, \tag{27}$$

then, x is a root, that is, $x = r_k (k = 1, 2, ..., m)$. When $f(x) \neq 0$ we have

$$\begin{aligned}(U_f|\psi\rangle_d - |\psi\rangle_d) + (U_f|\psi\rangle_e - |\psi\rangle_e) = & \int_{-p}^{+p} dx\{\omega(d)^{|f(x)|} - 1\}|x\rangle|\phi_d\rangle \\ & + \int_{-p}^{+p} dx\{\omega(e)^{|f(x)|} - 1\}|x\rangle|\phi_e\rangle \quad (\neq \mathbf{0}),\end{aligned} \tag{28}$$

then, x is not a root. We investigate the quantum state $((U_f|\psi\rangle_d - |\psi\rangle_d) + (U_f|\psi\rangle_e - |\psi\rangle_e))$ in the interval $[-p, +p]$. Then we can pick up the m roots. Thus, we find the m roots in the mathematical logical sense.

IV. A SIMPLE CONCRETE EXAMPLE OF OUR ALGORITHM

The Abel–Ruffini theorem states that there is no solution in radicals to general polynomial equations of degree five or higher with arbitrary coefficients. Here we solve partly the mathematical problem by finding the roots of the following example polynomial function $f(x) = (x + 2)(x - 3)(x - 4)(x - 6)(x - 8)$ without knowing the five roots.

As a very simple concrete example we give a quantum algorithm for finding the roots of a polynomial function $f(x) = (x + 2)(x - 3)(x - 4)(x - 6)(x - 8)$. We search, for example, for roots in the interval $[-5, +5], (p = 5)$, that is, $-2, 3$, and 4.

Let us consider a quantum algorithm for finding the roots (except for 6 and 8) of a polynomial function $f(x) = (x + 2)(x - 3)(x - 4)(x - 6)(x - 8)$. We use necessary and sufficient conditions for finding the three roots of the polynomial function. See the theorem (10). Here all the roots are in the real numbers $\mathbf{R}$. All the roots are different numbers and the number of the roots is three. That is, $-5 < -2 < 3 < 4 < +5$. Here the problem is of searching quantum algorithm for finding the roots of the polynomial function.

We define a quantum state $|\phi_2\rangle$ as:

$$|\phi_2\rangle = \frac{1}{\sqrt{2}} \int_0^2 dj\omega(2)^{2-j}|j\rangle, \tag{29}$$

where $\omega(2) = e^{i\pi} = -1$. By the phase kickback [32] (See Appendix A) we have the following relation:

$$U_f|x\rangle|\phi_2\rangle = \omega(2)^{|f(x)|}|x\rangle|\phi_2\rangle. \tag{30}$$

Here, we define the input state as:

$$|\psi\rangle_2 = \int_{-5}^{+5} dx|x\rangle|\phi_2\rangle. \tag{31}$$

By applying U_f, to $|\psi\rangle_2$, we obtain the following output state by the phase kickback:

$$U_f|\psi\rangle_2 = \int_{-5}^{+5} dx\omega(2)^{|f(x)|}|x\rangle|\phi_2\rangle. \tag{32}$$

Thus, by looking at the state $U_f|\psi\rangle_2$, we see the phase factor $\omega(2)^{|f(x)|}$. If r is a root of $f(x)$, then $f(r) = 0$. Thus, by looking at the state $U_f|\psi\rangle_2$, we do not see the phase factor $\omega(2)^{|f(r)|}$. There are three points, in the interval $[-5, +5]$, such that

$$U_f|\psi\rangle_2 = |\psi\rangle_2. \tag{33}$$

Similarly, we have, using $3(=e)$,

$$U_f|\psi\rangle_3 = |\psi\rangle_3. \tag{34}$$

Hence we have

$$(U_f|\psi\rangle_2 - |\psi\rangle_2) + (U_f|\psi\rangle_3 - |\psi\rangle_3) = \begin{cases} \int_{-5}^{+5} dx\{\omega(2)^{|f(x)|} - 1\}|x\rangle|\phi_2\rangle & \\ + \int_{-5}^{+5} dx\{\omega(3)^{|f(x)|} - 1\}|x\rangle|\phi_3\rangle & \text{if } \ f(x) \neq 0, \\ \mathbf{0} & \text{if } \ f(x) = 0. \end{cases} \tag{35}$$

Therefore, we determine the three roots by evaluating the relation (35). As a result, we find a simple relation (35) to the root-finding problem. Let r be a root. Then $f(r) = 0$, thus the oracle becomes the identity operator I. That is, $U_f = U_0 = I$ if f is zero. In more detail, the phase kickback does not occur if f is zero. The phase kickback occurs if f is not zero. When $f(x) = 0$ we have

$$(U_f|\psi\rangle_2 - |\psi\rangle_2) + (U_f|\psi\rangle_3 - |\psi\rangle_3) = \mathbf{0}, \tag{36}$$

then, x is a root, that is, $x = -2, 3$, and 4. When $f(x) \neq 0$ we have

$$\begin{aligned}(U_f|\psi\rangle_2 - |\psi\rangle_2) + (U_f|\psi\rangle_3 - |\psi\rangle_3) = & \int_{-5}^{+5} dx\{\omega(2)^{|f(x)|} - 1\}|x\rangle|\phi_2\rangle \\ & + \int_{-5}^{+5} dx\{\omega(3)^{|f(x)|} - 1\}|x\rangle|\phi_3\rangle \ \ (\neq \mathbf{0}),\end{aligned} \tag{37}$$

then, x is not a root. Let us investigate the quantum state $((U_f|\psi\rangle_2 - |\psi\rangle_2) + (U_f|\psi\rangle_3 - |\psi\rangle_3))$ in the interval $[-5, +5], (p = 5)$.

- If $x = -2$ then we have

$$(U_f|\psi\rangle_2 - |\psi\rangle_2) + (U_f|\psi\rangle_3 - |\psi\rangle_3) = \mathbf{0}. \tag{38}$$

- If $x = 3$ then we have

$$(U_f|\psi\rangle_2 - |\psi\rangle_2) + (U_f|\psi\rangle_3 - |\psi\rangle_3) = \mathbf{0}. \tag{39}$$

- If $x = 4$ then we have

$$(U_f|\psi\rangle_2 - |\psi\rangle_2) + (U_f|\psi\rangle_3 - |\psi\rangle_3) = \mathbf{0}. \tag{40}$$

- If $(x \neq -2, 3, \text{ and } 4) \wedge (-5 \leq x \leq +5)$ then we have

$$(U_f|\psi\rangle_2 - |\psi\rangle_2) + (U_f|\psi\rangle_3 - |\psi\rangle_3) \neq \mathbf{0}. \tag{41}$$

Then we can pick up the three roots $(-2, 3, \text{ and } 4)$. Thus, we find the three roots in the mathematical logical sense.

We find the remained roots 6 and 8 using the quadratic formula for solving equations. Finally, we find all the roots $-2, 3, 4, 6$, and 8.

V. CONCLUSIONS

We first have proposed necessary and sufficient conditions for the root-finding problem. A quantum algorithm for finding the roots of a polynomial function $f(x) = x^m + a_{m-1}x^{m-1} + ... + a_1x + a_0$ has been studied in term of the phase kickback as an application of the necessary and sufficient condition. As a result, we have found a simple relation to the root-finding problem. Here all the roots have been in the real numbers $\mathbf{R}$. All the roots have been different numbers and the number of the roots is m. The essence of our deep desire to describe our honest view has been a new insight into root-finding problem for future studies over the academic world as we discovered straightly the importance of the phase kickback in the root-finding problems. As a very simple concrete example we have given a quantum algorithm for finding the roots (except for 6 and 8) of a polynomial function $f(x) = (x + 2)(x - 3)(x - 4)(x - 6)(x - 8)$. We have found the remained roots 6 and 8 using the quadratic formula for solving equations. Finally, we have found all the roots $-2, 3, 4, 6$, and 8.

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Ethical approval

The authors are in an applicable thought to ethical approval.

Competing interests

The authors state that there is no conflict of interest.

Author contributions

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Data availability

No data associated in the manuscript.

Appendix A: The phase kickback

We have the following equation by the phase kickback [32]:

$$U_f|x\rangle|\phi_d\rangle = \omega(d)^{|f(x)|}|x\rangle|\phi_d\rangle. \tag{A1}$$

where $\omega(d) = e^{2\pi i/d}$ and $|f(x)|$ is the absolute value of $f(x)$. In what follows, we discuss the rationale behind the above relation (A1). Consider the action of the U_f gate on the state $|x\rangle|\phi_d\rangle$. Each term in $|\phi_d\rangle$ is of the form $\omega^{d-j}|j\rangle$. We observe that

$$U_f\omega^{d-j}|x\rangle|j\rangle = \omega^{d-j}|x\rangle|(|f(x)| + j) \bmod d\rangle. \tag{A2}$$

A variable k is introduced such that $|f(x)| + j = k$, from which it follows that $d - j = d + |f(x)| - k$. Thus, (A2) becomes

$$U_f\omega^{d-j}|x\rangle|j\rangle = \omega^{|f(x)|}\omega^{d-k}|x\rangle|k \bmod d\rangle. \tag{A3}$$

If $k < d$ we have that $|k \bmod d\rangle = |k\rangle$ and thus the terms in $|\phi_d\rangle$ for which $k < d$ are transformed as follows:

$$U_f\omega^{d-j}|x\rangle|j\rangle = \omega^{|f(x)|}\omega^{d-k}|x\rangle|k\rangle. \tag{A4}$$

On the other hand, as both $|f(x)|$ and j are bounded from above by d, k is strictly less than $2d$. Thus, when $d \leq k < 2d$, we have $|k \bmod d\rangle = |k - d\rangle$. Let $k - d = m$. We have

$$\begin{aligned}&\omega^{|f(x)|}\omega^{d-k}|x\rangle|k \bmod d\rangle = \omega^{|f(x)|}\omega^{-m}|x\rangle|m\rangle \\ &= \omega^{|f(x)|}\omega^{d-m}|x\rangle|m\rangle.\end{aligned} \tag{A5}$$

Hence, the terms in $|\phi_d\rangle$ for which $k \geq d$ are transformed as follows:

$$U_f \omega^{d-j} |x\rangle|j\rangle = \omega^{|f(x)|} \omega^{d-m} |x\rangle|m\rangle. \tag{A6}$$

Finally, regarding (A4) and (A6), we have

$$U_f |x\rangle|\phi_d\rangle = \omega^{|f(x)|} |x\rangle|\phi_d\rangle. \tag{A7}$$

Therefore, the relation (A1) holds.

REFERENCES

[1] R. P. Feynman, R. B. Leighton, and M. Sands, "Lectures on Physics, Volume III, Quantum Mechanics," Addison-Wesley Publishing Company (1965).
[2] J. J. Sakurai, "Modern Quantum Mechanics," (Addison-Wesley Publishing Company, 1995), Revised ed.
[3] A. Peres, "Quantum Theory: Concepts and Methods," (Kluwer Academic, Dordrecht, The Netherlands, 1993).
[4] M. Redhead, "Incompleteness, Nonlocality, and Realism," (Clarendon Press, Oxford, 1989), 2nd ed.
[5] J. von Neumann, "Mathematical Foundations of Quantum Mechanics," (Princeton University Press, Princeton, New Jersey, 1955).
[6] M. A. Nielsen and I. L. Chuang, "Quantum Computation and Quantum Information," (Cambridge University Press, 2000).
[7] A. S. Holevo, "Quantum Systems, Channels, Information, A Mathematical Introduction," (De Gruyter, 2012). https://doi.org/10.1515/9783110273403
[8] K. Nagata, D. N. Diep, A. Farouk, and T. Nakamura, "Simplified Quantum Computing with Applications," (IOP Publishing, Bristol, UK, 2022). https://iopscience.iop.org/book/mono/978-0-7503-4700-6
[9] R. Rennie (Editor), "Oxford dictionary of physics," (Oxford University Press, 2015), Seventh ed.
[10] D. Deutsch, "Quantum theory, the Church-Turing principle and the universal quantum computer," Proc. R. Soc. Lond. A **400**, 97 (1985). https://doi.org/10.1098/rspa.1985.0070
[11] D. Deutsch and R. Jozsa, "Rapid solution of problems by quantum computation," Proc. R. Soc. Lond. A **439**, 553 (1992). https://doi.org/10.1098/rspa.1992.0167
[12] R. Cleve, A. Ekert, C. Macchiavello, and M. Mosca, "Quantum algorithms revisited," Proc. R. Soc. Lond. A **454**, 339 (1998). https://doi.org/10.1098/rspa.1998.0164
[13] E. Bernstein and U. Vazirani, "Quantum complexity theory," Proceedings of 25th Annual ACM Symposium on Theory of Computing (STOC '93), p. 11 (1993). https://doi.org/10.1145/167088.167097
[14] E. Bernstein and U. Vazirani, "Quantum Complexity Theory," SIAM J. Comput. **26**, 1411 (1997). https://doi.org/10.1137/S0097539796300921
[15] K. Nagata, H. Geurdes, S. K. Patro, S. Heidari, A. Farouk, and T. Nakamura, "Generalization of the Bernstein-Vazirani algorithm beyond qubit systems," Quantum Stud.: Math. Found. **7**, 17 (2020). https://doi.org/10.1007/s40509-019-00196-4
[16] D. R. Simon, "On the power of quantum computation," Proceedings of 35th IEEE Annual Symposium on Foundations of Computer Science, p. 116 (1994). https://doi.org/10.1109/SFCS.1994.365701
[17] P. W. Shor, "Algorithms for quantum computation: discrete logarithms and factoring," Proceedings of 35th IEEE Annual Symposium on Foundations of Computer Science, p. 124 (1994). https://doi.org/10.1109/SFCS.1994.365700
[18] L. K. Grover, "A fast quantum mechanical algorithm for database search," Proceedings of 28th Annual ACM Symposium on Theory of Computing, p. 212 (1996). https://doi.org/10.1145/237814.237866
[19] K. Nagata and T. Nakamura, "Some Theoretically Organized Algorithm for Quantum Computers," Int. J. Theor. Phys. **59**, 611 (2020). https://doi.org/10.1007/s10773-019-04354-7
[20] T. Nakamura and K. Nagata, "Physics' Evolution Toward Computing," Int. J. Theor. Phys. **60**, 70 (2021). https://doi.org/10.1007/s10773-020-04661-4
[21] S. L. Braunstein, "Error Correction for Continuous Quantum Variables," Phys. Rev. Lett. **80**, 4084 (1998). https://doi.org/10.1103/PhysRevLett.80.4084
[22] S. L. Braunstein, "Quantum error correction for communication with linear optics," Nature (London) **394**, 47 (1998). https://doi.org/10.1038/27850
[23] S. Lloyd and S. L. Braunstein, "Quantum Computation over Continuous Variables," Phys. Rev. Lett. **82**, 1784 (1999). https://doi.org/10.1103/PhysRevLett.82.1784
[24] T. C. Ralph, "Continuous variable quantum cryptography," Phys. Rev. A **61**, 010303(R) (1999). https://doi.org/10.1103/PhysRevA.61.010303
[25] M. Hillery, "Quantum cryptography with squeezed states," Phys. Rev. A **61**, 022309 (2000). https://doi.org/10.1103/PhysRevA.61.022309
[26] M. D. Reid, "Quantum cryptography with a predetermined key, using continuous-variable Einstein-Podolsky-Rosen correlations," Phys. Rev. A **62**, 062308 (2000). https://doi.org/10.1103/PhysRevA.62.062308

[27] D. Gottesman and J. Preskill, "Secure quantum key distribution using squeezed states," Phys. Rev. A **63**, 022309 (2001). https://doi.org/10.1103/PhysRevA.63.022309
[28] F. Grosshans and P. Grangier, "Continuous Variable Quantum Cryptography Using Coherent States," Phys. Rev. Lett. **88**, 057902 (2002). https://doi.org/10.1103/PhysRevLett.88.057902
[29] S. D. Bartlett, B. C. Sanders, S. L. Braunstein, and K. Nemoto, "Efficient Classical Simulation of Continuous Variable," Quantum Information Processes. Phys. Rev. Lett. **88**, 097904 (2002). https://doi.org/10.1103/PhysRevLett.88.097904
[30] D. N. Diep, K. Nagata, and R. Wong, "Continuous-variable quantum computing and its applications to cryptography," Int. J. Theor. Phys. **59**, 3184 (2020). https://doi.org/10.1007/s10773-020-04571-5
[31] K. Nagata and T. Nakamura "Quantum algorithm for the root-finding problem," Quantum Studies: Mathematics and Foundations, Volume 6, Issue 1 (2019), pp. 135–139. https://doi.org/10.1007/s40509-018-0171-0
[32] K. Nagata, H. Geurdes, S. K. Patro, S. Heidari, A. Farouk, and T. Nakamura, "Quantum Algorithm for Determining a Complex Number String," Int. J. Theor. Phys. **58**, 3694 (2019). https://doi.org/10.1007/s10773-019-04239-9