BEYOND GÖDEL

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ABSTRACT

In 1931, Gödel wrote a landmark paper exhibiting a formal system that contains statements that cannot be proven. However, the deficiency of formal systems is much greater. In this paper starting with the same set, we develop two logically valid arguments simultaneously, proving a statement to be both true and false. This result is possible because we validly draw a false conclusion from true premises. This result is profound. It is no wonder that the most lucid arguments still sometimes result in hung juries, and earnest people can disagree on the most fundamental issues. Truth is a much deeper concept than logical validity. Inconsistency in mathematical systems has a long heritage. From the earliest times division by zero led to many contradictions, which could only be rectified by banning division by zero. Gottlob Frege developed the first formal system of arithmetic at the beginning of the twentieth century. However, it contains a contradictory self-referential set. Gödel proved that any formal system containing arithmetic (although not necessarily inconsistent) is incomplete.

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1 Introduction

Inconsistency in mathematical systems has a long heritage. From the earliest times division by zero led to many contradictions [1], which could only be rectified by banning division by zero.

$$\text{(start with) } x = 1 \tag{1}$$

$$\text{(multiply by x) } x^2 = x \tag{2}$$

$$\text{(subtract 1) } x^2 - 1 = x - 1 \tag{3}$$

$$\text{(divide by x - 1 or 0) } x + 1 = 1 \tag{4}$$

$$\text{(subtract 1 and end with) } x = 0 \tag{5}$$

Gottlob Frege developed the earliest formal system of arithmetic at the beginning of the twentieth century. However, it contained a contradictory self-referential set [2]: the set of all sets that do not contain themselves. Does it contain itself? If it does, then it does not. If it does not, then it does. Gödel proved that any formal system containing arithmetic (though not necessarily inconsistent) is incomplete [3].

We make a logically valid argument that sets of rationals $(0, a)$ with $a < 100$ have largest elements. Many people will find it repugnant that a false statement has been validly deduced from previous true statements. However, perhaps people will become more tolerant when they realize that in discussing anything there may be a logically valid argument using the same premises arriving at an opposite conclusion. This also shows that truth is a much deeper concept than logical validity.

2 Inconsistency

For rational numbers *a* in (*0, 100*) let the collection of R*a* sets be $\{y$ is a rational number $\mid 0 \leq y < a\}$

Argument#1: No R*a* set contains a largest element.
1) Suppose there is a largest element *b* in some individual R*a*.
2) $b < (a+b)/2 < a$.
3) Let *c = (a + b) / 2.*
4) Then *c* is in R*a* and $b < c$.
5) Therefore, no R*a* set contains a largest element.
Argument#2: Each R*a* set contains a largest element.
1) Below each R*a* for all rationals $x < a$ is a collection of R*x* subsets $\{z$ is a rational number $\mid 0 \leq z < x\}$.
2) Each R*a* set and its collection of R*x* subsets comprise a nested descending set hierarchy with R*a* at the top.
3) Any chosen R*a* set contains the x indices missing from all the R*x* subsets below it in the set hierarchy.
4) Since any chosen R*a* set contains the x indices missing from all the R*x* subsets below it, the union of the R*x* subsets of a chosen R*a* set does not contain all elements of the chosen set.
5) There is at least one element $s \geq$ all elements in the subsets of the chosen set.
6) Let *e* and *f* be two elements of the chosen set with $e > f$.
7) f is an element of R*e*, which is a proper subset of the chosen set.
8) For any two chosen set elements the smaller element is contained in an R*x* subset of the chosen set.
9) By Steps 6), 7), and 8), there is at most one chosen set element missing from all its R*x* subsets.
10) By Steps 5) and 9), the chosen set contains a largest element *b* not in any *Rx* subset below it in the hierarchy.
11) There is no *c = (a + b) / 2* as in Argument#1. It would be a second element not in an Rx subset below the chosen set, which we know from step 8) is not possible.

3 Conclusion

Argument#1 is correct and its conclusion is true. In argument#2 the first three statements of are true, but the fourth statement is false. No infinite set of rationals in an open interval can have a largest element. Most people dismiss argument#2. They are unable to accept that a false statement can be a valid logical deduction drawn from true statements that directly proceed it. If the rationals in *(0, 100)* are replaced by natural numbers, argument#2 remains exactly the same and no one questions its validity. We use the first three statements in argument#2 on the Ra sets of natural numbers, and we conclude that the union of the R*x* set collection does not equal R*a*. This is true, since the Ra sets of natural numbers are finite. However, the first three statements in argument#2 are equally valid for the Ra sets of rational numbers. If statement#4 is a valid deduction for the Ra sets of natural numbers, then it is an equally valid (but false) deduction for the Ra sets of rational numbers.

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