COMPLEX CIRCLES OF PARTITION AND THE ASYMPTOTIC BINARY GOLDBACH CONJECTURE

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Abstract. In this work, we continue the complex circle of partition development that was started in our foundational study [3]. With regard to a cCoP and its embedding circle, we define interior and exterior points. On this foundation, we expand the concept of point density, established in [2], to include complex circles of partition. We propose the idea of a quotient complex circle of partition and investigate some of its features in analogy to the quotient group in group theory. With this notion we can prove an asymptotic version of the Binary Goldbach Conjecture.

1. Introduction and Preliminaries

The Goldbach conjecture was born in 1742 through a correspondence between the German mathematician Christian Goldbach and the Swiss mathematician Leonard Euler. There are two known versions of the problem: the binary case and the ternary situation. The binary version ask whether every even number greater than 6 can be represented as the sum of two primes, whereas the ternary version ask whether every odd number greater than 7 can be expressed as the sum of three primes. The ternary version, however, was very recently solved in the preprint [4] that compiled and build on several chain of works. Although the binary problem has not been solved yet, significant strides have been made on its variations. The first significant step in this direction can be found in (see [7]), which demonstrates that every even number can be expressed as the sum of at most C primes, where C is a practically computable constant. In the early twentieth century, G.H Hardy and J.E Littlewood assuming the Generalized Riemann hypothesis (see [9]), showed that the number of even numbers $\leq X$ and violating the binary Goldbach conjecture is much less than $X^{\frac{1}{2}+c}$, where c is a small positive constant. Using sieve theory techniques, Jing-run Chen [5] showed that every even number can either be written as a sum of two prime numbers or a prime number and a number which is a product of two primes. It is well known that almost all even numbers can be expressed as the sum of two prime numbers, with the density of even numbers representable in this fashion being one [8], [1]. It is also known that there exists a constant K such that any even number can be expressed as the sum of two prime numbers and a maximum of K powers of two, where $K = 13$ [6].

We devised a method that we believe could be a useful tool and a recipe for analyzing issues pertaining to the partition of numbers in designated subsets of $\mathbb{N}$

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in our work [2], which was partially inspired by the binary Goldbach conjecture and its variants. The technique is fairly simple, and it is similar to how the points on a geometric circle can be arranged. In [3], we have improved this strategy by switching from integer base sets to special complex number subsets. As a result, the *complex circle of partition* structure was defined (cCoP). The interior and exterior points of cCoPs as well as various applications are now introduced as we continue this work.

In an effort to make our work more self-contained, we have chosen to provide a little background of the method of circles of partition in the following sequel

Definition 1.1. Let $n \in \mathbb{N}$ and $\mathbb{M} \subseteq \mathbb{N}$. We denote with

$$\mathcal{C}(n,\mathbb{M}) = \{[x] \mid x,y \in \mathbb{M}, n = x+y\}$$

the **Circle of Partition** generated by n with respect to the subset $\mathbb{M}$. We will abbreviate this in the further text as CoP. We call members of $\mathcal{C}(n,\mathbb{M})$ as points and denote them by $[x]$. For the special case $\mathbb{M} = \mathbb{N}$ we denote the CoP shortly as $\mathcal{C}(n)$. We denote with $\|[x]\| := x$ the **weight** of the point $[x]$ and correspondingly the weight set of points in the CoP $\mathcal{C}(n,\mathbb{M})$ as $\|\mathcal{C}(n,\mathbb{M})\|$. Obviously holds

$$\|\mathcal{C}(n)\| = \{1,2,\ldots,n-1\}.$$

Definition 1.2. We denote the line $\mathbb{L}_{[x],[y]}$ joining the point $[x]$ and $[y]$ as an axis of the CoP $\mathcal{C}(n,\mathbb{M})$ if and only if $x+y=n$. We say the axis point $[y]$ is an axis partner of the axis point $[x]$ and vice versa. We do not distinguish between $\mathbb{L}_{[x],[y]}$ and $\mathbb{L}_{[y],[x]}$, since it is essentially the the same axis. The point $[x] \in \mathcal{C}(n,\mathbb{M})$ such that $2x = n$ is the **center** of the CoP. If it exists then we call it as a **degenerated axis** $\mathbb{L}_{[x]}$ in comparison to the **real axes** $\mathbb{L}_{[x],[y]}$. We denote the assignment of an axis $\mathbb{L}_{[x],[y]}$ to a CoP $\mathcal{C}(n,\mathbb{M})$ as

$$\mathbb{L}_{[x],[y]} \,\hat{\in}\, \mathcal{C}(n,\mathbb{M}) \text{ which means } [x],[y] \in \mathcal{C}(n,\mathbb{M}) \text{ with } x+y=n.$$

In the following we consider only real axes. Therefore we abstain from the attribute *real* in the sequel.

Proposition 1.3. *Each axis is uniquely determined by points* $[x] \in \mathcal{C}(n,\mathbb{M})$.

Proof. Let $\mathbb{L}_{[x],[y]}$ be an axis of the CoP $\mathcal{C}(n,\mathbb{M})$. Suppose as well that $\mathbb{L}_{[x],[z]}$ is also an axis with $z \neq y$. Then it follows by Definition 1.2 that we must have $n = x+y = x+z$ and therefore $y = z$. This cannot be and the claim follows immediately. ☐

Corollary 1.4. *Each point of a CoP* $\mathcal{C}(n,\mathbb{M})$ *except its center has exactly one axis partner.*

Proof. Let $[x] \in \mathcal{C}(n,\mathbb{M})$ be a point without an axis partner being not the center of the CoP. Then holds for every point $[y] \neq [x]$ except the center

$$x+y \neq n.$$

This is a contradiction to the Definition 1.1. Due to Proposition 1.3 the case of more than one axis partners is impossible. This completes the proof. ☐

Notation. We denote by

$$\mathbb{N}_n = \{m \in \mathbb{N} \mid \ m \leq n\} \tag{1.1}$$

the **sequence** of the first n natural numbers. We denote the assignment of an axis $\mathbb{L}_{[x],[y]}$ resp. $\mathbb{L}_{[x]}$ to a CoP $\mathcal{C}(n,\mathbb{M})$ as

$\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M})$ which means $[x],[y] \in \mathcal{C}(n,\mathbb{M})$ and $x+y=n$ resp.

$\mathbb{L}_{[x]} \hat{\in} \mathcal{C}(n,\mathbb{M})$ which means $[x] \in \mathcal{C}(n,\mathbb{M})$ and $2x = n$

and the number of real axes of a CoP as

$$\nu(n,\mathbb{M}) := \#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M}) \mid x < y\}.$$

Obviously holds

$$\nu(n,\mathbb{M}) = \left\lfloor \frac{k}{2} \right\rfloor, \text{ if } |\mathcal{C}(n,\mathbb{M})| = k.$$

For any $f,g : \mathbb{N} \longrightarrow \mathbb{N}$, we write $f(n) \sim g(n)$ if and only if $\lim\limits_{n\longrightarrow\infty} \frac{f(n)}{g(n)} = 1$. We also write $f(n) = o(1)$ if and only if $\lim\limits_{n\longrightarrow\infty} f(n) = 0$.

The *complex circle of partition* approach is an extension of the method of *circle of partition* which is based on the following definition.

Definition 1.5. Let $\mathbb{M} \subseteq \mathbb{N}$ and

$$\mathbb{C}_{\mathbb{M}} := \{z = x + iy \mid x \in \mathbb{M}, y \in \mathbb{R}\} \subset \mathbb{C}$$

be a subset of the complex numbers where the real part is from $\mathbb{M} \subseteq \mathbb{N}$. Then a CoP with a special requirement

$$\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}}) = \{[z] \mid z, n-z \in \mathbb{C}_M, \Im(z)^2 = \Re(z)\,(n - \Re(z))\}$$

will be denoted as a **complex Circle of Partition**, abbreviated as **cCoP**. The special requirement will be called as the *circle condition*.

The components x and y we will call as *real weight* resp. *imaginary weight*. The CoP $\mathcal{C}(n,\mathbb{M})$ will be called as the *source CoP*.

In order to distinguish between points $[z]$ of cCoPs and points z in the complex plane $\mathbb{C}$ we denote the latter as *complex points*.

In the sequel we give a short outline about the basic properties of complex circles of partition.

The most important property is that all members of a cCoP are located on a circle in the complex plane $\mathbb{C}$ that has its center on the real axis at $\frac{n}{2}$ and has a diameter n. This is at once the length of each axis of $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$.

To each axis of a cCoP there exists a conjugate axis. For axis partners holds

$$\Im(z) = -\Im(n - z). \tag{1.2}$$

The circle in the complex plane with center on the real axis at $\frac{n}{2}$ and diameter n is called as the *embedding circle* $\mathfrak{C}_n$ of the cCop $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$. Any two, as well as all, embedding circles have only the origin as common point. Therefore any two cCoPs have no common point.

The length of a chord between any two points $[z_1] = [x_1+iy_1]$ and $[z_2] = [x_2+iy_2]$ of a cCoP $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ is given by

$$|\mathcal{L}_{[z_1],[z_2]}| = \Gamma([z_1],[z_2]) = |\sqrt{x_1(n-x_2)} \pm \sqrt{x_2(n-x_1)}|, \tag{1.3}$$

whereby $''-''$ will be taken if $\mathrm{sign}(y_1) = \mathrm{sign}(y_2)$ and $''+''$ else. The chord turns into an axis with length n if $[z_1], [z_2]$ are axis partners.

If $\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ is a non-empty cCoP and $\mathfrak{I}_n$ resp. $\mathfrak{X}_n$ all complex points inside resp. outside of the embedding circle $\mathfrak{C}_n$, then for all complex points of $\mathfrak{I}_n \cap \mathbb{C}_\mathbb{M}$ holds that their distances to each point of $\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ is less than n

$$|z - w| < n \text{ for all } z \in ||\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})|| \text{ and all } w \in \mathfrak{I}_n \cap \mathbb{C}_\mathbb{M}. \tag{1.4}$$

And vice versa holds also

$$|z - w| > n \text{ for some } z \in ||\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})|| \text{ and all } w \in \mathfrak{X}_n \cap \mathbb{C}_\mathbb{M}. \tag{1.5}$$

2. Interior and Exterior Points of Complex Circles of Partition

In this section we introduce and develop the notion of **interior** and **exterior** points of complex circles of partition.

Definition 2.1. Since $\mathfrak{I}_n, \mathfrak{X}_n$ are defined in Definition 2.8 in [3] as all complex points **inside** resp. **outside** of the embedding circle $\mathfrak{C}_n$, we call the points $z \in \mathfrak{I}_n \cap \mathbb{C}_\mathbb{M}$ as **interior** points with respect to $\mathfrak{C}_n$ and denote the set of all such points as $\mathrm{Int}[\mathfrak{C}_n]$.

Correspondingly, we call the complex points $z \in \mathfrak{X}_n \cap \mathbb{C}_\mathbb{M}$ as **exterior** points with respect to $\mathfrak{C}_n$ and denote the set of all these points as $\mathrm{Ext}[\mathfrak{C}_n]$.

Obviously holds

$$\mathrm{Int}[\mathfrak{C}_n] = \mathfrak{I}_n \cap \mathbb{C}_\mathbb{M} \text{ and } \mathrm{Ext}[\mathfrak{C}_n] = \mathfrak{X}_n \cap \mathbb{C}_\mathbb{M}.$$

Definition 2.2. Let $\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ be a non-empty cCoP and $\mathfrak{C}_n$ its embedding circle. Then we call the complex points $z \in \mathrm{Int}[\mathfrak{C}_n]$ as **interior** points with respect to the cCoP $\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ and denote the set of all these points as $\mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})]$ if and only if for **all** points $[w] \in \mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ holds $|z - w| < n$[1].

Correspondingly, we call the complex points $z \in \mathrm{Ext}[\mathfrak{C}_n]$ as **exterior** points with respect to $\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ and denote the set of all these points as $\mathrm{Ext}[\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})]$ if and only if for **some** points $[w] \in \mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ holds $|z - w| > n$.

Let $n_o \in \mathbb{N}$ be the least generator for all cCoPs. If $n > n_o$ and $\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ is an empty cCoP, then $\mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})]$ and $\mathrm{Ext}[\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})]$ are empty too.

Theorem 2.3. *If $\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ is a non-empty cCoP then holds*

$$\begin{gathered}\mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})] = \mathrm{Int}[\mathfrak{C}_n] = \mathfrak{I}_n \cap \mathbb{C}_\mathbb{M} \\ \textit{and} \\ \mathrm{Ext}[\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})] = \mathrm{Ext}[\mathfrak{C}_n] = \mathfrak{X}_n \cap \mathbb{C}_\mathbb{M}.\end{gathered} \tag{2.1}$$

Proof. It suffices to prove that the distances of all complex points z of $\mathfrak{I}_n \cap \mathbb{C}_\mathbb{M}$ to all points $[w] \in \mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ are less than n resp. of $\mathfrak{X}_n \cap \mathbb{C}_\mathbb{M}$ to some points $[w] \in \mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ are greater than n. But this has already been proven in Theorem 3.3 in [3] for $\mathfrak{I}_n$ resp. $\mathfrak{X}_n$ instead of $\mathfrak{I}_n \cap \mathbb{C}_\mathbb{M}$ resp. $\mathfrak{X}_n \cap \mathbb{C}_\mathbb{M}$. Hence the claim is proved. □

[1] $|.|$ means the usual distance between the points z and w in the complex plane $\mathbb{C}$.

Corollary 2.4. *If* $\mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})] \neq \emptyset$ *then* $\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})$ *is non-empty too since there is at least an axis* $\mathbb{L}_{[w],[n-w]} \hat{\in} \mathcal{C}^o(n,\mathbb{C}_\mathbb{M})$ *such that the distances from both axis points to all complex points of* $\mathrm{Int}[\mathfrak{C}_n]$ *are less than* n*.*

Proposition 2.5. *Let* $\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})$ *and* $\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})$ *be two non-empty cCoPs. If and only if* $m < n$ *holds*

$$\mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})] \subset \mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})] \text{ and } \mathrm{Ext}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})] \subset \mathrm{Ext}[\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})].$$

Proof. Let $m < n$, then since (2.1) holds

$$\begin{aligned}\mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})] &= \mathfrak{I}_m \cap \mathbb{C}_\mathbb{M} \text{ and since (2.4) in [3]}\\ &\subset \mathfrak{I}_n \cap \mathbb{C}_\mathbb{M} = \mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})].\end{aligned}$$

Vice versa holds

$$\begin{aligned}\mathrm{Ext}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})] &= \mathfrak{X}_n \cap \mathbb{C}_\mathbb{M} \text{ and since (2.4) in [3]}\\ &\subset \mathfrak{X}_m \cap \mathbb{C}_\mathbb{M} = \mathrm{Ext}[\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})].\end{aligned}$$

On the other hand from $\mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})] \subset \mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})]$ follows $\mathfrak{I}_m \cap \mathbb{C}_\mathbb{M} \subset \mathfrak{I}_n \cap \mathbb{C}_\mathbb{M}$, which is only with $m < n$ solvable. Analogously follows from $\mathrm{Ext}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})] \subset \mathrm{Ext}[\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})]$ also $m < n$. □

Proposition 2.6. *Let* $\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})$ *and* $\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})$ *be two non-empty cCoPs. If and only if* $m < n$ *holds*

$$||\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})|| \subset \mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})] \text{ and } ||\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})|| \subset \mathrm{Ext}[\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})].$$

Proof. Let $m < n$, then since (2.4) in [3] and $||\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})|| \subset \mathbb{C}_\mathbb{M}$ holds

$$\begin{aligned}||\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})|| &\subset \mathfrak{C}_m \cap \mathbb{C}_\mathbb{M}\\ &\subset (\mathfrak{C}_m \cap \mathbb{C}_\mathbb{M}) \cup \mathfrak{I}_m\\ &\subset (\mathfrak{C}_m \cup \mathfrak{I}_n) \cap \mathbb{C}_\mathbb{M} \text{ and since } \mathfrak{C}_m \subset \mathfrak{I}_n\\ &= \mathfrak{I}_n \cap \mathbb{C}_\mathbb{M} \text{ and because of (2.1)}\\ &= \mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})].\end{aligned}$$

In a similar manner $||\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})|| \subset \mathrm{Ext}[\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})]$ can be proved.

On the other hand, the embedding $||\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})|| \subset \mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})]$ implies $\mathfrak{I}_m \cap \mathbb{C}_\mathbb{M} \subset \mathfrak{I}_n \cap \mathbb{C}_\mathbb{M}$, which is only with $m < n$ solvable. Analogously follows from $\mathrm{Ext}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})] \subset \mathrm{Ext}[\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})]$ also $m < n$. □

Definition 2.7. Let $\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})$ be a non-empty cCoP and $[z_1],[z_2] \in \mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})]$. Then we say the line $\mathcal{L}_{[z_1],[z_2]} \in \mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})]$ if and only if it joins the points $[z_1],[z_2] \in \mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})]$.

Next we show that we can use information about the length of an axis of a cCoP and an interior point to determine an exterior point. We summarize this criterion in the following proposition.

Proposition 2.8. *Let* $\mathcal{C}^o(n,\mathbb{C}_\mathbb{M}) \neq \emptyset$*. If* $[z_1],[z_2]$ *are axis partners of the cCoP* $\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})$ *and* $|\mathbb{L}_{[z_1],[z_2]}| = m > n$*, then* $z_2 \in \mathrm{Ext}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})]$*.*

Proof. From the requirement $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ with $m > n$ and Proposition 2.5, it follows that

$$||\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})|| \subset \mathrm{Ext}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})] \text{ and therefore}$$
$$z_2 \in \mathrm{Ext}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})].$$

□

An important feature that governs the landscape of the complex circles of partition is the interplay between the points on the cCoP and their corresponding interior and exterior points. It is always plausible to find an interior with respect to a cCoP that is non-empty. In fact the interior with respect to a non-empty cCoP constitute the entire space bounded by the cCoP. On the other hand, if the interior (resp. exterior) is empty then the cCoP by itself is empty.

Proposition 2.9. *Let $\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}}) \neq \emptyset$. If $\mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})] \subset \mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})]$, then $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}}) \neq \emptyset$.*

Proof. The conditions above with Definition 2.1 implies that $\mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})] \neq \emptyset$ and $\mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})] \supset \emptyset$, and hence $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}}) \neq \emptyset$. □

We state a sort of converse of the above result in the following theorem.

Theorem 2.10. *Let $\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}}), \mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}}) \neq \emptyset$. If $m < n$, then there exists a chord $\mathcal{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ such that the complex points $z_1, z_2 \notin \mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})]$.*

Proof. By virtue of Definition 2.8 in [3] holds $\mathfrak{C}_n \cap \mathfrak{I}_n = \emptyset$ and $||\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})|| \subset \mathfrak{C}_n$, it follows easily that $\mathfrak{I}_n \cap ||\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})|| = \emptyset$. Since $\mathcal{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$, we have $z_1, z_2 \notin \mathfrak{I}_n$ and because of $m < n$ holds $\mathfrak{I}_m \subset \mathfrak{I}_n$ and hence

$$z_1, z_2 \notin \mathfrak{I}_n \supset \mathfrak{I}_m \supset \mathfrak{I}_m \cap \mathbb{C}_{\mathbb{M}} = \mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})].$$

□

3. Quotient Complex Circles of Partition

In this section we introduce and develop the notion of the **quotient** complex circles of partition. This notion is akin to and parallels the notion of quotient groups in group theory.

Definition 3.1. Let $\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}}), \mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}}) \neq \emptyset$ with $\mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})] \subset \mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})]$. Then by the **quotient** cCoP $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})/_z\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ induced by $[z] \in \mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$, we mean the collection of all cCoPs

$$\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})/_z\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}}) := \{\mathcal{C}^o(n_j,\mathbb{C}_{\mathbb{M}}) \mid j = 1,\ldots,k\}$$

determined by the generators

$$n_j = \Re(z) + u_j \mid u_j \in ||\mathcal{C}(m,\mathbb{M})||, j = 1,\ldots,k$$

with $\mathcal{C}(m,\mathbb{M})$ as the source CoP of $\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ and $k = |\mathcal{C}(m,\mathbb{M})|$.

We call the total number of all distinct cCoPs belonging to the **quotient** cCoP $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})/_z\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ induced by the point $[z] \in \mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ the **index** of the $\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ in $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ induced by $[z]$

$$\mathrm{Ind}_z[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}}) : \mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})].$$

We call the union

$$\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})/\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}}) := \bigcup_{[\Re(z)]\in\mathcal{C}(n,\mathbb{M})} \mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})/_z\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$$

a **complete quotient** cCoP. We call the total number of all *distinct* cCoPs in $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})/\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ the **index** of the cCoP $\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ in $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$

$$\mathrm{Ind}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}}) : \mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})].$$

Obviously each member of the collection $\{\mathcal{C}^o(n_j,\mathbb{C}_{\mathbb{M}}) \mid j=1,\ldots,k\}$ has an axis

$$\mathbb{L}_{[z],[w_j]} \hat{\in} \mathcal{C}^o(n_j,\mathbb{C}_{\mathbb{M}}) \text{ with } w_j = u_j + i\Im(w_j) \in ||\mathcal{C}^o(n_j,\mathbb{C}_{\mathbb{M}})||.$$

Lemma 3.2 (The squeeze principle)**.** *Let $\mathcal{C}^o(m,\mathbb{C}_{\mathbb{B}}),\mathcal{C}^o(m+t,\mathbb{C}_{\mathbb{B}}) \neq \emptyset$ with*

$$\mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})] \subset \mathrm{Int}[\mathcal{C}^o(m+t,\mathbb{C}_{\mathbb{M}})]$$

for $t \geq 4$. If $m < s < m+t$ such that s,m,t are of the same parity and $\mathbb{B} \subset \mathbb{M}$ with

$$\{u \in ||\mathcal{C}(m,\mathbb{M})|| \mid u \in \mathbb{B}\} \subseteq \{u \in ||\mathcal{C}(m+t,\mathbb{M})|| \mid u \in \mathbb{B}\}$$

and

$$||\mathcal{C}(m,\mathbb{M})|| \subset ||\mathcal{C}(m+t,\mathbb{M})||$$

and there exists $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m+t,\mathbb{M})$ with $x \in \mathbb{B}$ and $x < y$ such that

$$y > w = \max\{u \in ||\mathcal{C}(m,\mathbb{M})|| \mid u \in \mathbb{B}\} \tag{3.1}$$

and $x > m-w$, then there exists

$$\mathcal{C}^o(s,\mathbb{C}_{\mathbb{M}}) \in \mathcal{C}^o(m+t,\mathbb{C}_{\mathbb{M}})/\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$$

such that

$$\mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})] \subset \mathrm{Int}[\mathcal{C}^o(s,\mathbb{C}_{\mathbb{M}})] \subset \mathrm{Int}[\mathcal{C}^o(m+t,\mathbb{C}_{\mathbb{M}})].$$

Proof. In virtue of (3.1) holds $w \in \mathbb{B}$. As required the axis $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m+t,\mathbb{M})$ exists with $x \in \mathbb{B}$ such that $m-w < x < y$. Then under the requirement

$$\{u \in ||\mathcal{C}(m,\mathbb{M})|| \mid u \in \mathbb{B}\} \subseteq \{u \in ||\mathcal{C}(m+t,\mathbb{M})|| \mid u \in \mathbb{B}\}$$

and

$$\{u \in ||\mathcal{C}(m,\mathbb{M})||\} \subset \{u \in ||\mathcal{C}(m+t,\mathbb{M})||\}$$

we have the inequality

$$\begin{aligned} m = w + (m-w) < w + x = w + (m+t-y) &= m+t+(w-y) \\ &< m+t, \text{ since } y > w \end{aligned} \tag{3.2}$$

and $m-w < x = m+t-y$ holds $y-w < t$. With $w+x=s$ there is an axis $\mathbb{L}_{[x],[w]} \hat{\in} \mathcal{C}(s,\mathbb{B})$ and it follows that $\mathcal{C}(s,\mathbb{B}) \neq \emptyset$ and hence $\mathcal{C}^o(s,\mathbb{C}_{\mathbb{B}}) \neq \emptyset$ with

$$\mathcal{C}^o(s,\mathbb{C}_{\mathbb{M}}) \in \mathcal{C}^o(m+t,\mathbb{C}_{\mathbb{M}})/\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$$

by virtue of our construction and

$$\mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})] \subset \mathrm{Int}[\mathcal{C}^o(s,\mathbb{C}_{\mathbb{M}})] \subset \mathrm{Int}[\mathcal{C}^o(m+t,\mathbb{C}_{\mathbb{M}})]$$

since $\mathcal{C}^o(s,\mathbb{C}_{\mathbb{B}}) \subset \mathcal{C}^o(s,\mathbb{C}_{\mathbb{M}})$ and Proposition 2.6. This completes the proof. ∎

Lemma 3.2 can be viewed as a basic tool-box for studying the possibility of partitioning numbers of a particular parity with components belonging to a special subset of the integers. It works by choosing two non-empty cCoPs with the same base set and finding further non-empty cCoPs with generators trapped in between these two generators. This principle can be used in an ingenious manner to study the broader question concerning the feasibility of partitioning numbers with each summand belonging to the same subset of the positive integers. We launch the following proposition as an outgrowth of Lemma 3.2.

Proposition 3.3 (The interval binary Goldbach partition detector)**.** *Let $\mathbb{P}$ be the set of all prime numbers and $\mathcal{C}^o(m,\mathbb{C}_\mathbb{P}),\mathcal{C}^o(m+t,\mathbb{C}_\mathbb{P})\neq\emptyset$ by $t\geq 4$. If $m<s<m+t$ such that $s,m,t\equiv 0\,(\mathrm{mod}\,2)$ and there exists $\mathbb{L}_{[x],[y]}\,\hat{\in}\,\mathcal{C}(m+t,\mathbb{N})$ with $x\in\mathbb{P}$ and $x<y$ such that*

$$y>w=\max\{u\in||\mathcal{C}(m,\mathbb{N})||\mid u\in\mathbb{P}\}\tag{3.3}$$

and $x>m-w$ then there must exists $m<s<m+t$ such that $\mathcal{C}^o(s,\mathbb{C}_\mathbb{P})\neq\emptyset$.

Proof. This is a consequence of Lemma 3.2 by taking $\mathbb{M}=\mathbb{N}$ and $\mathbb{B}=\mathbb{P}$ since its requirements are satisfied with

$$\begin{aligned}\{u\in||\mathcal{C}(m,\mathbb{N})||\mid u\in\mathbb{P}\}&=\{u\in\mathbb{P}\mid 3\leq u\leq m-1\}\\&\subseteq\{u\in\mathbb{P}\mid 3\leq u\leq m+t-1\}\\&=\{u\in||\mathcal{C}(m+t,\mathbb{N})||\mid u\in\mathbb{P}\}\end{aligned}$$

and

$$||\mathcal{C}(m,\mathbb{N})||=\{1,2,\ldots,m-1\}\subset||\mathcal{C}(m+t,\mathbb{N})||=\{1,2,\ldots,m-1+t\}.$$

And in virtue of Proposition 2.6 due to $m<m+t$ holds also

$$\mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_\mathbb{P})]\subset\mathrm{Int}[\mathcal{C}^o(m+t,\mathbb{C}_\mathbb{P})].$$

□

Proposition 3.4 (Interval Goldbach partition)**.** *Let $\mathbb{P}$ be the set of all prime numbers and $\mathcal{C}^o(m,\mathbb{C}_\mathbb{P}),\mathcal{C}^o(m+t,\mathbb{C}_\mathbb{P})\neq\emptyset$ for $t\geq 4$. If $m-1\in\mathbb{P}$ then there exist some $s\equiv 0\,(\mathrm{mod}\,2)$ with $m<s<m+t$ such that $\mathcal{C}(s,\mathbb{P})\neq\emptyset$.*

Proof. Under the requirements $\mathcal{C}^o(m,\mathbb{C}_\mathbb{P}),\mathcal{C}^o(m+t,\mathbb{C}_\mathbb{P})\neq\emptyset$ for $t\geq 4$ and with w in virtue of (3.3), we choose $\mathbb{L}_{[3],[y]}\,\hat{\in}\,\mathcal{C}(m+t,\mathbb{N})$ so that $w=m-1$ and $y>w$ since $y=m+t-3>m$ for $t\geq 4$ and $m-1\in\mathbb{P}$. The inequality holds

$$y-w=y-(m-1)\leq(m+t-3)-(m-1)<t$$

and the conditions in Proposition 3.3 are satisfied, so that there exists some $s\equiv 0\,(\mathrm{mod}\,2)$ with $m<s<m+t$ such that $\mathcal{C}(s,\mathbb{P})\neq\emptyset$, f.i. $s=3+m-1=m+2$ with $\mathbb{L}_{[3],[m-1]}\,\hat{\in}\,\mathcal{C}(m+2,\mathbb{P})$. □

Theorem 3.5. *Let $\mathbb{P}$ be the set of all prime numbers and $\mathcal{C}^o(m,\mathbb{C}_\mathbb{P}),\mathcal{C}^o(m+t,\mathbb{C}_\mathbb{P})\neq\emptyset$ for $t\geq 4$ such that $m-1\in\mathbb{P}$. Then there are finitely many $s\equiv 0\pmod 2$ with $m<s<m+t$ such that $\mathcal{C}(s,\mathbb{P})\neq\emptyset$.*

Proof. The result is obtained by iterating repeatedly on the generators $s\equiv 0\pmod 2$ with $m<s<m+t$ such that $\mathcal{C}(s,\mathbb{P})\neq\emptyset$. □

Theorem 3.6 (Conditional Goldbach)**.** *Let $\mathbb{P}$ be the set of all prime numbers and $m \in 2\mathbb{N}$ such that $\mathcal{C}(m,\mathbb{P}) \neq \emptyset$ for m **sufficiently** large. If for **all** $t \geq 4$ there exists $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m+t,\mathbb{N})$ with $x \in \mathbb{P}$ and $x < y$ such that*

$$y > w = \max\{u \in ||\mathcal{C}(m,\mathbb{N})|| \mid u \in \mathbb{P}\}$$

and $m - w < x$, then there are CoPs $\mathcal{C}(s,\mathbb{P}) \neq \emptyset$ for all (sufficiently large) $s \in 2\mathbb{N} \mid s > m$.

Proof. It is known that there are infinitely many even numbers that can be written as the sum of two primes, so that for $m \in 2\mathbb{N}$ **sufficiently** large with $\mathcal{C}(m,\mathbb{P}) \neq \emptyset$ then $t \geq 4$ can be chosen **arbitrarily** large such that $\mathcal{C}(m+t,\mathbb{P}) \neq \emptyset$. Under the requirements and appealing to Proposition 3.3 there must exist some $s \equiv 0 \pmod 2$ with $m < s < m+t$ such that $\mathcal{C}(s,\mathbb{P}) \neq \emptyset$. Now we continue our arguments on the intervals of generators $[m,s]$ and $[s,s+r]$. If there exist some $u,v \in 2\mathbb{N}$ such that $m < u < s$ and $s < v < s+r$, then we repeat the argument under the requirements (for arbitrary t) to deduce that $\mathcal{C}(u,\mathbb{P}) \neq \emptyset$ and $\mathcal{C}(v,\mathbb{P}) \neq \emptyset$. We can iterate the process repeatedly so long as there exists some even generators trapped in the following sub-intervals of generators $[m,u],[u,s],[s,v],[v,v+r]$ where $v+r = m+t$ for $t \geq 4$. Since t can be chosen arbitrarily so that $\mathcal{C}(m+t,\mathbb{P}) \neq \emptyset$, the assertion follows immediately. □

To this end and in reference to Theorem 3.5, questions arise pertaining to the best possible strategy to adopt to cover all even numbers in the interval $m < s < m+t$ so that their corresponding CoPs for which they are generators are non-empty. Since there must always exist an even number (many) such that $m-1$ is prime, the strategy above can be used in a fairly ingenious manner to cover all generators, which one would consider to be a full justification of the binary Goldbach conjecture. In the following sequel, we study possible ways of covering even generators that may not be covered in the interval $(m,m+t)$ for $t \geq 4$.

4. Application to the binary Goldbach conjecture

In this section we apply the notion of the quotient complex circles of partition and the squeeze principle to study the binary Goldbach conjecture in the very large. Despite Estermann's 1938 proof (see [1]) that the binary Goldbach conjecture is true for nearly all positive integers, we can use our tool to independently establish the binary Goldbach conjecture in an asymptotical sense. We lay down the following elementary results which will feature prominently in our arguments.

Lemma 4.1 (The prime number theorem)**.** *Let $\pi(m)$ denotes the number of prime numbers less than or equal to m and $p_{\pi(m)}$ denotes the $\pi(m)^{th}$ prime number. Then we have the asymptotic*

$$p_{\pi(m)} \sim m\left(1 - \frac{\log\log m}{\log m}\right).$$

Proof. This is an easy consequence by combining the two versions of the prime number theorem

$$\pi(m) \sim \frac{m}{\log m} \quad \textbf{and} \quad p_k \sim k\log k$$

where p_k denotes the k^{th} prime number. Since with $k = \pi(m)$ we get

$$\begin{aligned} p_k = p_{\pi(m)} &\sim \frac{m}{\log m} \log\left(\frac{m}{\log m}\right) \\ &= \frac{m}{\log m}(\log m - \log\log m) \\ &= m\left(1 - \frac{\log\log m}{\log m}\right). \end{aligned}$$

□

Obviously holds with the variable denotations from the previous section

$$w = \max\{u \in ||\mathcal{C}(m,\mathbb{N}) \mid u \in \mathbb{P}\} = p_{\pi(m)}. \tag{4.1}$$

Lemma 4.2 (Bertrand's postulate)**.** *There exists a prime number in the interval $(k, 2k)$ for all $k > 1$.*

The formula in Lemma 4.1 obviously suggests that the $\pi(m)^{th}$ prime number satisfies and implies the asymptotic relation $p_{\pi(m)} \sim m$. While this is valid in practice, it does not actually help in measuring the asymptotic of the discrepancy between the maximum prime number less than m and m. It gives the misleading impression that this discrepancy has absolute difference tending to zero in the very large. We reconcile this potentially nudging flaw by doing things slightly differently.

Lemma 4.3 (The little lemma)**.** *Let $\mathbb{P}$ be the set of all prime numbers and $m \in \mathbb{N}$ be* ***sufficiently*** *large such that $\mathcal{C}(m,\mathbb{P}) \neq \emptyset$. Then for all $x \in \mathbb{P}$ satisfying $\frac{m\log\log m}{\log m} < x < \frac{m\log(\log m)^2}{\log m}$ the asymptotic and inequalities*

$$m - w \sim \frac{m\log\log m}{\log m}$$

and

$$0 \lesssim |w - (m + t - x)| \lesssim t$$

hold for $t \geq 4$.

Proof. Appealing to the prime number theorem, we obtain with (4.1) the asymptotic inequalities

$$\begin{aligned} m - w &= m - p_{\pi(m)} \\ &\sim m - m(1 - \frac{\log\log m}{\log m}) \\ &= \frac{m\log\log m}{\log m} \end{aligned}$$

for all sufficiently large $m \in 2\mathbb{N}$ and

$$\begin{aligned} m+t-x &> m+t-\frac{m\log(\log m)^2}{\log m} \\ &= m(1-\frac{\log(\log m)^2}{\log m})+t \\ &\sim m+t > p_{\pi(m)} = w \end{aligned}$$

and

$$\begin{aligned} |w-(m+t-x)| &= |m+t-x-p_{\pi(m)}| \\ &< |m+t-\frac{m\log\log m}{\log m}-p_{\pi(m)}| \\ &\sim |m+t-\frac{m\log\log m}{\log m}-m(1-\frac{\log\log m}{\log m})| \\ &= t \end{aligned}$$

for $t \geq 4$. □

We are now ready to prove the binary Goldbach conjecture for all **sufficiently** large even numbers. The following result is a culmination and - to a larger extent - a mishmash of ideas espoused in this paper.

Theorem 4.4 (Asymptotic Goldbach theorem)**.** *Every* ***sufficiently*** *large even number can be written as the sum of two prime numbers.*

Proof. The claim is equivalent to the statement:

For every sufficiently large even number n holds $\mathcal{C}(n,\mathbb{P}) \neq \emptyset$.

It is known that there are infinitely many even numbers $m > 0$ with $\mathcal{C}(m,\mathbb{P}) \neq \emptyset$. Let us choose $m \in 2\mathbb{N}$ **sufficiently** large such that $\mathcal{C}(m,\mathbb{P}) \neq \emptyset$ and choose $t \geq 4$ such that $\mathcal{C}(m+t,\mathbb{P}) \neq \emptyset$. Let us choose a prime number $x < \frac{m\log(\log m)^2}{\log m}$ such that $x > \frac{m\log\log m}{\log m}$, since by Bertrand's postulate (Lemma 4.2) there exists a prime number x such that $x \in (k,2k)$ for every $k > 1$. Then we get for the axis partner $[y]$ of the axis point $[x]$ of $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m+t,\mathbb{N})$ the inequality

$$\begin{aligned} y = m+t-x &> m+t-\frac{m\log(\log m)^2}{\log m} \\ &= m(1-\frac{\log(\log m)^2}{\log m})+t \\ &\sim m+t > p_{\pi(m)} = w \end{aligned}$$

for $t \geq 4$ and by appealing to Lemma 4.3 also the following asymptotic inequalities

$$m-w \sim \frac{m\log\log m}{\log m} < x$$

and

$$|y-w| = |(m+t-x)-w| = |m-w+t-x| \lesssim |x+t-x| = t.$$

Then the requirements in Theorem 3.6 are fulfilled **asymptotically** with

$$y \gtrsim w \text{ and } x \gtrsim m-w \text{ and } 0 \lesssim |y-w| \lesssim t$$

and the result follows by arbitrarily choosing $t \geq 4$ so that $\mathcal{C}(m+t, \mathbb{P}) \neq \emptyset$ and adapting the proof in Theorem 3.6. $\square$

Theorem 4.4 is equivalent to the statement: there must exist some positive constant N such that for all $m \geq N$, then it is always possible to partition every even number m as a sum of two prime numbers. This result - albeit constructive to some extent - looses it's constructive flavour so that we cannot carry out this construction to cover all even numbers, since we are unable to obtain any quantitative (lower) bound for the threshold N. At least, we are able to get a handle on the conjecture **asymptotically**.

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