Proof of Riemann hypothesis

By Toshihiko Ishiwata

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Abstract. This paper is a trial to prove Riemann hypothesis according to the following process. 1. We make the infinite number of infinite series from one equation that gives $\zeta(s)$ analytic continuation to $Re(s) > 0$ and 2 formulas ($1/2 + a + bi$, $1/2 - a - bi$) which show zero point of $\zeta(s)$. 2. We find that the value of $F(a)$ (that is the infinite series regarding a) must be zero from the above infinite number of infinite series. 3. We find that $F(a) = 0$ has the only solution of $a = 0$. 4. Zero point of $\zeta(s)$ must be $1/2 \pm bi$ because a cannot have any value but zero.

1. Introduction

The following (1) gives Riemann zeta function $\zeta(s)$ analytic continuation to $Re(s) > 0$. "$+\cdots\cdots$" means infinite series in all equations in this paper.

$$1 - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - 6^{-s} + \cdots\cdots = (1 - 2^{1-s})\zeta(s) \tag{1}$$

The following (2) shows the zero point of the left side of (1) and also non-trivial zero point of $\zeta(s)$.

$$S_0 = 1/2 + a + bi \tag{2}$$

The range of a is $0 \leq a < 1/2$ by the critical strip of $\zeta(s)$. The range of b is $b > 14$ due to the following reasons. And i is $\sqrt{-1}$.

1.1 [Conjugate complex number of S_0] $= 1/2 + a - bi$ is also zero point of $\zeta(s)$. Therefore $b \geq 0$ is necessary and sufficient range for investigation.

1.2 The range of b of zero points found until now is $b > 14$.

The following (3) also shows zero point of $\zeta(s)$ by the functional equation of $\zeta(s)$.

$$S_1 = 1 - S_0 = 1/2 - a - bi \tag{3}$$

We have the following (4) and (5) by substituting S_0 for s in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero respectively.

$$1 = \frac{\cos(b\log 2)}{2^{1/2+a}} - \frac{\cos(b\log 3)}{3^{1/2+a}} + \frac{\cos(b\log 4)}{4^{1/2+a}} - \frac{\cos(b\log 5)}{5^{1/2+a}} + \cdots\cdots \tag{4}$$

$$0 = \frac{\sin(b\log 2)}{2^{1/2+a}} - \frac{\sin(b\log 3)}{3^{1/2+a}} + \frac{\sin(b\log 4)}{4^{1/2+a}} - \frac{\sin(b\log 5)}{5^{1/2+a}} + \cdots\cdots \tag{5}$$

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We also have the following (6) and (7) by substituting S_1 for s in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero respectively.

$$1 = \frac{\cos(b\log 2)}{2^{1/2-a}} - \frac{\cos(b\log 3)}{3^{1/2-a}} + \frac{\cos(b\log 4)}{4^{1/2-a}} - \frac{\cos(b\log 5)}{5^{1/2-a}} + \cdots\cdots \tag{6}$$

$$0 = \frac{\sin(b\log 2)}{2^{1/2-a}} - \frac{\sin(b\log 3)}{3^{1/2-a}} + \frac{\sin(b\log 4)}{4^{1/2-a}} - \frac{\sin(b\log 5)}{5^{1/2-a}} + \cdots\cdots \tag{7}$$

2. Infinite number of infinite series

We define $f(n)$ as follows.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geq 0 \qquad (n = 2, 3, 4, 5, \cdots\cdots) \tag{8}$$

We have the following (9) from (4) and (6) with the method shown in item 1.1 of [Appendix 1: Equation construction].

$$0 = f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5) + \cdots\cdots \tag{9}$$

We also have the following (10) from (5) and (7) with the method shown in item 1.2 of [Appendix 1].

$$0 = f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5) + \cdots\cdots \tag{10}$$

We can have the following (11) (which is the function of real number x) from the above (9) and (10) with the method shown in item 1.3 of [Appendix 1]. And the value of (11) is always zero at any value of x.

$$\begin{aligned}0 &\equiv \cos x\{\text{right side of (9)}\} + \sin x\{\text{right side of (10)}\}\\ &= \cos x\{f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5) + \cdots\cdots\}\\ &\quad + \sin x\{f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5) + \cdots\cdots\}\\ &= f(2)\cos(b\log 2 - x) - f(3)\cos(b\log 3 - x) + f(4)\cos(b\log 4 - x)\\ &\quad - f(5)\cos(b\log 5 - x) + f(6)\cos(b\log 6 - x) - \cdots\cdots \end{aligned} \tag{11}$$

We have the following (12-1) by substituting $b\log 1$ for x in (11).

$$\begin{aligned}0 = &f(2)\cos(b\log 2 - b\log 1) - f(3)\cos(b\log 3 - b\log 1) + f(4)\cos(b\log 4 - b\log 1)\\ &- f(5)\cos(b\log 5 - b\log 1) + f(6)\cos(b\log 6 - b\log 1) - \cdots\cdots \end{aligned} \tag{12-1}$$

We have the following (12-2) by substituting $b\log 2$ for x in (11).

$$\begin{aligned}0 = &f(2)\cos(b\log 2 - b\log 2) - f(3)\cos(b\log 3 - b\log 2) + f(4)\cos(b\log 4 - b\log 2)\\ &- f(5)\cos(b\log 5 - b\log 2) + f(6)\cos(b\log 6 - b\log 2) - \cdots\cdots \end{aligned} \tag{12-2}$$

We have the following (12-3) by substituting $b\log 3$ for x in (11).

$$0 = f(2)\cos(b\log 2 - b\log 3) - f(3)\cos(b\log 3 - b\log 3) + f(4)\cos(b\log 4 - b\log 3)$$

$$- f(5)\cos(b\log 5 - b\log 3) + f(6)\cos(b\log 6 - b\log 3) - \cdots\cdots \tag{12-3}$$

In the same way as above we can have the following (12-N) by substituting $b\log N$ for x in (11). $(N = 4, 5, 6, 7, 8, \cdots\cdots)$

$$\begin{aligned} 0 =& f(2)\cos(b\log 2 - b\log N) - f(3)\cos(b\log 3 - b\log N) + f(4)\cos(b\log 4 - b\log N) \\ & - f(5)\cos(b\log 5 - b\log N) + f(6)\cos(b\log 6 - b\log N) - \cdots\cdots \end{aligned} \tag{12-N}$$

3. Verification of $F(a) = 0$

We define $g(k, N)$ as follows. $(k = 2, 3, 4, 5, \cdots\cdots)$

$$\begin{aligned} g(k, N) &= \cos(b\log k - b\log 1) + \cos(b\log k - b\log 2) + \cdots + \cos(b\log k - b\log N) \\ &= \cos(b\log 1 - b\log k) + \cos(b\log 2 - b\log k) + \cdots + \cos(b\log N - b\log k) \\ &= \cos(b\log 1/k) + \cos(b\log 2/k) + \cos(b\log 3/k) + \cdots + \cos(b\log N/k) \end{aligned} \tag{13}$$

We can have the following (14) from the equations of (12-1), (12-2), (12-3), $\cdots\cdots$, (12-N) with the method shown in item 1.4 of [Appendix 1].

$$\begin{aligned} 0 =& f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cdots + \cos(b\log 2 - b\log N)\} \\ & - f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cdots + \cos(b\log 3 - b\log N)\} \\ & + f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cdots + \cos(b\log 4 - b\log N)\} \\ & - f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cdots + \cos(b\log 5 - b\log N)\} \\ & + \cdots\cdots \\ =& f(2)g(2, N) - f(3)g(3, N) + f(4)g(4, N) - f(5)g(5, N) + \cdots\cdots \end{aligned} \tag{14}$$

Here we define $F(a)$ as follows.

$$F(a) = f(2) - f(3) + f(4) - f(5) + f(6) - \cdots\cdots \tag{15}$$

We can have the following (16) by deviding the above (14) by $g(2, N)$. Because $g(2, N) \neq 0$ is true in $N_0 \leq N$ as shown in [Appendix 2: Proof of $g(2, N) \neq 0$]. N_0 is the large natural number that holds (29) in [Appendix 2].

$$0 = f(2) - \frac{f(3)g(3, N)}{g(2, N)} + \frac{f(4)g(4, N)}{g(2, N)} - \frac{f(5)g(5, N)}{g(2, N)} + \cdots\cdots \qquad (N_0 \leq N) \tag{16}$$

We can have the following (17) from the above (16) by performing $N \to \infty$. Because $\lim_{N\to\infty} \frac{g(k, N)}{g(2, N)} = 1$ $(k = 3, 4, 5, 6, 7 \cdots\cdots)$ is true as shown in [Appendix 3: Proof of $\lim_{N\to\infty} \frac{g(k, N)}{g(2, N)} = 1$].

$$\begin{aligned} 0 =& \lim_{N\to\infty} \{f(2) - \frac{f(3)g(3, N)}{g(2, N)} + \frac{f(4)g(4, N)}{g(2, N)} - \frac{f(5)g(5, N)}{g(2, N)} + \cdots\cdots\} \\ =& f(2) - f(3) + f(4) - f(5) + f(6) - \cdots\cdots = F(a) \qquad (N_0 \leq N) \end{aligned} \tag{17}$$

4. Conclusion

$F(a) = 0$ has the only solution of $a = 0$ as shown in [Appendix 4: Solution for $F(a) = 0$]. a has the range of $0 \leq a < 1/2$ by the critical strip of $\zeta(s)$. However, a cannot have any value but zero because a is the solution for $F(a) = 0$. Due to $a = 0$ non-trivial zero point of Riemann zeta function $\zeta(s)$ shown by (2) and (3) must be $1/2 \pm bi$ and other zero point does not exist. Therefore Riemann hypothesis which says " All non-trivial zero points of Riemann zeta function $\zeta(s)$ exist on the line of $Re(s) = 1/2$. " is true.

Appendix 1. : Equation construction

We can construct (9), (10), (11) and (14) by applying the following Theorem 1[1].

Theorem 1

On condition that the following (Series 1) and (Series 2) converge respectively, the following (Series 3) and (Series 4) are true.

$$\text{(Series 1)} = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots\cdots = A$$
$$\text{(Series 2)} = b_1 + b_2 + b_3 + b_4 + b_5 + \cdots\cdots = B$$
$$\text{(Series 3)} = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + (a_4 + b_4) + \cdots\cdots = A + B$$
$$\text{(Series 4)} = (a_1 - b_1) + (a_2 - b_2) + (a_3 - b_3) + (a_4 - b_4) + \cdots\cdots = A - B$$

1.1. Construction of (9)

We can have the following (9) as (Series 4) by regarding (6) and (4) as (Series 1) and (Series 2) respectively.

$$\text{(Series 1)} = \frac{\cos(b\log 2)}{2^{1/2-a}} - \frac{\cos(b\log 3)}{3^{1/2-a}} + \frac{\cos(b\log 4)}{4^{1/2-a}} - \frac{\cos(b\log 5)}{5^{1/2-a}} + \cdots\cdots = 1 \qquad (6)$$
$$\text{(Series 2)} = \frac{\cos(b\log 2)}{2^{1/2+a}} - \frac{\cos(b\log 3)}{3^{1/2+a}} + \frac{\cos(b\log 4)}{4^{1/2+a}} - \frac{\cos(b\log 5)}{5^{1/2+a}} + \cdots\cdots = 1 \qquad (4)$$
$$\text{(Series 4)} = f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5) + \cdots\cdots = 1 - 1 = 0 \qquad (9)$$

Here $f(n)$ is defined as follows.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geq 0 \qquad (n = 2, 3, 4, 5, \cdots\cdots) \qquad (8)$$

1.2. Construction of (10)

We can have the following (10) as (Series 4) by regarding (7) and (5) as (Series 1) and (Series 2) respectively.

$$\text{(Series 1)} = \frac{\sin(b\log 2)}{2^{1/2-a}} - \frac{\sin(b\log 3)}{3^{1/2-a}} + \frac{\sin(b\log 4)}{4^{1/2-a}} - \frac{\sin(b\log 5)}{5^{1/2-a}} + \cdots\cdots = 0 \qquad (7)$$
$$\text{(Series 2)} = \frac{\sin(b\log 2)}{2^{1/2+a}} - \frac{\sin(b\log 3)}{3^{1/2+a}} + \frac{\sin(b\log 4)}{4^{1/2+a}} - \frac{\sin(b\log 5)}{5^{1/2+a}} + \cdots\cdots = 0 \qquad (5)$$

$$(\text{Series } 4) = f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5) + \cdots\cdots = 0 - 0 \quad (10)$$

1.3. Construction of (11)

We can have the following (11) as (Series 3) by regarding the following equations as (Series 1) and (Series 2).

$$\begin{aligned}(\text{Series } 1) &= \cos x\{\text{right side of (9)}\}\\ &= \cos x\{f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5)\\ &\quad + \cdots\cdots\} \equiv 0\\ (\text{Series } 2) &= \sin x\{\text{right side of (10)}\}\\ &= \sin x\{f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5)\\ &\quad + \cdots\cdots\} \equiv 0\\ (\text{Series } 3) &= f(2)\cos(b\log 2 - x) - f(3)\cos(b\log 3 - x) + f(4)\cos(b\log 4 - x)\\ &\quad - f(5)\cos(b\log 5 - x) + \cdots\cdots \equiv 0 + 0 \qquad (11)\end{aligned}$$

1.4. Construction of (14)

1.4.1 We can have the following (12-1*2) as (Series 3) by regarding (12-1) and (12-2) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}(\text{Series } 1) = &f(2)\cos(b\log 2 - b\log 1) - f(3)\cos(b\log 3 - b\log 1)\\ &+ f(4)\cos(b\log 4 - b\log 1) - f(5)\cos(b\log 5 - b\log 1)\\ &+ f(6)\cos(b\log 6 - b\log 1) - \cdots\cdots = 0 \qquad (12\text{-}1)\\ (\text{Series } 2) = &f(2)\cos(b\log 2 - b\log 2) - f(3)\cos(b\log 3 - b\log 2)\\ &+ f(4)\cos(b\log 4 - b\log 2) - f(5)\cos(b\log 5 - b\log 2)\\ &+ f(6)\cos(b\log 6 - b\log 2) - \cdots\cdots = 0 \qquad (12\text{-}2)\\ (\text{Series } 3) = &f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2)\}\\ &- f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2)\}\\ &+ f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2)\}\\ &- f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2)\}\\ &+ \cdots\cdots = 0 + 0 \qquad (12\text{-}1{*}2)\end{aligned}$$

1.4.2 We can have the following (12-1*3) as (Series 3) by regarding (12-1*2) and (12-3) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}(\text{Series } 2) = &f(2)\cos(b\log 2 - b\log 3) - f(3)\cos(b\log 3 - b\log 3)\\ &+ f(4)\cos(b\log 4 - b\log 3) - f(5)\cos(b\log 5 - b\log 3)\\ &+ f(6)\cos(b\log 6 - b\log 3) - \cdots\cdots = 0 \qquad (12\text{-}3)\\ (\text{Series } 3) = &f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cos(b\log 2 - b\log 3)\}\\ &- f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cos(b\log 3 - b\log 3)\}\\ &+ f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cos(b\log 4 - b\log 3)\}\\ &- f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cos(b\log 5 - b\log 3)\}\end{aligned}$$

$$+\cdots\cdots=0+0 \tag{12-1*3}$$

1.4.3 We can have the following (12-1*4) as (Series 3) by regarding (12-1*3) and (12-4) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}(\text{Series 2}) &= f(2)\cos(b\log 2-b\log 4)-f(3)\cos(b\log 3-b\log 4)\\ &\quad+f(4)\cos(b\log 4-b\log 4)-f(5)\cos(b\log 5-b\log 4)\\ &\quad+f(6)\cos(b\log 6-b\log 4)-\cdots\cdots=0 \end{aligned}\tag{12-4}$$

$$\begin{aligned}&(\text{Series 3})\\ &=f(2)\{\cos(b\log 2-b\log 1)+\cos(b\log 2-b\log 2)+\cdots+\cos(b\log 2-b\log 4)\}\\ &\quad-f(3)\{\cos(b\log 3-b\log 1)+\cos(b\log 3-b\log 2)+\cdots+\cos(b\log 3-b\log 4)\}\\ &\quad+f(4)\{\cos(b\log 4-b\log 1)+\cos(b\log 4-b\log 2)+\cdots+\cos(b\log 4-b\log 4)\}\\ &\quad-f(5)\{\cos(b\log 5-b\log 1)+\cos(b\log 5-b\log 2)+\cdots+\cos(b\log 5-b\log 4)\}\\ &\quad+\cdots\cdots=0+0\end{aligned}\tag{12-1*4}$$

1.4.4 In the same way as above we can have the following (12-1*N)=(14) as (Series 3) by regarding (12-1*N-1) and (12-N) as (Series 1) and (Series 2) respectively. ($N=5,6,7,8,\cdots\cdots$) $g(k,N)$ is defined in page 3. ($k=2,3,4,5,\cdots\cdots$)

$$\begin{aligned}&f(2)\{\cos(b\log 2-b\log 1)+\cos(b\log 2-b\log 2)+\cdots+\cos(b\log 2-b\log N)\}\\ &-f(3)\{\cos(b\log 3-b\log 1)+\cos(b\log 3-b\log 2)+\cdots+\cos(b\log 3-b\log N)\}\\ &+f(4)\{\cos(b\log 4-b\log 1)+\cos(b\log 4-b\log 2)+\cdots+\cos(b\log 4-b\log N)\}\\ &-f(5)\{\cos(b\log 5-b\log 1)+\cos(b\log 5-b\log 2)+\cdots+\cos(b\log 5-b\log N)\}\\ &+\cdots\cdots\\ &=f(2)g(2,N)-f(3)g(3,N)+f(4)g(4,N)-f(5)g(5,N)+f(6)g(6,N)-\cdots\cdots\\ &=0+0\end{aligned}\tag{12-1*N}$$

Appendix 2. : Proof of $g(2, N) \neq 0$

2.1. Investigation of $g(k, N)$

2.1.1 We define G and H as follows.

$$\begin{aligned}G&=\lim_{N\to\infty}\frac{1}{N}\{\cos(b\log\frac{1}{N})+\cos(b\log\frac{2}{N})+\cos(b\log\frac{3}{N})+\cdots+\cos(b\log\frac{N}{N})\}\\ &=\int_0^1\cos(b\log x)dx\end{aligned}\tag{20-1}$$

$$\begin{aligned}H&=\lim_{N\to\infty}\frac{1}{N}\{\sin(b\log\frac{1}{N})+\sin(b\log\frac{2}{N})+\sin(b\log\frac{3}{N})+\cdots+\sin(b\log\frac{N}{N})\}\\ &=\int_0^1\sin(b\log x)dx\end{aligned}\tag{20-2}$$

We calculate G and H by Integration by parts.

$$G=[x\cos(b\log x)]_0^1+bH=1+bH$$

$$H = [x\sin(b\log x)]_0^1 - bG = -bG$$

Then we can have the values of G and H from the above equations as follows.

$$G = \frac{1}{1+b^2} \qquad H = \frac{-b}{1+b^2} \tag{21}$$

2.1.2 We define as follows.

$$\frac{\cos(b\log\frac{1}{N}) + \cos(b\log\frac{2}{N}) + \cos(b\log\frac{3}{N}) + \cdots + \cos(b\log\frac{N}{N})}{N} - G = E_c(N) \tag{22-1}$$

$$\frac{\sin(b\log\frac{1}{N}) + \sin(b\log\frac{2}{N}) + \sin(b\log\frac{3}{N}) + \cdots + \sin(b\log\frac{N}{N})}{N} - H = E_s(N) \tag{22-2}$$

From the definition of (20-1), (20-2), (22-1) and (22-2) we have the following (23).

$$\lim_{N\to\infty} E_c(N) = 0 \qquad \lim_{N\to\infty} E_s(N) = 0 \tag{23}$$

2.1.3 From (13) we can calculate $g(k,N)$ as follows.

$$\begin{aligned}
g(k,N) &= \cos(b\log 1/k) + \cos(b\log 2/k) + \cos(b\log 3/k) + \cdots + \cos(b\log N/k) \\
&= N\frac{1}{N}\{\cos(b\log\frac{1}{N}\frac{N}{k}) + \cos(b\log\frac{2}{N}\frac{N}{k}) + \cos(b\log\frac{3}{N}\frac{N}{k}) + \cdots + \cos(b\log\frac{N}{N}\frac{N}{k})\} \\
&= N\frac{1}{N}\{\cos(b\log\frac{1}{N} + b\log\frac{N}{k}) + \cos(b\log\frac{2}{N} + b\log\frac{N}{k}) + \cos(b\log\frac{3}{N} + b\log\frac{N}{k}) \\
&\quad + \cdots\cdots + \cos(b\log\frac{N}{N} + b\log\frac{N}{k})\} \\
&= N\frac{1}{N}\{\cos(b\log\frac{N}{k})\}\{\cos(b\log\frac{1}{N}) + \cos(b\log\frac{2}{N}) + \cos(b\log\frac{3}{N}) + \cdots + \cos(b\log\frac{N}{N})\} \\
&\quad - N\frac{1}{N}\{\sin(b\log\frac{N}{k})\}\{\sin(b\log\frac{1}{N}) + \sin(b\log\frac{2}{N}) + \sin(b\log\frac{3}{N}) + \cdots + \sin(b\log\frac{N}{N})\} \\
&= N\{\cos(b\log\frac{N}{k})\}G + N\{\cos(b\log\frac{N}{k})\}\{\frac{\cos(b\log\frac{1}{N}) + \cos(b\log\frac{2}{N}) + \cdots + \cos(b\log\frac{N}{N})}{N} - G\} \\
&\quad - N\{\sin(b\log\frac{N}{k})\}H - N\{\sin(b\log\frac{N}{k})\}\{\frac{\sin(b\log\frac{1}{N}) + \sin(b\log\frac{2}{N}) + \cdots + \sin(b\log\frac{N}{N})}{N} - H\}
\end{aligned} \tag{24-1}$$

$$\begin{aligned}
&= N\{\cos(b\log\frac{N}{k})\}G + N\{\cos(b\log\frac{N}{k})\}E_c(N) \\
&\quad - N\{\sin(b\log\frac{N}{k})\}H - N\{\sin(b\log\frac{N}{k})\}E_s(N)
\end{aligned} \tag{24-2}$$

$$\begin{aligned}
&= N\{\cos(b\log\frac{N}{k})\}\frac{1}{1+b^2} + N\{\cos(b\log\frac{N}{k})\}E_c(N) \\
&\quad + N\{\sin(b\log\frac{N}{k})\}\frac{b}{1+b^2} - N\{\sin(b\log\frac{N}{k})\}E_s(N)
\end{aligned} \tag{24-3}$$

$$= \frac{N\sin(b\log N/k + \tan^{-1} 1/b)}{\sqrt{1+b^2}} - N\sqrt{E_c(N)^2 + E_s(N)^2}\sin\{b\log N/k - \tan^{-1} E_c(N)/E_s(N)\} \tag{24-4}$$

$$= NR(1)\sin\{b\log N/k + \theta(1)\} - NR(2)\sin\{b\log N/k - \theta(2)\} \tag{24-5}$$
$$= NR(3)\sin\{b\log N/k + \theta(3)\} \tag{24-6}$$

2.1.4 From (22-1), (22-2) and (24-1) we have (24-2).

2.1.5 From (21) and (24-2) we have (24-3). From (24-3) and (23) we have the following (25).

$$g(k,N) = (24\text{-}3)$$
$$= N\{\cos(b\log\frac{N}{k})\frac{1}{1+b^2} + \cos(b\log\frac{N}{k})E_c(N) + \sin(b\log\frac{N}{k})\frac{b}{1+b^2} - \sin(b\log\frac{N}{k})E_s(N)\}$$
$$\sim \quad N\{\cos(b\log\frac{N}{k})\frac{1}{1+b^2} + \sin(b\log\frac{N}{k})\frac{b}{1+b^2}\}$$
$$= \frac{N\sin(b\log N/k + \tan^{-1}1/b)}{\sqrt{1+b^2}} \qquad (N\to\infty) \tag{25}$$

2.1.6 We define as follows. From (24-4), the following (26-1) and (26-2) we have (24-5).

$$R(1) = 1/\sqrt{1+b^2} \qquad \theta(1) = \tan^{-1}1/b \tag{26-1}$$
$$R(2) = \sqrt{E_c(N)^2 + E_s(N)^2} \qquad \theta(2) = \tan^{-1}E_c(N)/E_s(N) \tag{26-2}$$

2.1.7 We can calculate the following (27-1) and (27-2) from the following (FIgure 1). $R(3)$ can be calculated by Cosine theorem. We have (24-6) from (24-5), (27-1) and (27-2).

$$R(3) = \sqrt{R(1)^2 + R(2)^2 - 2R(1)R(2)\cos\{\theta(1)+\theta(2)\}} \tag{27-1}$$
$$\theta(3) = \tan^{-1}\frac{R(1)\sin\theta(1) + R(2)\sin\theta(2)}{R(1)\cos\theta(1) - R(2)\cos\theta(2)} \tag{27-2}$$

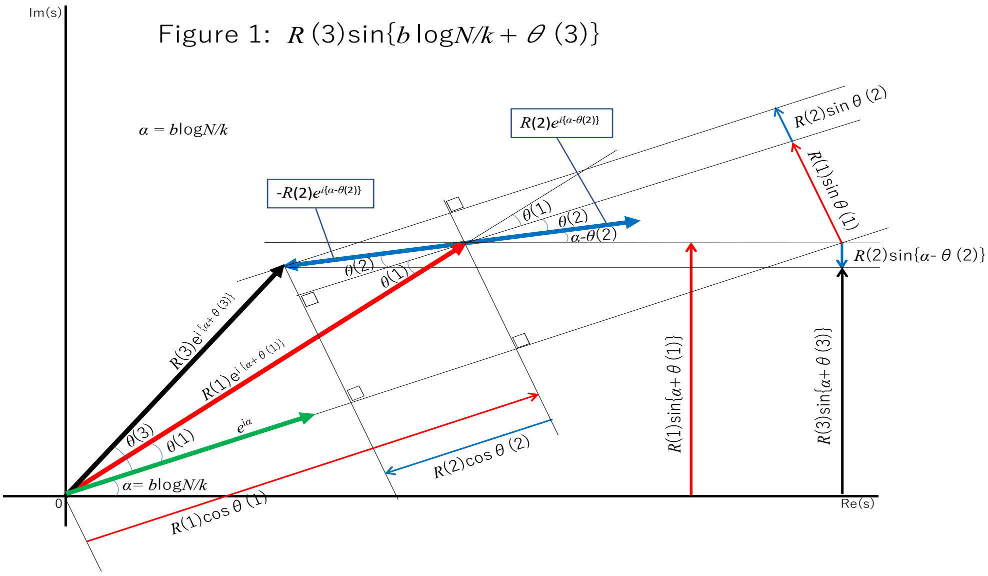


Figure 1: $R(3)\sin\{b\log N/k + \theta(3)\}$

2.1.8 The condition of $R(3) = 0$ is as follows.

$$R(1) = 1/\sqrt{1+b^2} = \sqrt{E_c(N)^2 + E_s(N)^2} = R(2) \tag{28-1}$$
$$\theta(1) = \tan^{-1} 1/b = -\tan^{-1} E_c(N)/E_s(N) = -\theta(2) \tag{28-2}$$

There is the large natural number N_0 that holds the following (29) because of $\lim_{N\to\infty} \sqrt{E_c(N)^2 + E_s(N)^2} = 0$.

$$1/\sqrt{1+b^2} > \sqrt{E_c(N)^2 + E_s(N)^2} > 0 \qquad (N_0 \leq N) \tag{29}$$

From the above (28-1) and (29) the following (30) holds.

$$R(3) \neq 0 \qquad (N_0 \leq N) \tag{30}$$

2.2. Verification of $\sin\{b \log N/2 + \theta(3)\} \neq 0$

If we assume that $\sin\{b \log N/2 + \theta(3)\} = 0 \quad (N = 3, 4, 5, 6, 7, \cdots\cdots)$ is true, the following (31) is supposed to be true.

$$b \log N/2 + \theta(3) = K\pi \qquad (K = 2, 3, 4, \cdots\cdots) \tag{31}$$

The range of b is $14 < b$ as shown in page 1. We have $\log 3/2 = 0.405$ and $-\pi/2 < \theta(3) < \pi/2$ from (27-2). Then we have $K > 1.3$ from $[14 * 0.405 - \pi/2 = 4.09 < K\pi]$. Therefore $(K = 2, 3, 4, \cdots\cdots)$ holds.
From (31) we have the following (32-1) and (32-2).

$$\log N/2 = \frac{K\pi - \theta(3)}{b} = M > 0 \tag{32-1}$$
$$N = 2e^M \tag{32-2}$$

We have $M > 0$ from $K \geq 2$ and $\theta(3) < \pi/2$. (32-2) has an impossible formation like (natural number) = (irrational number). Therefore (32-2) is false and (31) (which is the original equation of (32-2)) is also false. Now we can have the following (33).

$$\sin\{b \log N/2 + \theta(3)\} \neq 0 \qquad (N = 3, 4, 5, 6, 7, \cdots\cdots) \tag{33}$$

2.3. Verification of $g(2, N) \neq 0$

We have the following (25-1) from (25) in item 2.1.5 and the following (34) from (24-6) in item 2.1.3, (30) and (33).

$$g(2, N) \quad \sim \quad \frac{N \sin(b \log N/2 + \tan^{-1} 1/b)}{\sqrt{1+b^2}} \qquad (N \to \infty) \tag{25-1}$$
$$g(2, N) = NR(3) \sin\{b \log N/2 + \theta(3)\} \neq 0 \qquad (N_0 \leq N) \tag{34}$$

We can confirm that $g(2, N)$ does not have the value of zero in $N_0 \leq N$. N_0 is the large natural number that holds (29) in item 2.1.8.

Appendix 3. : Proof of $\lim_{N\to\infty}\frac{g(k,N)}{g(2,N)}=1$

We can confirm $\lim_{N\to\infty}\frac{g(k,N)}{g(2,N)}=1$ according to the following process. $(k=3,4,5,\cdots\cdots)$

3.1 We can have the following (35) from (25) and (25-1) in [Appendix 2].

$$\frac{g(k,N)}{g(2,N)} \sim \frac{\frac{N}{\sqrt{1+b^2}}\sin(b\log\frac{N}{k}+\tan^{-1}\frac{1}{b})}{\frac{N}{\sqrt{1+b^2}}\sin(b\log\frac{N}{2}+\tan^{-1}\frac{1}{b})}=\frac{\sin(b\log\frac{N}{k}+\tan^{-1}\frac{1}{b})}{\sin(b\log\frac{N}{2}+\tan^{-1}\frac{1}{b})} \quad (N\to\infty) \tag{35}$$

3.2 We can have the following (36) from the following (37).

$$\begin{aligned}\frac{\sin(b\log\frac{N}{k}+\tan^{-1}\frac{1}{b})}{\sin(b\log\frac{N}{2}+\tan^{-1}\frac{1}{b})}&=\frac{\sin\{\frac{b\log N/k+\tan^{-1}1/b}{b\log N/2+\tan^{-1}1/b}(b\log\frac{N}{2}+\tan^{-1}\frac{1}{b})\}}{\sin(b\log\frac{N}{2}+\tan^{-1}\frac{1}{b})}\\ &\sim \frac{\sin(b\log\frac{N}{2}+\tan^{-1}\frac{1}{b})}{\sin(b\log\frac{N}{2}+\tan^{-1}\frac{1}{b})}=1 \qquad (N\to\infty)\end{aligned} \tag{36}$$

$$\lim_{N\to\infty}\frac{b\log\frac{N}{k}+\tan^{-1}\frac{1}{b}}{b\log\frac{N}{2}+\tan^{-1}\frac{1}{b}}=\lim_{N\to\infty}\frac{1-\frac{\log k}{\log N}+\frac{\tan^{-1}1/b}{b\log N}}{1-\frac{\log 2}{\log N}+\frac{\tan^{-1}1/b}{b\log N}}=1 \tag{37}$$

3.3 $\frac{g(k,N)}{g(2,N)}$ approaches to $\frac{\sin(b\log\frac{N}{k}+\tan^{-1}\frac{1}{b})}{\sin(b\log\frac{N}{2}+\tan^{-1}\frac{1}{b})}$ infinitely with $N\to\infty$ as shown in the above (35). And $\frac{\sin(b\log\frac{N}{k}+\tan^{-1}\frac{1}{b})}{\sin(b\log\frac{N}{2}+\tan^{-1}\frac{1}{b})}$ converges to 1 with $N\to\infty$ as shown in the above (36). Therefore $\frac{g(k,N)}{g(2,N)}$ also converges to 1 with $N\to\infty$. From (35) and (36) we have the following (38).

$$\lim_{N\to\infty}\frac{g(k,N)}{g(2,N)}=\lim_{N\to\infty}\frac{\sin(b\log\frac{N}{k}+\tan^{-1}\frac{1}{b})}{\sin(b\log\frac{N}{2}+\tan^{-1}\frac{1}{b})}=1 \tag{38}$$

Appendix 4. : Solution for $F(a)=0$

4.1. Preparation for verification of $F(a)>0$

4.1.1. Investigation of $f(n)$

$$f(n)=\frac{1}{n^{1/2-a}}-\frac{1}{n^{1/2+a}}\geq 0 \qquad (n=2,3,4,5,\cdots\cdots) \tag{8}$$

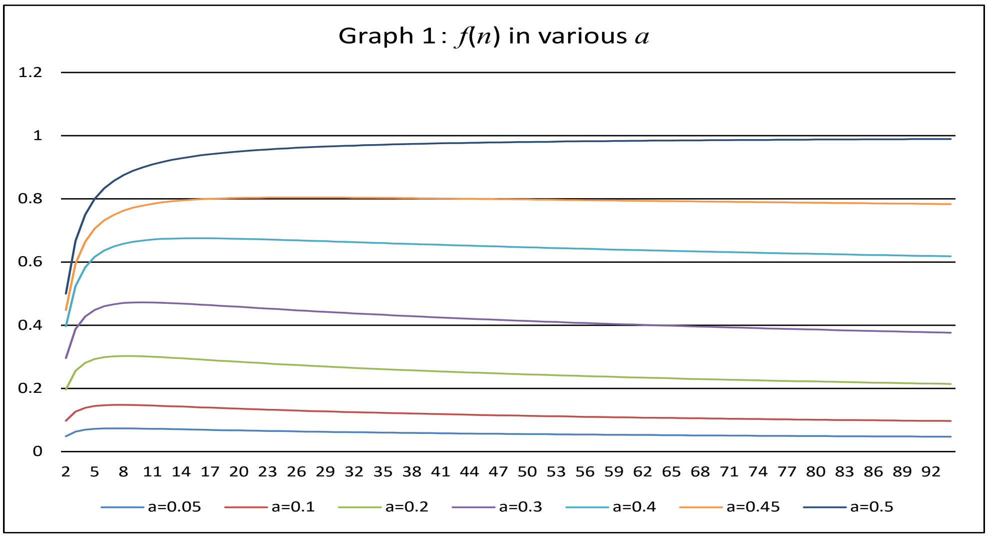
$$F(a)=f(2)-f(3)+f(4)-f(5)+f(6)-\cdots\cdots \tag{15}$$

$a=0$ is the solution for $F(a)=0$ due to $f(n)\equiv 0$ at $a=0$. Hereafter we define the range of a as $0<a<1/2$ to verify $F(a)>0$. The alternating series $F(a)$ converges due to $\lim_{n\to\infty}f(n)=0$.

We have the following (41) by differentiating $f(n)$ regarding n.

$$\frac{df(n)}{dn} = \frac{1/2+a}{n^{a+3/2}} - \frac{1/2-a}{n^{3/2-a}} = \frac{1/2+a}{n^{a+3/2}}\{1-(\frac{1/2-a}{1/2+a})n^{2a}\} \tag{41}$$

The value of $f(n)$ increases with increase of n and reaches the maximum value $f(n_{max})$ at $n = n_{max}$. Afterward $f(n)$ decreases to zero with $n \to \infty$. n_{max} is one of the 2 consecutive natural numbers that sandwich $(\frac{1/2+a}{1/2-a})^{\frac{1}{2a}}$. (Graph 1) shows $f(n)$ in various value of a. At $a = 1/2$ $f(n)$ does not have $f(n_{max})$ and increases to 1 with $n \to \infty$ due to $n_{max} = \infty$.



4.1.2. Verification method for $F(a) > 0$

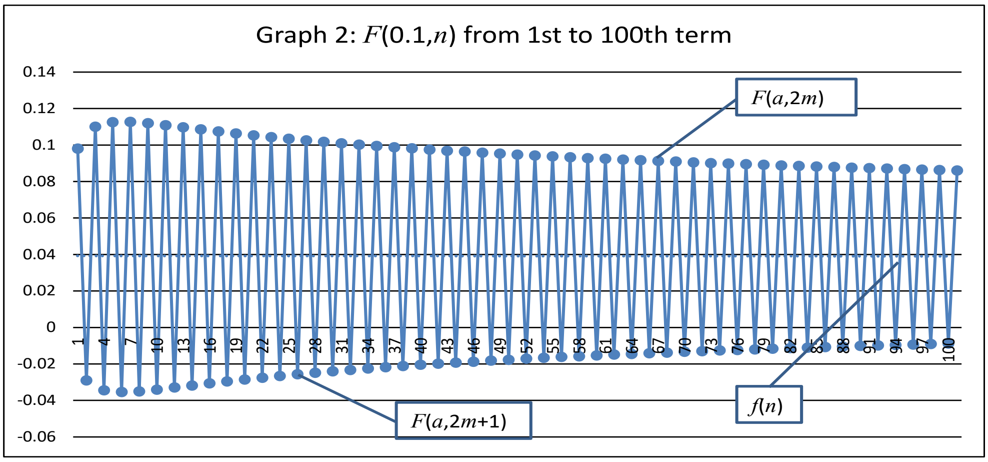
We define $F(a, n)$ as the following (42).

$$F(a,n) = f(2) - f(3) + f(4) - f(5) + \cdots + (-1)^n f(n) \qquad (n = 2, 3, 4, 5, \cdots\cdots) \tag{42}$$

$$\lim_{n\to\infty} F(a,n) = F(a) \tag{43}$$

$F(a)$ is an alternating series. So $F(a, n)$ repeats increase and decrease by $f(n)$ with increase of n as shown in (Graph 2). In (Graph 2) upper points mean $F(a, 2m)$ $(m = 1, 2, 3, \cdots\cdots)$ and lower points mean $F(a, 2m+1)$. $F(a, 2m)$ decreases and converges to $F(a)$ with $m \to \infty$. $F(a, 2m+1)$ increases and also converges to $F(a)$ with $m \to \infty$ due to $\lim_{n\to\infty} f(n) = 0$. We can have the following (44).

$$\lim_{m\to\infty} F(a,2m) = \lim_{m\to\infty} F(a,2m+1) = F(a) \tag{44}$$



Graph 2: $F(0.1,n)$ from 1st to 100th term

We define $F1(a)$ and $F1(a, 2m+1)$ as follws.

$$F1(a) = \{f(2) - f(3)\} + \{f(4) - f(5)\} + \{f(6) - f(7)\} + \cdots\cdots \tag{45}$$

$$F1(a, 2m+1) = \{f(2) - f(3)\} + \{f(4) - f(5)\} + \cdots + \{f(2m) - f(2m+1)\}$$

$$= f(2) - f(3) + f(4) - f(5) + \cdots + f(2m) - f(2m+1) = F(a, 2m+1) \tag{46}$$

$$\lim_{m\to\infty} F1(a, 2m+1) = F1(a) \tag{47}$$

From the above (44), (46) and (47) we have $F(a) = F1(a)$. We can use $F1(a)$ instead of $F(a)$ to verify $F(a) > 0$.
We enclose 2 terms of $F(a)$ each from the first term with { } as follows. If n_{max} is p or $p+1$ （p: odd number）, the inside sum of { } from $f(2)$ to $f(p)$ has negative value and the inside sum of { } after $f(p+1)$ has positive value.

$$F(a) = f(2) - f(3) + f(4) - f(5) + f(6) - f(7) + \cdots\cdots$$

$$= \{f(2) - f(3)\} + \{f(4) - f(5)\} + \cdots + \{f(p-1) - f(p)\} + \{f(p+1) - f(p+2)\} + \cdots\cdots$$

(inside sum of { }) $< 0 \longleftarrow|\longrightarrow$(inside sum of { }) > 0
(total sum of { }) $= -B \longleftarrow|\longrightarrow$(total sum of { }) $= A$

We define as follows.

[the partial sum from $f(2)$ to $f(p)$] $= -B < 0$
[the partial sum from $f(p+1)$ to $f(\infty)$] $= A > 0$

$$F(a) = A - B \tag{48}$$

So we can verify $F(a) > 0$ by verifying $A > B$.

4.1.3. Investigation of $\{f(n) - f(n+1)\}$

We have the following (49) by differentiating $\{f(n) - f(n+1)\}$ regarding n.

$$\frac{df(n)}{dn} - \frac{df(n+1)}{dn} = \frac{1/2+a}{n^{3/2+a}}\{1 - (\frac{n}{n+1})^{3/2+a}\} - \frac{1/2-a}{n^{3/2-a}}\{1 - (\frac{n}{n+1})^{3/2-a}\}$$

$$= C(n) - D(n) \tag{49}$$

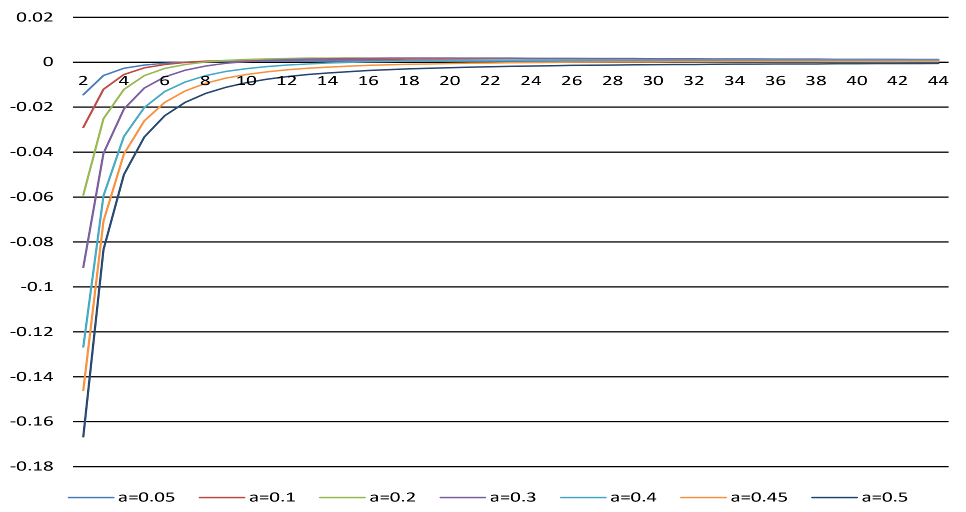
When n is a small natural number the value of $\{f(n)-f(n+1)\}$ increases with increase of n due to $C(n) > D(n)$. With increase of n the value reaches the maximum value $\{q_{max}\}$ at $C(n) \fallingdotseq D(n)$. (n is a natural number. The situation cannot be $C(n) = D(n)$.) After that the situation changes to $C(n) < D(n)$ and the value decreases to zero with $n \to \infty$. (Graph 3) shows the value of $\{f(n) - f(n+1)\}$ in various value of a. (Graph 4) shows the value of $\{f(n) - f(n+1)\}$ at $a = 0.1$. We can find the following from (Graph 3) and (Graph 4).

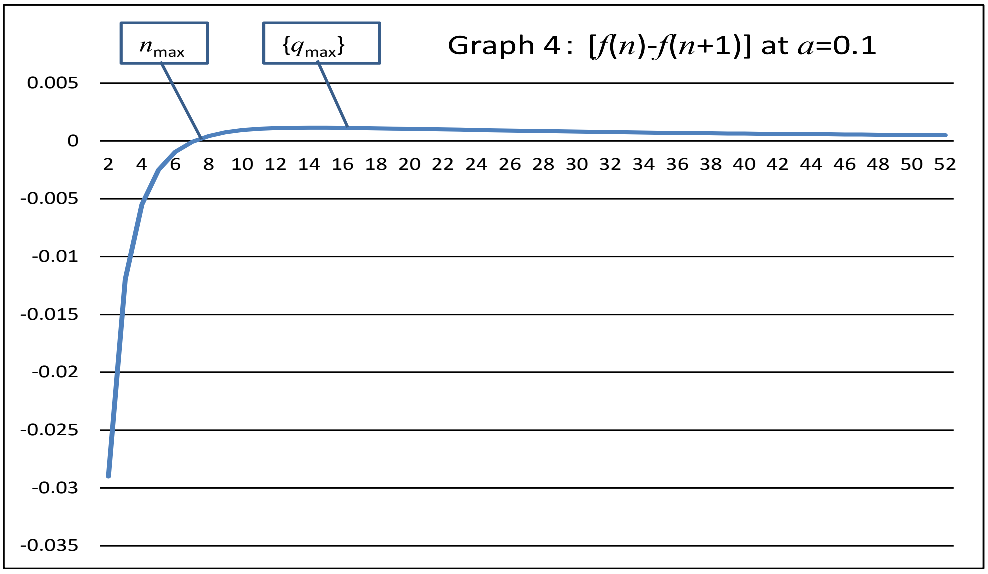
4.1.3.1 When $\left|\frac{df(n)}{dn}\right|$ becomes the maximum value $|f(n) - f(n+1)|$ also becomes the maximum value at same value of a. From (Graph 1) we can find that $\left|\frac{df(n)}{dn}\right|$ becomes the maximum value at $n = 2$. Therefore the maximum value of $|f(n) - f(n+1)|$ is $\{f(3) - f(2)\}$ at same value of a as shown in (Graph 3).

4.1.3.2 With increase of n the sign of $\{f(n) - f(n+1)\}$ changes from minus to plus at $n = n_{max}$ ($n = n_{max} + 1$) when n_{max} is even(odd) number as shown in (Graph 4).

4.1.3.3 After that the value reaches the maximum value $\{q_{max}\}$ and the value decreases to zero with $n \to \infty$ as shown in (Graph 4).

Graph 3 : [$f(n)$-$f(n$+1)] in various a





4.2. Verification of $A > B$ (n_{max} is odd number.)

n_{max} is odd number as follows.

$$F(a) = f(2) - f(3) + f(4) - f(5) + f(6) - \cdots\cdots$$
$$= \{f(2) - f(3)\} + \{f(4) - f(5)\} + \cdots + \{f(n_{max} - 3) - f(n_{max} - 2)\} + \{f(n_{max} - 1) - f(n_{max})\}$$
$$+\{f(n_{max} + 1) - f(n_{max} + 2)\} + \{f(n_{max} + 3) - f(n_{max} + 4)\} + \{f(n_{max} + 5) - f(n_{max} + 6)\} + \cdots\cdots$$

We can have A and B as follows.

$$B = \{f(3) - f(2)\} + \{f(5) - f(4)\} + \{f(7) - f(6)\} + \cdots + \{f(n_{max} - 2) - f(n_{max} - 3)\} + \{f(n_{max}) - f(n_{max} - 1)\}$$
$$A = \{f(n_{max} + 1) - f(n_{max} + 2)\} + \{f(n_{max} + 3) - f(n_{max} + 4)\} + \{f(n_{max} + 5) - f(n_{max} + 6)\} + \cdots\cdots$$

4.2.1. Condition for B

We define as follows.

{ (yellow) } : the term which is included within B.

{ (gray) } : the term which is not included within B.

We have the following (50).

$$f(n_{max}) - f(2) = \{f(n_{max}) - f(n_{max} - 1)\} + \{f(n_{max} - 1) - f(n_{max} - 2)\} + \{f(n_{max} - 2) - f(n_{max} - 3)\}$$
$$+ \cdots + \{f(7) - f(6)\} + \{f(6) - f(5)\} + \{f(5) - f(4)\} + \{f(4) - f(3)\} + \{f(3) - f(2)\} \quad (50)$$

And we have the following inequalities from (Graph 3) and (Graph 4).

$$\{f(3) - f(2)\} > \{f(4) - f(3)\} > \{f(5) - f(4)\} > \{f(6) - f(5)\} > \{f(7) - f(6)\} > \cdots\cdots$$
$$> \{f(n_{max} - 2) - f(n_{max} - 3)\} > \{f(n_{max} - 1) - f(n_{max} - 2)\} > \{f(n_{max}) - f(n_{max} - 1)\} > 0$$

From the above (50) we have the following (51).

$$f(n_{max}) - f(2) + \{f(3) - f(2)\}$$

$$
\begin{array}{l}
= \{\colorbox{yellow}{$f(3)-f(2)$}\} + \{\colorbox{yellow}{$f(5)-f(4)$}\} + \{\colorbox{yellow}{$f(7)-f(6)$}\} + \cdots + \{\colorbox{yellow}{$f(n_{max}-2)-f(n_{max}-3)$}\} + \{\colorbox{yellow}{$f(n_{max})-f(n_{max}-1)$}\} \\
\qquad \| \qquad\qquad\qquad \wedge \qquad\qquad\qquad \wedge \qquad\qquad\qquad\qquad \wedge \quad \leftarrow \text{Value comparison} \rightarrow \quad \wedge \\
+\{\colorbox{yellow}{$f(3)-f(2)$}\} + \{\colorbox{lightgray}{$f(4)-f(3)$}\} + \{\colorbox{lightgray}{$f(6)-f(5)$}\} + \cdots + \{\colorbox{lightgray}{$f(n_{max}-3)-f(n_{max}-4)$}\} + \{\colorbox{lightgray}{$f(n_{max}-1)-f(n_{max}-2)$}\} \\
> 2B
\end{array}
\tag{51}
$$

Due to [Total sum of upper row of the above (51) $= B <$ Total sum of lower row of (51)] we have the following (52).

$$
f(n_{max}) - f(2) + \{f(3) - f(2)\} > 2B \tag{52}
$$

4.2.2. Condition for A ($\{q_{max}\}$ is included within A.)

We abbreviate $\{f(n_{max} + q) - f(n_{max} + q + 1)\}$ to $\{q\}$ for easy description. ($q = 0, 1, 2, 3, \cdots\cdots$) All $\{q\}$ has positive value as shown in item 4.1.2.
We define as follows.

$\{\colorbox{yellow}{$$}\}$: the term which is included within A.

$\{\colorbox{lightgray}{$$}\}$: the term which is not included within A.

$\{q_{max}\}$ has the maximum value in all $\{q\}$. And $\{q_{max}\}$ is included within A. Then value comparison of $\{q\}$ is as follows.

$$
\{\colorbox{yellow}{1}\} < \{\colorbox{lightgray}{2}\} < \{\colorbox{yellow}{3}\} < \cdots < \{\colorbox{lightgray}{$q_{max}-3$}\} < \{\colorbox{yellow}{$q_{max}-2$}\} < \{\colorbox{lightgray}{$q_{max}-1$}\} < \{\colorbox{yellow}{q_{max}}\} > \{\colorbox{lightgray}{$q_{max}+1$}\} > \{\colorbox{yellow}{$q_{max}+2$}\} > \{\colorbox{lightgray}{$q_{max}+3$}\} > \cdots\cdots
$$

We have the following (53).

$$
\begin{array}{l}
f(n_{max}+1) = \{\colorbox{yellow}{$f(n_{max}+1)-f(n_{max}+2)$}\} + \{\colorbox{lightgray}{$f(n_{max}+2)-f(n_{max}+3)$}\} + \{\colorbox{yellow}{$f(n_{max}+3)-f(n_{max}+4)$}\} \\
\qquad\qquad + \{\colorbox{lightgray}{$f(n_{max}+4)-f(n_{max}+5)$}\} + \cdots\cdots \\
= \{\colorbox{yellow}{1}\} + \{\colorbox{lightgray}{2}\} + \{\colorbox{yellow}{3}\} + \{\colorbox{lightgray}{4}\} + \cdots + \{\colorbox{lightgray}{$q_{max}-3$}\} + \{\colorbox{yellow}{$q_{max}-2$}\} + \{\colorbox{lightgray}{$q_{max}-1$}\} + \{\colorbox{yellow}{q_{max}}\} + \{\colorbox{lightgray}{$q_{max}+1$}\} + \{\colorbox{yellow}{$q_{max}+2$}\} + \{\colorbox{lightgray}{$q_{max}+3$}\} + \cdots\cdots
\end{array}
\tag{53}
$$

From the above (53) we have the following (54).

$$
\begin{array}{l}
f(n_{max}+1) - \{\colorbox{lightgray}{$q_{max}-1$}\} \\
= \{\colorbox{yellow}{1}\} + \{\colorbox{lightgray}{2}\} + \{\colorbox{yellow}{3}\} + \{\colorbox{lightgray}{4}\} + \cdots + \{\colorbox{lightgray}{$q_{max}-3$}\} + \{\colorbox{yellow}{$q_{max}-2$}\} + \{\colorbox{yellow}{q_{max}}\} + \{\colorbox{lightgray}{$q_{max}+1$}\} + \{\colorbox{yellow}{$q_{max}+2$}\} + \{\colorbox{lightgray}{$q_{max}+3$}\} + \cdots\cdots \\
\leftarrow \cdots\cdots\cdots\cdots \text{Range 1} \cdots\cdots\cdots\cdots \rightarrow | \leftarrow \cdots\cdots\cdots \text{Range 2} \cdots\cdots\cdots
\end{array}
\tag{54}
$$

(Range 1) and (Range 2) are determined as above. In (Range 1) value comparison is as follows.

$$
\{\colorbox{yellow}{1}\} < \{\colorbox{lightgray}{2}\} < \{\colorbox{yellow}{3}\} < \{\colorbox{lightgray}{4}\} < \cdots < \{\colorbox{yellow}{$q_{max}-4$}\} < \{\colorbox{lightgray}{$q_{max}-3$}\} < \{\colorbox{yellow}{$q_{max}-2$}\}
$$

And we can find the following.

$$
\begin{array}{l}
\text{Total sum of } \{\colorbox{yellow}{}\} = \{\colorbox{yellow}{1}\} + \{\colorbox{yellow}{3}\} + \{\colorbox{yellow}{5}\} + \{\colorbox{yellow}{7}\} + \cdots + \{\colorbox{yellow}{$q_{max}-4$}\} + \{\colorbox{yellow}{$q_{max}-2$}\} \\
\qquad\qquad\qquad\qquad\qquad \vee \qquad \vee \qquad \vee \qquad\qquad\quad \vee \qquad\qquad\quad \vee \quad \leftarrow \text{Value comparison} \\
\text{Total sum of } \{\colorbox{lightgray}{}\} = \qquad\quad \{\colorbox{lightgray}{2}\} + \{\colorbox{lightgray}{4}\} + \{\colorbox{lightgray}{6}\} + \cdots + \{\colorbox{lightgray}{$q_{max}-5$}\} + \{\colorbox{lightgray}{$q_{max}-3$}\}
\end{array}
$$

Therefore [Total sum of $\{\colorbox{yellow}{$$}\}$ > Total sum of $\{\colorbox{lightgray}{$$}\}$] holds.
In (Range 2) value comparison is as follows.

$$
\{\colorbox{yellow}{q_{max}}\} > \{\colorbox{lightgray}{$q_{max}+1$}\} > \{\colorbox{yellow}{$q_{max}+2$}\} > \{\colorbox{lightgray}{$q_{max}+3$}\} > \{\colorbox{yellow}{$q_{max}+4$}\} > \{\colorbox{lightgray}{$q_{max}+5$}\} > \{\colorbox{yellow}{$q_{max}+6$}\} > \cdots\cdots
$$

And we can find the following.

$$\text{Total sum of }\{\colorbox{yellow}{$$}\} = \{\colorbox{yellow}{q_{max}}\}+\{\colorbox{yellow}{$q_{max}+2$}\}+\{\colorbox{yellow}{$q_{max}+4$}\}+\{\colorbox{yellow}{$q_{max}+6$}\}+\cdots\cdots$$
$$\vee \qquad \vee \qquad \vee \qquad \vee \qquad \leftarrow\text{Value comparison}$$
$$\text{Total sum of }\{\colorbox{lightgray}{$$}\} = \{\colorbox{lightgray}{$q_{max}+1$}\}+\{\colorbox{lightgray}{$q_{max}+3$}\}+\{\colorbox{lightgray}{$q_{max}+5$}\}+\{\colorbox{lightgray}{$q_{max}+7$}\}+\cdots\cdots$$

Therefore [Total sum of $\{\colorbox{yellow}{$$}\}$ > Total sum of $\{\colorbox{lightgray}{$$}\}$] holds.
In (Range 1)+(Range 2) we have [Total sum of $\{\colorbox{yellow}{$$}\} = A >$ Total sum of $\{\colorbox{lightgray}{$$}\}$].
We have the following (55).

$$f(n_{max}+1) - \{q_{max}-1\} < 2A \tag{55}$$

4.2.3. Condition for A ($\{q_{max}\}$ is not included within A.)

We have the following (56). $\{q_{max}\}$ is not included within A.

$$\begin{aligned} f(n_{max}+1) &= \{\colorbox{yellow}{$f(n_{max}+1)-f(n_{max}+2)$}\}+\{\colorbox{lightgray}{$f(n_{max}+2)-f(n_{max}+3)$}\}+\{\colorbox{yellow}{$f(n_{max}+3)-f(n_{max}+4)$}\} \\ &\quad +\{\colorbox{lightgray}{$f(n_{max}+4)-f(n_{max}+5)$}\}+\cdots\cdots \\ &= \{\colorbox{yellow}{1}\}+\{\colorbox{lightgray}{2}\}+\{\colorbox{yellow}{3}\}+\{\colorbox{lightgray}{4}\}+\cdots+\{\colorbox{yellow}{$q_{max}-3$}\}+\{\colorbox{lightgray}{$q_{max}-2$}\}+\{\colorbox{yellow}{$q_{max}-1$}\}+\{\colorbox{lightgray}{q_{max}}\} \\ &\quad +\{\colorbox{yellow}{$q_{max}+1$}\}+\{\colorbox{lightgray}{$q_{max}+2$}\}+\{\colorbox{yellow}{$q_{max}+3$}\}+\cdots\cdots \end{aligned} \tag{56}$$

From the above (56) we have the following (57).

$$\begin{aligned} &f(n_{max}+1) - \{\colorbox{lightgray}{q_{max}}\} \\ &= \{\colorbox{yellow}{1}\}+\{\colorbox{lightgray}{2}\}+\{\colorbox{yellow}{3}\}+\{\colorbox{lightgray}{4}\}+\cdots+\{\colorbox{yellow}{$q_{max}-3$}\}+\{\colorbox{lightgray}{$q_{max}-2$}\}+\{\colorbox{yellow}{$q_{max}-1$}\}+\{\colorbox{yellow}{$q_{max}+1$}\}+\{\colorbox{lightgray}{$q_{max}+2$}\}+\{\colorbox{yellow}{$q_{max}+3$}\}+\cdots\cdots \end{aligned} \tag{57}$$
$$\leftarrow \cdots\cdots\cdots\cdots \text{ Range 1 } \cdots\cdots\cdots\cdots \rightarrow | \leftarrow \cdots\cdots \text{ Range 2 } \cdots\cdots$$

(Range 1) and (Range 2) are determined as above. In (Range 1) value comparison is as follows.

$$\{\colorbox{yellow}{1}\}<\{\colorbox{lightgray}{2}\}<\{\colorbox{yellow}{3}\}<\{\colorbox{lightgray}{4}\}<\cdots<\{\colorbox{lightgray}{$q_{max}-4$}\}<\{\colorbox{yellow}{$q_{max}-3$}\}<\{\colorbox{lightgray}{$q_{max}-2$}\}<\{\colorbox{yellow}{$q_{max}-1$}\}$$

And we can find the following.

$$\text{Total sum of }\{\colorbox{yellow}{$$}\} = \{\colorbox{yellow}{1}\}+\{\colorbox{yellow}{3}\}+\{\colorbox{yellow}{5}\}+\{\colorbox{yellow}{7}\}+\cdots+\{\colorbox{yellow}{$q_{max}-3$}\}+\{\colorbox{yellow}{$q_{max}-1$}\}$$
$$\vee \qquad \vee \qquad \vee \qquad \vee \qquad \vee \qquad \leftarrow\text{Value comparison}$$
$$\text{Total sum of }\{\colorbox{lightgray}{$$}\} = \{\colorbox{lightgray}{2}\}+\{\colorbox{lightgray}{4}\}+\{\colorbox{lightgray}{6}\}+\cdots+\{\colorbox{lightgray}{$q_{max}-4$}\}+\{\colorbox{lightgray}{$q_{max}-2$}\}$$

Therefore [Total sum of $\{\colorbox{yellow}{$$}\}$ > Total sum of $\{\colorbox{lightgray}{$$}\}$] holds.
In (Range 2) value comparison is as follows.

$$\{\colorbox{yellow}{$q_{max}+1$}\}>\{\colorbox{lightgray}{$q_{max}+2$}\}>\{\colorbox{yellow}{$q_{max}+3$}\}>\{\colorbox{lightgray}{$q_{max}+4$}\}>\{\colorbox{yellow}{$q_{max}+5$}\}>\{\colorbox{lightgray}{$q_{max}+6$}\}>\{\colorbox{yellow}{$q_{max}+7$}\}>\cdots\cdots$$

And we can find the following.

$$\text{Total sum of }\{\colorbox{yellow}{$$}\} = \{\colorbox{yellow}{$q_{max}+1$}\}+\{\colorbox{yellow}{$q_{max}+3$}\}+\{\colorbox{yellow}{$q_{max}+3$}\}+\{\colorbox{yellow}{$q_{max}+7$}\}+\cdots\cdots$$
$$\vee \qquad \vee \qquad \vee \qquad \vee \qquad \leftarrow\text{Value comparison}$$
$$\text{Total sum of }\{\colorbox{lightgray}{$$}\} = \{\colorbox{lightgray}{$q_{max}+2$}\}+\{\colorbox{lightgray}{$q_{max}+4$}\}+\{\colorbox{lightgray}{$q_{max}+6$}\}+\{\colorbox{lightgray}{$q_{max}+8$}\}+\cdots\cdots$$

Therefore [Total sum of $\{\colorbox{yellow}{$$}\}$ > Total sum of $\{\colorbox{lightgray}{$$}\}$] holds.
In (Range 1)+(Range 2) we have [Total sum of $\{\colorbox{yellow}{$$}\} = A >$ Total sum of $\{\colorbox{lightgray}{$$}\}$].
We have the following (58).

$$f(n_{max}+1) - \{q_{max}\} < 2A \tag{58}$$

4.2.4. Condition for $A > B$

From (55) and (58) we have the following inequality.

$$f(n_{max}+1) - [\{q_{max}\} \text{ or } \{q_{max}-1\}] < 2A$$

As shown in item 4.1.3.1 $\{f(3)-f(2)\}$ is the maximum in all $|f(n)-f(n+1)|$. Then the following holds.

$$\{f(3)-f(2)\} > [\{q_{max}\} \text{ or } \{q_{max}-1\}]$$
$$\{f(3)-f(2)\} > f(n_{max}) - f(n_{max}+1)$$

We have the following inequality from the above 3 inequalities.

$$\begin{aligned} 2A >& f(n_{max}+1) - [\{q_{max}\} \text{ or } \{q_{max}-1\}] > f(n_{max}+1) - \{f(3)-f(2)\} \\ >& f(n_{max}) - \{f(3)-f(2)\} - \{f(3)-f(2)\} = f(n_{max}) - 2\{f(3)-f(2)\} \end{aligned} \tag{59}$$

We have the following (60) for $A > B$ from (52) and (59).

$$2A > f(n_{max}) - 2\{f(3)-f(2)\} > f(n_{max}) - f(2) + \{f(3)-f(2)\} > 2B \tag{60}$$

From (60) we can have the final condition for $A > B$ as follows.

$$(4/3)f(2) > f(3) \tag{61}$$

(Graph 5) shows $(4/3)f(2) - f(3) = (4/3)(\frac{1}{2^{1/2-a}} - \frac{1}{2^{1/2+a}}) - (\frac{1}{3^{1/2-a}} - \frac{1}{3^{1/2+a}})$.

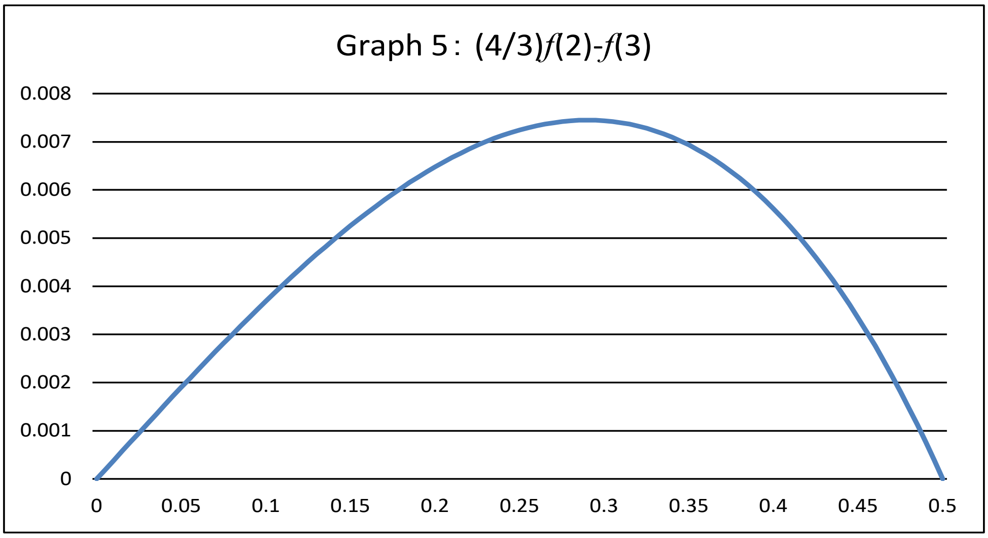


Table 1 : The values of $(4/3)f(2) - f(3)$

a	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
(4/3)f(2)-f(3)	0	0.001903	0.003694	0.005257	0.00648	0.007246	0.007437	0.006933	0.005611	0.003343	0

(Graph 6) shows [differentiated $\{(4/3)f(2) - f(3)\}$ regarding a] i.e. $(4/3)f'(2) - f'(3) = (4/3)\{\log 2(\frac{1}{2^{1/2-a}} + \frac{1}{2^{1/2+a}})\} - \{\log 3(\frac{1}{3^{1/2-a}} + \frac{1}{3^{1/2+a}})\}$.

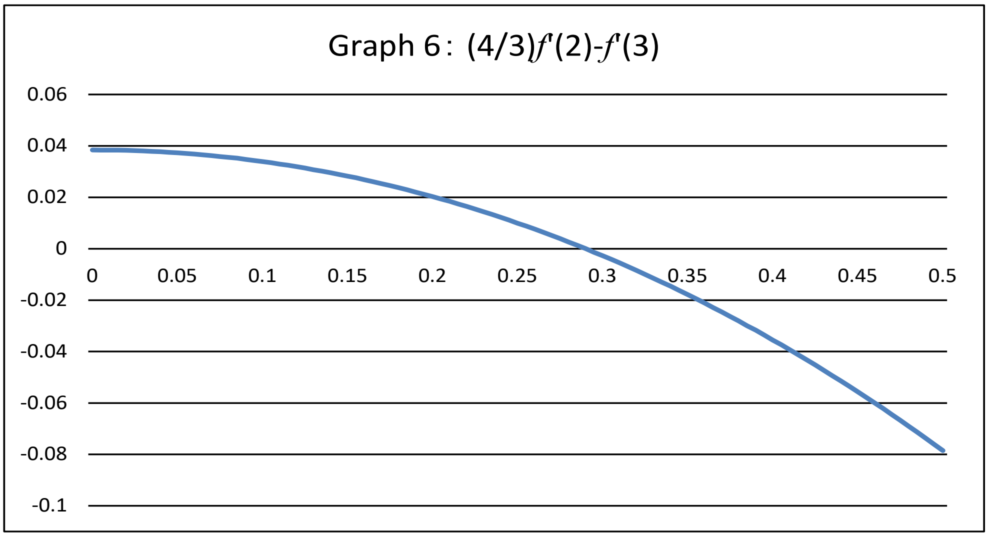


Table 2：The values of $(4/3)f'(2) - f'(3)$

a	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
(4/3)f'(2)−f'(3)	0.038443	0.037313	0.033921	0.02825	0.020277	0.009967	−0.00272	−0.01785	−0.03547	−0.05567	−0.07852

From (Graph 5) and (Graph 6) we can find $[(4/3)f(2) - f(3) > 0$ in $0 < a < 1/2]$ that means $A > B$ i.e. $F(a) > 0$ in $0 < a < 1/2$.

4.3. Verification of $A > B$ (n_{max} is even number.)

n_{max} is even number as follows.

$$F(a) = f(2) - f(3) + f(4) - f(5) + f(6) - \cdots\cdots$$
$$= \{f(2) - f(3)\} + \{f(4) - f(5)\} + \cdots + \{f(n_{max} - 4) - f(n_{max} - 3)\} + \{f(n_{max} - 2) - f(n_{max} - 1)\}$$
$$+\{f(n_{max}) - f(n_{max} + 1)\} + \{f(n_{max} + 2) - f(n_{max} + 3)\} + \{f(n_{max} + 4) - f(n_{max} + 5)\} + \cdots\cdots$$

We can have A and B as follows.

$$B = \{f(3) - f(2)\} + \{f(5) - f(4)\} + \{f(7) - f(6)\} + \cdots + \{f(n_{max} - 3) - f(n_{max} - 4)\} + \{f(n_{max} - 1) - f(n_{max} - 2)\}$$
$$A = \{f(n_{max}) - f(n_{max} + 1)\} + \{f(n_{max} + 2) - f(n_{max} + 3)\} + \{f(n_{max} + 4) - f(n_{max} + 5)\} + \cdots\cdots$$

$$f(n_{max}) = \{f(n_{max}) - f(n_{max} + 1)\} + \{f(n_{max} + 1) - f(n_{max} + 2)\} + \{f(n_{max} + 2) - f(n_{max} + 3)\}$$
$$+\{f(n_{max} + 3) - f(n_{max} + 4)\} + \cdots\cdots$$
$$= \{0\} + \{1\} + \{2\} + \{3\} + \{4\} + \cdots + \{q_{max} - 3\} + \{q_{max} - 2\} + \{q_{max} - 1\} + \{q_{max}\} + \{q_{max} + 1\} + \{q_{max} + 2\} + \{q_{max} + 3\} + \cdots\cdots$$

After the same process as in item 4.2.1 we can have the following (62).

$$f(n_{max} - 1) - f(2) + \{f(3) - f(2)\} > 2B \qquad (62)$$

As shown in item 4.1.3.1 $\{f(3)-f(2)\}$ is the maximum in all $|f(n)-f(n+1)|$. Then the following holds.

$$\begin{aligned}\{f(3)-f(2)\} >& [\{q_{max}\} \text{ or } \{q_{max}-1\}]\\ f(n_{max}) >& f(n_{max}-1)\end{aligned}$$

We have the following (63) from the above inequalities and the same process as in item 4.2.2 and item 4.2.3.

$$\begin{aligned}2A >& f(n_{max}) - [\{q_{max}\} \text{ or } \{q_{max}-1\}] > f(n_{max}) - \{f(3)-f(2)\}\\ >& f(n_{max}-1) - \{f(3)-f(2)\}\end{aligned} \tag{63}$$

We have the following (64) for $A > B$ from (62) and (63).

$$2A > f(n_{max}-1) - \{f(3)-f(2)\} > f(n_{max}-1) - f(2) + \{f(3)-f(2)\} > 2B \tag{64}$$

From (64) we can have the final condition for $A > B$ as follows.

$$(3/2)f(2) > f(3) \tag{65}$$

In the inequality of $[(3/2)f(2) > (4/3)f(2) > f(3) > 0]$, $(3/2)f(2) > (4/3)f(2)$ is true self-evidently and in item 4.2.4 we already confirmed that the following (61) was true in $0 < a < 1/2$.

$$(4/3)f(2) > f(3) \tag{61}$$

Therefore the above (65) is true in $0 < a < 1/2$. Now we can confirm $F(a) > 0$ in $0 < a < 1/2$.

4.4. Conclusion

As shown in item 4.2 and item 4.3 $F(a) = 0$ has the only solution of $a = 0$ due to $[0 \le a < 1/2], [F(0) = 0]$ and $[F(a) > 0$ in $0 < a < 1/2]$.

4.5. Graph of $F(a)$

We can approximate $F(a)$ with the average of $\{F(a, n-1) + F(a, n)\}/2$. But we approximate $F(a)$ by the following (66) for better accuracy. (Graph 7) shows $F(a)_n$ calculated at 3 cases of $n = 500, 1000, 5000$.

$$\frac{\frac{F(a,n-1)+F(a,n)}{2} + \frac{F(a,n)+F(a,n+1)}{2}}{2} = F(a)_n \tag{66}$$

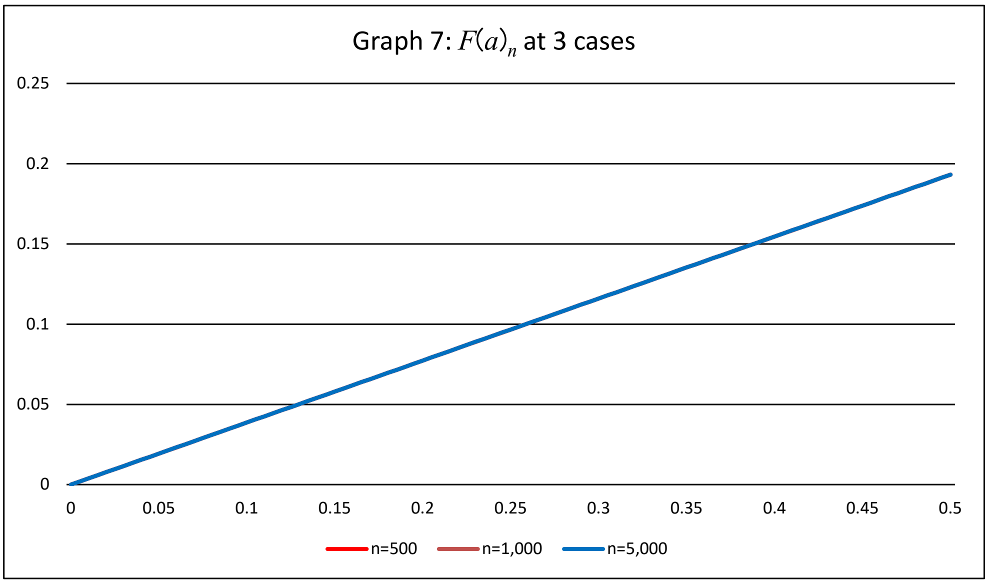


Table 3 : The values of $F(a)_n$ at 3 cases

a	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
n=500	0	0.01932876	0.03865677	0.05798326	0.0773074	0.09662832	0.11594507	0.13525658	0.15456168	0.17385904	0.19314718
n=1,000	0	0.01932681	0.03865282	0.05797725	0.0772993	0.09661821	0.11593325	0.13524382	0.15454955	0.17385049	0.19314743
n=5,000	0	0.01932876	0.03865676	0.05798324	0.07730738	0.09662829	0.11594504	0.13525655	0.15456165	0.17385902	0.19314718

3 line graphs overlapped. Because $F(a)_n$ calculated at 3 cases of $n = 500, 1000, 5000$ are equal to 4 digits after the decimal point. The range of a is $0 \leq a < 1/2$. $a = 1/2$ is not included in the range. But we added $F(1/2)_n$ to calculation due to the following reason. $[f(n)$ at $a = 1/2]$ is $(1-1/n)$ and $F(1/2)$ fluctuates due to $\lim_{n\to\infty} f(n) = 1$. But the value of the above (66) converges to the fixed value on the condition of $\lim_{n\to\infty} \{f(n+1)-f(n)\} = 0$. The condition holds due to $f(n+1) - f(n) = 1/(n + n^2)$.
$F(a)$ is a monotonically increasing function as shown in (Graph 7). So $F(a) = 0$ has the only solution and the solution must be $a = 0$ due to the following facts. Therefore Riemann hypothesis must be true.

4.5.1 In 1914 G. H.Hardy proved that there are infinite zero points on the line of $Re(s) = 1/2$.

4.5.2 All zero points found until now exist on the line of $Re(s) = 1/2$.

References

[1] Yukio Kusunoki, Introduction to infinite series, Asakura syoten, (1972), page 22, (written in Japanese)

Toshihiko ISHIWATA
E-mail: toshihiko.ishiwata@gmail.com