Exploring the patterns in the Dirichlet Eta Function

(And the theoretical proof of the Riemann Hypothesis)

By

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(Researched up to 2018 in Australia, released for public view in 2021)

This paper is dedicated to Srinivas Ramanujan

Disclaimer: The author believes that the scientific and mathematical knowledge should not be restricted to the experts in the field, so the author has made an attempt to write a very elementary proof in the simplest possible way any maths enthusiast or a general reader from outside the mathematical community may also understand. The paper has not yet been submitted to a paid journal but has been made public so that anyone who is interested in the Riemann Hypothesis can openly access the proof and review if they like.

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ABSTRACT:

A hidden beautiful pattern in Dirichlet Eta Series (and hence in Riemann Zeta Function) is discovered. A new 'Mewada' function is defined which forms a part of the Dirichlet Eta series, and helps to reduce the series to a new series in which the behaviour and the limits of the behaviour of the sum of the individual terms can be studied and compared with crystal clear clarity. It is shown that the Dirichlet Eta function $\eta_{(x-iy)}$ (where x,y ∈ $\mathbb{R}$) allows for 'Zero' at only one value of 'x' for '0<x<1', for any given value of 'y', and hence it can be shown that non-trivial 'Zeros' of Dirichlet Eta can only exist at x=0.5 if they fall within the critical strip 0<x<1. Consequently it can be concluded that non-trivial 'Zeros' of Riemann Zeta function can only be at x=0.5, thereby proving Riemann Hypothesis with absolute certainty. It is also shown that all the non-trivial 'Zeros' of the Riemann Zeta are simple 'Zeros'. It is then shown that the same method of proof can be generalised to the other Dirichlet L-Functions with suitable modifications, thereby proving the Generalised Riemann Hypothesis also to be true.

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Chapter: 1

OUTLINE OF THE PROOF IN BRIEF:

Background:

The Dirichlet Eta function η(s) where s=x−iy (for 0<x<1, 0<y<$\pm\infty$), being sum of non-geometric infinite convergent series, does not obey any meaningful equation to be able to predict it's behaviour, unlike say a polynomial equation or trigonometric equations or exponential equations, which can be studies using standard mathematical tools. To study the behaviour of η(s) through the range of 'x' from 0 to 1, for any given value of 'y', one needs to actually compute the value of η(s) for the range. Knowing how the η(s) behaves for some or even a lot of values of 'y' does not help much in predicating behaviour of the function at other values of 'y' in any meaningful way. Even if one studies the behaviour of η(s) for 'y' up to 10x10^{12} or even up to 10^{400} or so if future computer technology permits, does not guarantee that it may not behave differently at some much higher value. Which implies that even if all the non-trivial 'Zeros' of Riemann Zeta up to 10^{400} are found to be on x=0.5 line, it does not guarantee that there won't be any Zero off the critical line at some higher value of 'y'.

Also, none of the standard tools can be applied to these functions to calculate its maxima, minima, 'Zeros' etc., which generally tend to work with standard mathematical equations.

Also, the 'Zeros' of these functions behave like prime number, i.e. despite being unpredictable and having random-like nature, they are not purely random-functions either, so none of the randomness-based or statistics-based theorems can be used in deriving the proofs for the Zeros of Dirichlet Eta or Riemann Zeta.

This difficulty explains why the Riemann Hypothesis had so far remained unproven despite some of the most brilliant minds working on it for over a century.

However, if there is some underlying theoretical and logical reason why there just can't be more than 1 Zero of η(s) for 'x' in the range 0<x<1, for any given 'y', then Riemann Hypothesis can be proven.

A new approach:

A new approach, and still a pretty elementary approach, has been made as discussed in this paper to study some behaviour of Dirichlet Eta function, to ultimately prove the Riemann Hypothesis theoretically.

(The author does not claim sole credit for the proof presented, and thank all the people who provided useful insights to the author.)

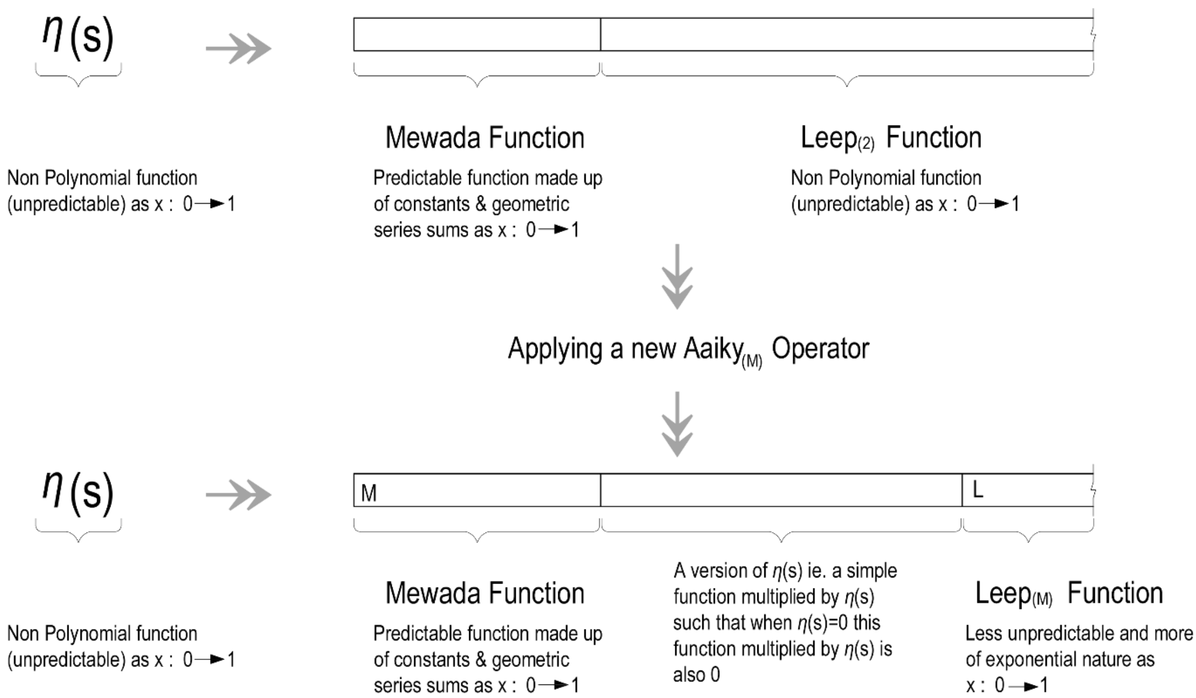
This paper does not rely on any functional equation of zeta, except to the extent of inferring that if there is a non-trivial 'Zero' for a given value of 'y' at only one value of 'x' in the range 0<x<1 then that 'Zero' can only be at x=0.5 and not anywhere else, to satisfy the Riemann Zeta functional equation.

This paper mostly relies on the actual behaviour of the Dirichlet Eta Series and the behaviour of the vectors of the individual terms of the series, and their behaviour in a group.

The author has derived a new operator $'Aaiky_{(M)}[\,]'$, which when applied to the convergent Dirichlet Eta Series η(x−iy), at 'Zeros' of the function, reduces it to a new convergent series, which can be represented as a sum of two new functions (convergent serieses) $'Mewada_{(M)}\,function'$ & $'Leep_{(x-iy,M)}\,function'$. The author shows that the $'Mewada_{(M)}'$ function is extremely well behaved and predictable for any 'y' and 0<x<1. The author then shows that the $'Leep_{(x-iy,M)}'$ function becomes more and more predictable and of the exponential nature as we increase the value of 'M' in the $'Aaiky_{(M)}[\,]'$ operator. Then, by comparing the graphs of the values of the two functions, we can see that they can intersect at maximum one point only, proving that there can only be maximum one Zero for 0<x<1 for any value of 'y'.

In the process of coming to the proof of Riemann Hypothesis, some very interesting patterns hidden inside the Dirichlet Eta Series are also observed, which will also be presented in this paper.

Consider Dirichlet Eta Function (a convergent series) to be made up of 2 separate converging serieses.



In the diagram above, for the baseline/main series of η(s) = 0 = $Mewada_{(x-iy)}$Function + $Leep_{(x-iy,2)}$ Function, no mathematical proof can derived for the conjecture that zero will be only at x=0.5

However, after applying $Aaiky_{(M)}$ operator to the series at η(s) = 0, we get a reduced series :

0 = $Mewada_{(x-iy)}$ + $Leep_{(x-iy,M)}$

At each 'Zero' of η(s), for any given 'y', the new equation must hold for each and every chosen value of 'M' i.e. the graphs of $Mewada_{(x-iy)}$ 'a fixed function' and $-Leep_{(x-iy,M)}$ 'a changing function' must intersect at all 'Zeros' at each and every value of chosen 'M'. This is obviously possible at only 1 point, like a curve rotating about a pivot. When we consider very high 'M' or as M→∞, we can mathematically prove that $-Leep_{(x-iy,\infty)}$ would be a vertical line, which can intersect $Mewada_{(x-iy)}$ at maximum 1 point only, thereby proving the hypothesis with absolute certainty.

Chapter: 2

THE R-FUNCTION AND THE MEWADA FUNCTION:

I define the 'R-function' $R_{(x-iy)}$ as a 'Geometric Progression Series' as follows:

R-Function $R_{(x-iy)}$ as sum of all the R-Terms, which are the terms where n=2,4,8,16,32,64,…

$$R_{(x-iy)} = \frac{1}{2^x}\angle y.log2 + \frac{1}{4^x}\angle y.log4 + \frac{1}{8^x}\angle y.log8 + \frac{1}{16^x}\angle y.log16 + \frac{1}{32^x}\angle y.log32 + \cdots to\ \infty$$

- Where 0 < x < 1 and 0 ≤ y < $\pm\infty$

i.e.

$$R_{(x-iy)} = \sum_{n=1}^{\infty} \frac{1}{(2^n)^x}\angle y.log(2^n)$$

… this is sum of the infinite number of polar vectors with values reducing at a fixed rate, where each vector is separated from previous vector by the same angle, exactly equal to $\angle y.log2$

$$R_{(x-iy)} = \frac{1}{2^x}\angle y.log2 \quad . \quad (1 + \frac{1}{2^x}\angle y.log2 + \frac{1}{4^x}\angle y.log4 + \frac{1}{8^x}\angle y.log8 + \frac{1}{16^x}\angle y.log16 + \cdots)$$

$$R_{(x-iy)} = \frac{1}{2^x}\angle y.log2 \quad . \quad (1 + R_{(x-iy)}) \quad = \quad \frac{1}{2^x}\angle y.log2 + \frac{1}{2^x}\angle y.log2 * R_{(x-iy)}$$

$$R_{(x-iy)} \quad . \quad (1 - \frac{1}{2^x}\angle y.log2) \quad = \quad \frac{1}{2^x}\angle y.log2$$

$$R_{(x-iy)} = \frac{1}{2^x\angle -y.log2 \quad -1}$$

Now I define 'Mewada Function' $Mewada_{(x-iy)}$ as $(1 - R_{(x-iy)})$:

$$Mewada_{(x-iy)} = 1 - \frac{1}{2^x\angle -y.log2 \quad -1} = \frac{2^x\angle -y.log2 \quad -1-1}{2^x\angle -y.log2 \quad -1} = \frac{2^x\angle -y.log2 \quad -2}{2^x\angle -y.log2 \quad -1}$$

At x = 1 – x = 0.5, this magically reduces to :

$$Mewada_{(0.5-iy)} = \sqrt{\mathbf{2}} \ \angle(2.\mathrm{Arctan}\left[\frac{\mathrm{Sin}(y.log2)}{\sqrt{2} - Cos(\mathrm{y.log2})}\right] - \pi + \mathrm{y.log2})$$

…which is very beautiful in the sense that no matter what the value of 'y' you chose, at x=0.5, the $Mewada_{(0.5-iy)}$ is always equal to $\sqrt{2}$, in value, with only the vector argument dependent on 'y'.

The 'value' of the Mewada Function generally depends on both 'x' and 'y', but when x = 0.5 , the 'value' magically becomes independent of 'y', and is fixed at √2.

Also note the following:

$Mewada_{(x-iy)}$ = 0   if and only if   x=1   and y.log2 = '0' or integer multiple of 2π

$Mewada_{(x-iy)}$ = ∞ if and only if   x=0   and y.log2 = 0 or integer multiple of 2π

We know that x=0 falls in the zero-free regions of Riemann Zeta, so we do not need to worry about the poles of Mewada function at x=0 & y.log2=integer multiples of 2π.

For anywhere within critical strip such that 0 < x < 1 (i.e. importantly at x ≠ 0, and x≠1) , the Mewada function is well-behaved, i.e. has no pole and no zero.

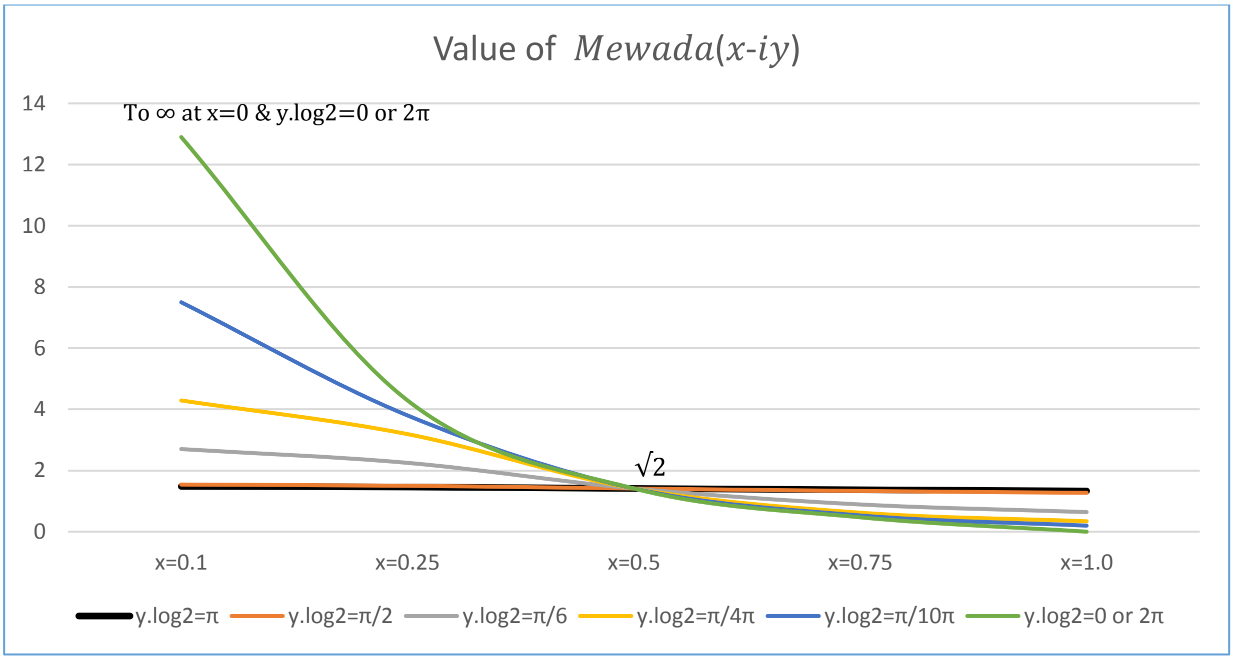
Mewada function has no maxima or minima for 0<x<1, so it does not reverse it's value fall …i.e. if 'x' increase from 0 to 1, value of Mewada function will only decrease in terms of absolute value of the polar vector, no matter what 'y' you choose.

Also, since 'Mewada' function is a function of 'x' and polar vector argument 'y.log2', it's cyclic at interval of 2π. So to study the behaviour of 'Mewada' function we only need to study it over 1 cycle of 2π, which covers the behaviour of the function for all the values of 'y' from –∞ to +∞ , unlike the zeta function or for eta function $\eta_{(x-iy)}$ for which one can't practically study the behaviour for all the values of 'y' up to 'infinity' within the critical strip 0<x<1.

Table M.1 : The values of $Mewada_{(x-iy)}$ for different values of 'y.log2' over a half cycle (it's symmetrical for the other half of the cycle)

y.Log 2 →	**0**	**0.1**	**π/4**	**1.5**	**π/2**	**1.7**	**3/4 π**	**3**	**π**
x = 0	∞	10 ∠-1.42	1.92 ∠-.678	1.59 ∠-0.34	1.58 ∠-.321	1.56 ∠-0.28	1.51 ∠-.137	1.5 ∠-0.023	1.5 ∠0
x = 0.1	12.9 ∠π	7.5 ∠-2.01	1.82 ∠-0.714	1.56 ∠-0.35	1.54 ∠-0.33	1.53 ∠-0.29	1.49 ∠-0.139	1.48 ∠-0..24	1.48 ∠0
x = 0.25	4.29 ∠π	3.8 ∠-2.42	1.67 ∠-0.76	1.5 ∠-0.36	1.5 ∠-0.335	1.49 ∠-0.295	1.46 ∠-0.14	1.46 ∠-0.024	1.46 ∠0
x = 0.4	2.1 ∠π	2.06 ∠-2.55	1.514 ∠-0.77	1.45 ∠-0.364	1.45 ∠-0.33	1.44 ∠-0.3	1.44 ∠-0.14	1.43 ∠-0.024	1.44 ∠0
x = 0.5	$\sqrt{2}$ ∠π	$\sqrt{2}$ ∠-2.57	$\sqrt{2}$ ∠-0.785	$\sqrt{2}$ ∠-0.364	$\sqrt{2}$ ∠-0.34	$\sqrt{2}$ ∠-0.3	$\sqrt{2}$ ∠-0.142	$\sqrt{2}$ ∠-0.024	$\sqrt{2}$ ∠0
x = 0.6	0.94 ∠π	0.97 ∠-2.56	1.32 ∠-0.78	1.38 ∠-0.363	1.38 ∠-0.34	1.38 ∠-0.3	1.4 ∠-0.141	1.4 ∠-0.024	1.4 ∠0
x = 0.75	0.47 ∠π	.53 ∠-2.42	1.145 ∠-0.76	1.33 ∠-0.36	1.33 ∠-0.335	1.34 ∠-0.295	1.37 ∠-0.141	1.37 ∠-0.024	1.37 ∠0
x = 0.9	0.15 ∠π	.27 ∠-2.01	1.09 ∠-0.714	1.28 ∠-0.35	1.29 ∠-0.33	1.3 ∠-0.29	1.34 ∠-0.14	1.35 ∠-0.029	1.35 ∠0
x = 1.0	0 ∠π	.198 ∠-1.42	1.04 ∠-0.68	1.26 ∠-0.34	1.27 ∠-0.32	1.28 ∠-0.28	1.32 ∠-0.137	1.33 ∠-0.02	1.33 ∠0

Diagram M.1 : Plot of the values of Mewada function for 0<x<1 and various 'y' over a half cycle



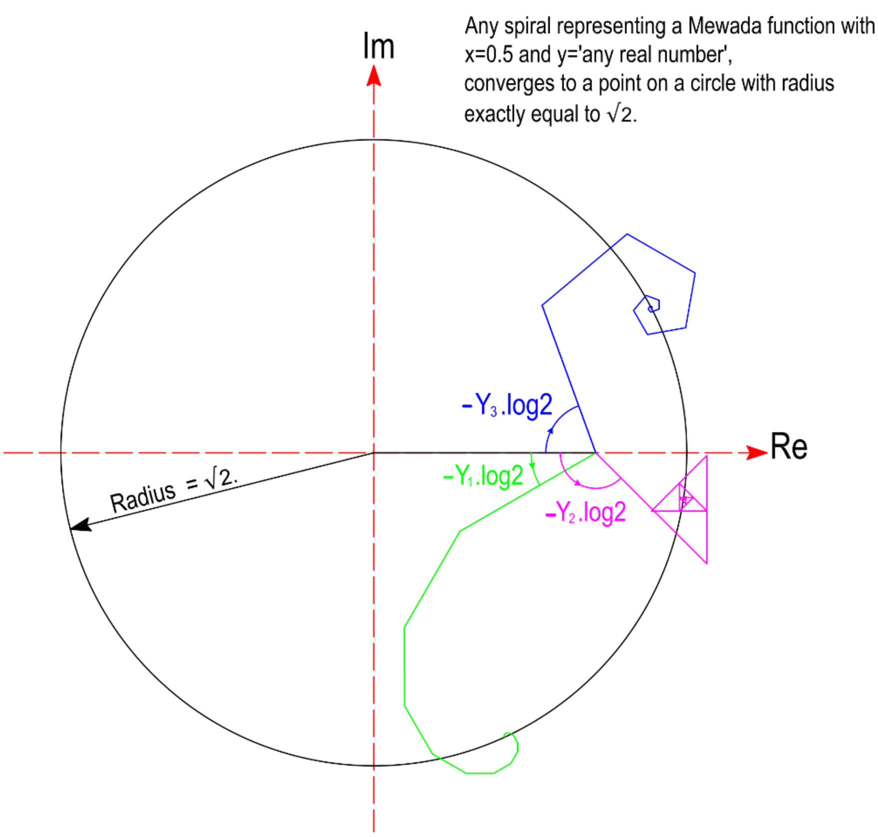


Diagram M.2 : Plot of spirals of vectors of function $Mewada_{(0.5 - iy)}$ for different values of 'y'

We will prove later in the paper that this 'Mewada' function is the beautiful function hidden in the "Zeros" of Dirichlet Eta series and hence in the "Zeros" of Riemann Zeta function.

Chapter: 3

THE DIRICHLET ETA FUNCTION/SERIES:

The **Dirichlet Eta Function** (aka Alternating Zeta Function) is defined by the following Dirichlet series, which converges for any complex number having real part > 0:

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots$$

…where s = (x - i y)… where x is 'real' part (>0) and 'iy' is imaginary part.
…where 'n' denotes the term number of the series.

This Dirichlet series is the alternating sum corresponding to the Dirichlet series expansion of the Riemann zeta function, $\zeta(s)$ — and for this reason the Dirichlet eta function is also known as the 'alternating zeta function', also denoted $\zeta^*(s)$. The following relation holds:

$$\eta(s) = \left(1 - 2^{1-s}\right)\zeta(s)$$

…where s = x–i y …where both 'x' & 'y' are real and 'i' is the imaginary operator

Zeros of Dirichlet Eta in critical strip (0<x<1) coincide with zeros of Riemann Zeta

$$\eta(s) = \frac{1}{1^{x-iy}} - \frac{1}{2^{x-iy}} + \frac{1}{3^{x-iy}} - \frac{1}{4^{x-iy}} + \frac{1}{5^{x-iy}} - \frac{1}{6^{x-iy}} + \ldots$$

This can be re-written to the following equivalent representation:

$$\eta(s) = \frac{1}{1^x}\angle y.log1 - \frac{1}{2^x}\angle y.log2 + \frac{1}{3^x}\angle y.log3 - \frac{1}{4^x}\angle y.log4 + \frac{1}{5^x}\angle y.log5 - \frac{1}{6^x}\angle y.log6 + \cdots to\ \infty$$

…all these individual terms of the series are equivalent to individual polar vectors with 'value' = $\frac{1}{n^x}$, and the vector-argument =y.log(n). Also, the –ve sign of a term would indicate that the vector is rotated by angle 'π'.

The value of $\eta(s)$ will be equal to the value of the resultant vector of sum of all the vectors of the series.

Chapter: 4

THE SUB-SERIESES OF DIRICHLET ETA SERIES AT 'ZEROS'

Consider the terms where n = 3, 6, 9, 12 , … (i.e. where 'n' is positive integer multiple of 3, or simply 3|n)

We got a mini series =>

⇨ $\frac{1}{3^x}\angle y.log3 - \frac{1}{6^x}\angle y.log6 + \frac{1}{9^x}\angle y.log9 - \frac{1}{12^x}\angle y.log12 + \frac{1}{15^x}\angle y.log15 - \cdots$

= $\frac{1}{3^x}\angle y.log3$. $(\frac{1}{1^x}\angle y.log1 - \frac{1}{2^x}\angle y.log2 + \frac{1}{3^x}\angle y.log3 - \frac{1}{4^x}\angle y.log4 + \frac{1}{5^x}\angle y.log5 - \cdots)$

= $\frac{1}{3^x}\angle y.log3$. η(s)

If the original series η(s) is 'convergent', then the mini-series/sub-series is also convergent, and shares the same progression pattern and same alternating signs as the original series.

So if η(s) = 0, i.e. at any root of Eta in critical strip, the sum of mini-series/sub-series of the terms where 'n' is integer multiple of 3, is also '0'.

Similarly we can observe that if η(s) = 0, sum of mini-series of the terms where 'n' is integer multiple of 5, is also '0'.

Similarly we can observe that if η(s) = 0, sum of mini-series of the terms where 'n' is integer multiple of any chosen odd prime number, is also '0'.

Similarly we can observe that if η(s) = 0, sum of mini-series of terms where 'n' is integer multiple of product of any 2 chosen odd prime numbers is also '0'.

For e.g.: Choose 3 & 5, (3 x 5 = 15) then the sum of series with only terms numbered 15, 30, 45, 60, 75, …∞ is also '0'

Choose 17 & 7, (17 x 7=11) then the sum of series with only terms numbered 119, 238, 357, 476, …∞ is also '0'

Similarly we can observe that if η(s) = 0, the sum of series of the terms where 'n' is integer multiple of product of any 3 chosen odd prime numbers, is also '0'.

For e.g.: Choose 3 & 5 & 11, (3 x 5 x 11=165) then the sum of series with only terms numbered 165, 330, 495, 660, …∞ is also '0'

Similarly we can observe that if η(s) = 0, the sum of series of the terms where 'n' is integer multiple of product of any number of chosen odd prime numbers, is also '0'.

For e.g.: Choose 5 & 7 & 13 & 23, (5x7x13x23=10465) then the sum of series with only terms numbered 10465, 20930, 31395, 41860…∞ is also '0'

i.e.

$\frac{1}{10465^x}\angle y.log10465 - \frac{1}{20930^x}\angle y.log20930 + \frac{1}{31395^x}\angle y.log31395 - \frac{1}{41860^x}\angle y.log41860 + \ldots\infty$ = '0'

… and so on …

[NOTE: While these theorems are very simple, and simple theoretical proof is given above, these have been computationally verified by the author, and any reader/scrutinizer may independently verify the same and/or request the details from the author on how these may be computationally verified]

<u>*Chapter: 5*</u>

THE 'AAIKY OPERATOR' & THE 'LEEP FUNCTION'

$\boldsymbol{Aaiky_{(M)}}[\,]$ & $\boldsymbol{Leep_{(\mathbf{x-iy},M)}}$

We can now introduce a new reduction operator $Aaiky_{(M)}[\,]$ *as follows:*

The $Aaiky_{(M)}[\,]$ *operator, when applied to Dirichlet Eta series, gets rid of all the terms up to n = M (M=any chosen +ve even integer) in the Dirichlet Eta series, and leaves only the following terms–*

- *'Term n=1' i.e.* $1/1^s$
- *'R-Terms with n=2,4,8,16,32,…' i.e.* $1/2^s$, $1/4^s$, $1/8^s$, $1/16^s$,…
- *Those terms which are either 'n=prime numbers > M', or 'n=(composite made of '2' and/or any prime-numbers > N)', all the way up to the ∞ , but excluding R-Terms as they are already listed above.*
  [There will be no terms where n=any integer multiple of 'any odd prime number up to M']

' $Aaiky_{(M)}[\,]$*' operator does not alter the 'zero value' of the equation if* η(s) *= 0 , i.e. as long as Aaiky operator is applied at* η(s) *= 0, i.e. at 'Zeros' of the Dirichlet Eta at any values of 'x' and 'y' that causes a 'Zero'.*

The proof is derived below:

Let's consider M=10, so we are going to apply $'Aaiky_{(10)}[\,]'$ operator to the η(s) at a 'zero', i.e. at (s) =0.

η(s) = 0 = $\frac{1}{1^x}\angle y.log1 - \frac{1}{2^x}\angle y.log2 + \frac{1}{3^x}\angle y.log3 - \frac{1}{4^x}\angle y.log4 + \frac{1}{5^x}\angle y.log5 - \frac{1}{6^x}\angle y.log6 + \cdots$

For simplicity we rename/re-assign the terms as follows:

T1 = $\frac{1}{1^x}\angle y.log1$

T2 = $\frac{1}{2^x}\angle y.log2$

T3 = $\frac{1}{3^x}\angle y.log3$ … and so on…

So now, at 'zero' we have,

η(s) = 0 = T1 – T2 + T3 – T4 + T5 – T6 + T7 – T8 + T9 – T10 + T11 – T12 + T13 – T14 + …

Now we know that if η(s) = 0, then the mini-series T3 – T6 + T9 – T12 + T15 – T18 +… = 0 (refer to chapter 4)

So let's get rid of all terms multiple of 3. Resultant value of the resultant series will also be zero.

[Addition/Subtraction of 2 Convergent Series each with value '0' results in a Convergent Series with value '0']

> [**<u>Note for general readers:</u>** We are adding a convergent series converging to '0' to another convergent series converging to '0', so we get a convergent series converging to '0'. This has no connection by any means to re-arranging a conditionally convergent series in any way that affects the progression rate of the series which may alter the value of the series. Not to be confused with rearrangement where we take out non-converging set of terms and reuse them in altered progression rate, which changes the value of the series. On the other hand, here, we are only adding 2 converging serieses without changing the value. Subtracting is same as adding, just with a –ve sign]

We now get

0 = T1 – T2 – T4 + T5 + T7 – T8 – T10 + T11 + T13 – T14 – T16 + T17 + T19 – T20 + …

…equation E0.3…

[NOTE: While these theorems are very simple, and simple theoretical proof is given above, these have been computationally verified by the author, and any reader/scrutinizer may independently verify the same and/or request the details from the author on how these may be computationally verified]

Now we also know that if η(s) = 0, then T5 – T10 + T15 – T20 + T25 – T30 +… = 0 (refer to chapter 4)

But we know that in the equation E0.3 above, we don't have terms +T15, -T30, +T45, -T60… because they got wiped out already while getting rid of all terms that were multiple of 3.

However we also know that if η(s) = 0, then T15 – T30 + T45 – T60 + T75 – T90 +… also = 0 (refer to chapter 4)

So, if we re-added the terms that are multiples of 15, back into equation E0.3, and then took out all terms that are multiple of 5, the resultant value = 0 will not change.

Now we get:

0 = T1 – T2 – T4 + T7 – T8 + T11 + T13 – T14 – T16 + T17 + T19 – T22 + T23 – T26 …

…Equation E0.5…

[NOTE: While the above equation is also very simple and simple theoretical proof is given above, this has been computationally verified by the author for various values of 'y', and any reader/scrutinizer may independently verify the same and/or request the details from the author on how these may be computationally verified]

Now we also know that if η(s) = 7, then T7 – T14 + T21 – T28 + T35 – T42 +… also = 0

But, we do not have terms that are multiples of 21 (i.e.3x7) or multiples of 35 (i.e.5x7) in the equation E0.5 because they already got wiped out earlier.

However the terms that are multiples of 21 total to '0' if η(s) = 0 (refer to chapter 4)

Also the terms that are multiples of 35 also total to '0' if η(s) = 0 (refer to chapter 4)

If we re-add terms that are multiples of 21 and also the terms that multiples of 35, back into the equation E.05, then it will not change the value of the equation = 0.

But, in this case we will have 2-copies of terms that are multiples of 105 (i.e. 3x5x7) added back to equation E0.5.

However that is also not a problem because all the terms that are multiples of 105 are also totalling to '0' at η(s) = 0. So we can get rid of these extra copies of those terms without altering the value of the equation = 0.

So we can get rid of all terms that are multiples of 7, from the equation E0.5, without altering η(s) = 0

Now we get:

η(s) = 0 = T1 – T2 – T4 – T8 + T11 + T13 – T16 + T17 + T19 – T22 + T23 – T26 …

…Equation E0.7…

[NOTE: While the above equation is very simple and simple theoretical proof is given above, this has been computationally verified by the author for various values of 'y', and any reader/scrutinizer may independently verify the same and/or request the details from the author on how these may be computationally verified]

In the same fashion, it can be shown that we can continue this operation up to any chosen value of 'M' without changing '0' value of the equation. This is only possible because of the beautiful relationships between sub-serieses of the Dirichlet Eta with the original series of the Dirichlet Eta. The link between the original series and any sub-serieses is the corresponding odd prime number.

…Interestingly such relationships within the series are not observed in various other Zeta functions (e.g. Epstein Zeta, Hurwitz Zeta except 2 cases, etc.) which may or may not have non-critical 'Zeros' off the critical line inside the critical strip, despite having similar 'functional equation'. So the proof derived in this paper is applicable to the Dirichlet Eta (Riemann Zeta), and to some other Dirichlet L-functions, but may not generally apply to all the other L-functions or to some other Zeta functions even if they may satisfy the functional equations… these will be discussed in later chapters.

Thus we see that now we only got following terms left in the equation, at η(s) = 0

(A) T1 i.e. 'n=1'
(B) T2, T4,T8,T16,T32,T64,...and so on... (I.e. all 'R-terms': n=2,4,8,16,32,...)
(C) All other terms that are numbered 'n'=prime number greater than 10 (for M=10)
(D) All other terms where n= composites of '2' and/or 'any prime numbers > 10'...except the 'R-Terms' already listed above.

(There will be no left-over terms where 'n' is an integer multiple of any 'odd prime <10')

(The + or – signs are maintained as per the original series, i.e. '+' for 'n=odd', '−' for 'n=even')

The Terms (A) and (B) above are collectively same as 'Mewada' function = $Mewada_{(x-iy)}$ (refer to chapter 2)

The Terms (C) and (D) above are now to be collectively called as a new 'Leep function' = $Leep_{(x-iy,M)}$, for M=10, it's = $Leep_{(x-iy,10)}$

So now, we define the 'Leep Function' $\boldsymbol{Leep_{(x-iy,M)}}$ as sum of the terms leftover in the Dirichlet Eta series for $\boldsymbol{\eta(s)}$ *or* $\boldsymbol{\eta(x-iy)}$, after applying $\boldsymbol{'Aaiky_{(M)}[\,]'}$ operator, such that 'n' is either any prime number > M, or 'n' is = (a composite number divisible by 'any prime numbers>M' and/or '2'), except the 'R-Terms' which are n=composite numbers made of only prime number '2'.

Thus, by applying operator ' $Aaiky_{(10)}[\,]$ ' to Dirichlet Eta series at η(s) = 0,

We get:-

$Aaiky_{(10)}$ [η(s)] = 0 = $Mewada_{(x-iy)}$ + $Leep_{(x-iy,10)}$

Thus, similarly if we apply operator $Aaiky_{(20)}[\,]$, i.e. $Aaiky_{(M)}[\,]$ with M=20, we get

$Aaiky_{(20)}$ [η(s)] = 0 = $Mewada_{(x-iy)}$ + $Leep_{(x-iy,20)}$

Similar we can show that if we apply operator $Aaiky_{(25)}[\,]$ for example, to Dirichlet Eta series at η(s) = 0,

We get

$Aaiky_{(25)}$ [η(s)] = 0 = $Mewada_{(x-iy)}$ + $Leep_{(x-iy,25)}$

While numerically it is a tedious task, it can be shown that we can apply $'Aaiky_{(M)}[\,]$' operator at any value of 'M', without any upper limit, all the way up to the infinity.

Thus for any chosen value of M, any positive even integer from M=2 up to M→∞,

We get

$Aaiky_{(M)}$ [η(s)] = 0 = $Mewada_{(x-iy)}$ + $Leep_{(x-iy,M)}$

...(The equation that will be used heavily in this paper)

Now let's look at how the Dirichlet Eta series look like after applying $Aaiky_{(M)}[\,]$ operator in normal terms…

If η(s) = 0, i.e. η(x-iy) = 0 for a given value of x and iy,

Applying $Aaiky_{(1000)}[\,]$ to Dirichlet Eta series, we get:

$Aaiky_{(1000)}$ [η(x-iy)] = 0 = $Mewada_{(x-iy)} + Leep_{(x-iy,1000)}$

$Aaiky_{(1000)}$ [η(x-iy)] = 0 = $Mewada_{(x-iy)} + \frac{1}{1009^x}\angle y.log1009 + \frac{1}{1013^x}\angle y.log1013 + \frac{1}{1019^x}\angle y.log1019 + \frac{1}{1021^x}\angle y.log1021 + \frac{1}{1031^x}\angle y.log1031 + \frac{1}{1033^x}\angle y.log1033 + \cdots - \frac{1}{2018^x}\angle y.log2018 - \frac{1}{2026^x}\angle y.log2026 + \frac{1}{2027^x}\angle y.log2027$ + …

Where…

$Leep_{(x-iy,1000)} = \frac{1}{1009^x}\angle y.log1009 + \frac{1}{1013^x}\angle y.log1013 + \frac{1}{1019^x}\angle y.log1019 + \frac{1}{1021^x}\angle y.log1021 + \frac{1}{1031^x}\angle y.log1031 + \frac{1}{1033^x}\angle y.log1033 + \cdots - \frac{1}{2018^x}\angle y.log2018 - \frac{1}{2026^x}\angle y.log2026 + \frac{1}{2027^x}\angle y.log2027$ + …

<u>Note1:</u> The $Leep_{(x-iy,1000)}$ does not contain any term with 'n' < 1000 (As we chose M=1000)

<u>Note2:</u> $Mewada_{(x-iy)}$ is constant w.r.t. 'M' for any given value of x & iy, i.e. it does not change with a change of 'M' in the operator $Aaiky_{(M)}[\,]$.

IMPORTANT NOTE: The value of function $Leep_{(x-iy,M)}$ generally change when we change 'M', however, at any 'Zero' of Dirichlet Eta, the function $Leep_{(x-iy,M)}$ does not change with 'M', and has to be always equal to - $Mewada_{(x-iy)}$. Otherwise it can't be a 'Zero' of Eta.

At 'Zero',

0 = $Mewada_{(x-iy)} + Leep_{(x-iy,M)}$ ..for each and every value of 'M'

So, at 'Zero' of Dirichlet Eta function,

$Leep_{(x-iy,M)} = -Mewada_{(x-iy)}$ ...for each & every value of 'M' … Even if we consider M → ∞

> [NOTE: While the above derivations are very simple and simple theoretical proof is given above, this has been computationally verified by the author for various values of 'y', for M=4, M=10, M=20, and any reader/scrutinizer may independently verify the same and/or request the author for details on how such computations may be done. If a reader/scrutinizer wants to independently verify the above by numerical computations method, for various values of 'y' for known non-trivial 'Zeros' of Eta/Zeta functions, and at reader's own chosen values of 'M', the author recommends to compute up to $'10^6$ x M' number of terms for medium accuracy of convergence, or, up to $'10^9$ x M' or higher number of terms for high accuracy of convergence, as obviously we can't compute up to infinite number terms with computers.]

ALTERNATIVE REPRESENTATION:

η(s) = $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\mathrm{n}^{x}} \angle y.log(\mathrm{n})$ = $\sum_{n=1}^{\infty} (-1)^{n+1}.\ T_n$ ...where $T_n = \frac{1}{\mathrm{n}^{x}} \angle y.log(\mathrm{n})$

At 'Zero',

η(s) = $\sum_{n=1}^{\infty} (-1)^{n+1}.\ T_n$ = 0 = $Mewada_{(x-iy)}$ + $Leep_{(x-iy,2)}$ ...i.e. baseline series, no terms eliminated

...Equation E0.0

Add following '0' value series to E0.0

$-\sum_{3|n}^{\infty} (-1)^{n+1}.\ T_n$ = 0 .... $(\text{i.e. all terms where } '3' \ divides\ 'n')$

we get $Aaiky_{(4)}$ [η(s)] = 0 = $Mewada_{(x-iy)}$ + $Leep_{(x-iy,4)}$

...Equation E0.3

Add following '0' value series to E0.3

$-\sum_{5|n}^{\infty} (-1)^{n+1}.\ T_n + \sum_{15|n}^{\infty} (-1)^{n+1}.\ T_n$ = 0

we get $Aaiky_{(6)}$ [η(s)] = 0 = $Mewada_{(x-iy)}$ + $Leep_{(x-iy,6)}$

...Equation E0.5

Add following '0' value series to E0.5

$-\sum_{7|n}^{\infty} (-1)^{n+1}.\ T_n + \sum_{21|n}^{\infty} (-1)^{n+1}.\ T_n + \sum_{35|n}^{\infty} (-1)^{n+1}.\ T_n - \sum_{105|n}^{\infty} (-1)^{n+1}.\ T_n$ = 0

we get $Aaiky_{(8)}$ [η(s)] = 0 = $Mewada_{(x-iy)}$ + $Leep_{(x-iy,8)}$

...Equation E0.7

Add following '0' value series to E0.7

$-\sum_{11|n}^{\infty} (-1)^{n+1}.\ T_n + \sum_{33|n}^{\infty} (-1)^{n+1}.\ T_n + \sum_{55|n}^{\infty} (-1)^{n+1}.\ T_n + \sum_{77|n}^{\infty} (-1)^{n+1}.\ T_n$

$-\sum_{165|n}^{\infty} (-1)^{n+1}.\ T_n - \sum_{231|n}^{\infty} (-1)^{n+1}.\ T_n - \sum_{385|n}^{\infty} (-1)^{n+1}.\ T_n + \sum_{1155|n}^{\infty} (-1)^{n+1}.\ T_n$ = 0

we get $Aaiky_{(12)}$ [η(s)] = 0 = $Mewada_{(x-iy)}$ + $Leep_{(x-iy,12)}$

...Equation E0.11

...And so on.

We get the general equation at η(s)=0, for any value of 'M':

$Aaiky_{(M)}$ [η(s)] = 0 = $Mewada_{(x-iy)}$ + $Leep_{(x-iy,M)}$

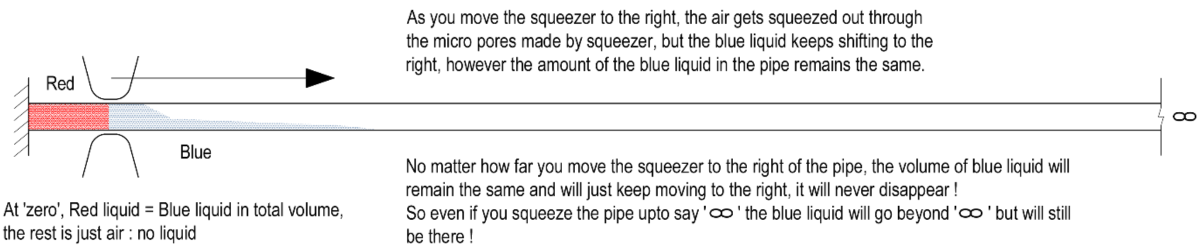
Interesting note/explanation for non-mathematician readers:

Question: What if someone hypothetically applies $Aaiky_{(\infty)}$ operator to the series at a 'Zero' of the η(s) ? Will it not get rid of all the terms from the Leep function/series because they are all multiples of odd prime numbers up to infinity ?

Answer: No, because no matter how high a number 'M' you chose, even if 'M→∞' the value of $Leep_{(x-iy,M)}$ at a 'Zero' will always remain same as $Mewada_{(x-iy)}$. While not mathematically accurate description of infinity, if you got rid of terms up to infinity, the terms beyond infinity will converge to the same value as $Mewada_{(x-iy)}$. In other words if you keep pushing the remainder to the right hand side of the series all the way to the infinity, the remainder will still have same value even at infinity.

In an attempt to give an intuitive illustration for the non-mathematician readers, consider the following illustrative picture. Imagine there is a tube that is infinitely long, i.e. with no end. If the tube contains a Red-liquid representing Mewada Function value, and a Blue-liquid representing Leep Function value, and the rest is just air representing other terms which converge to '0' value at Zero of η(s). If you separate Blue Liquid from Red liquid by squeezing the air out of the pipe and simultaneously moving the squeezer to the right, the Blue-liquid will keep moving in the tube towards the right, while the air is squeezed out of the tube though micro pores that filters out the air but not the liquid. Now, no matter how far you move the squeezer and the Blue-liquid to the right hand side, the amount of Blue-liquid in the tube will always remain same as the amount of Red-liquid in the pipe. Even if you move the squeezer toward infinity, the Blue-liquid will keep moving in the tube to the infinity but will still remain the same in volume. i.e. you just cannot get rid of the Blue-liquid by simply attempting to move the squeezer towards the infinity.

Similarly, at 'Zero' of η(s) the equation $Aaiky_{(M)}$ [η(s)] = 0 = $Mewada_{(x-iy)}$ + $Leep_{(x-iy,M)}$ will hold even if you consider M→∞ in the Aaiky operator. The value of Leep function will remain same as the value of Mewada function, no matter what 'M' one choses, at 'Zeros' of η(s), else there can't be a 'Zero' of η(s).



<u>***Chapter: 6***</u>

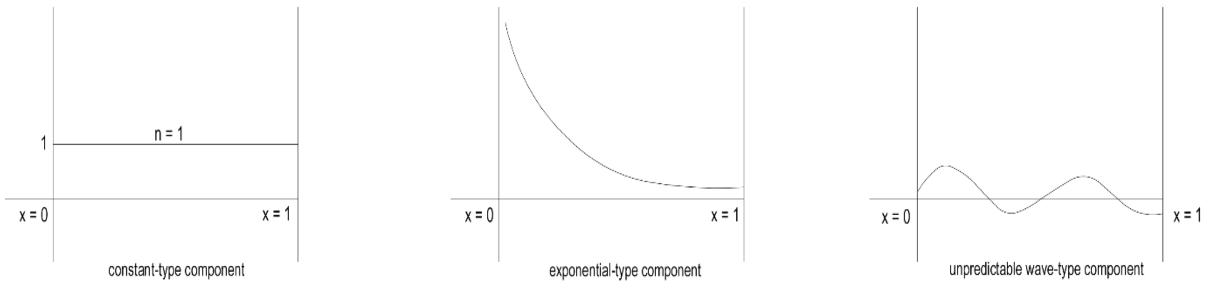
BUILDING THE THEORY FOR THE PROOF:

[This chapter is intended only to help with understanding how and why the proof works, and it's not the proof in itself]

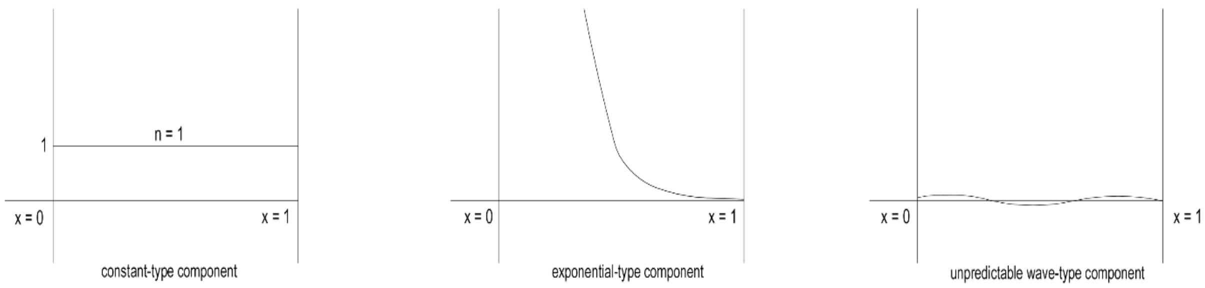
The Dirichlet Eta Function being the limit/sum of a convergent series of infinite number of reducing value vectors, not following any polynomial or exponential or trigonometric equations, can't be summarily studied in terms of any predictable or standard mathematical equations. We actually need to compute the value of the function for each different value of the variables. Despite having a degree of unpredictable behaviour, for sake of understanding, we can consider the function to be like a sum of 3 types of components: Constants, Exponentials, and Unpredictable wave-type.

Applying the Aaiky operator, increases the effect of the exponential components and reduces the effect of the unpredictable components. Higher the value of the 'M' of the Aaiky operator, lesser the effect of the unpredictable components w.r.t. the exponential components.

Components of η(s) at M = 2, ie. in baseline series

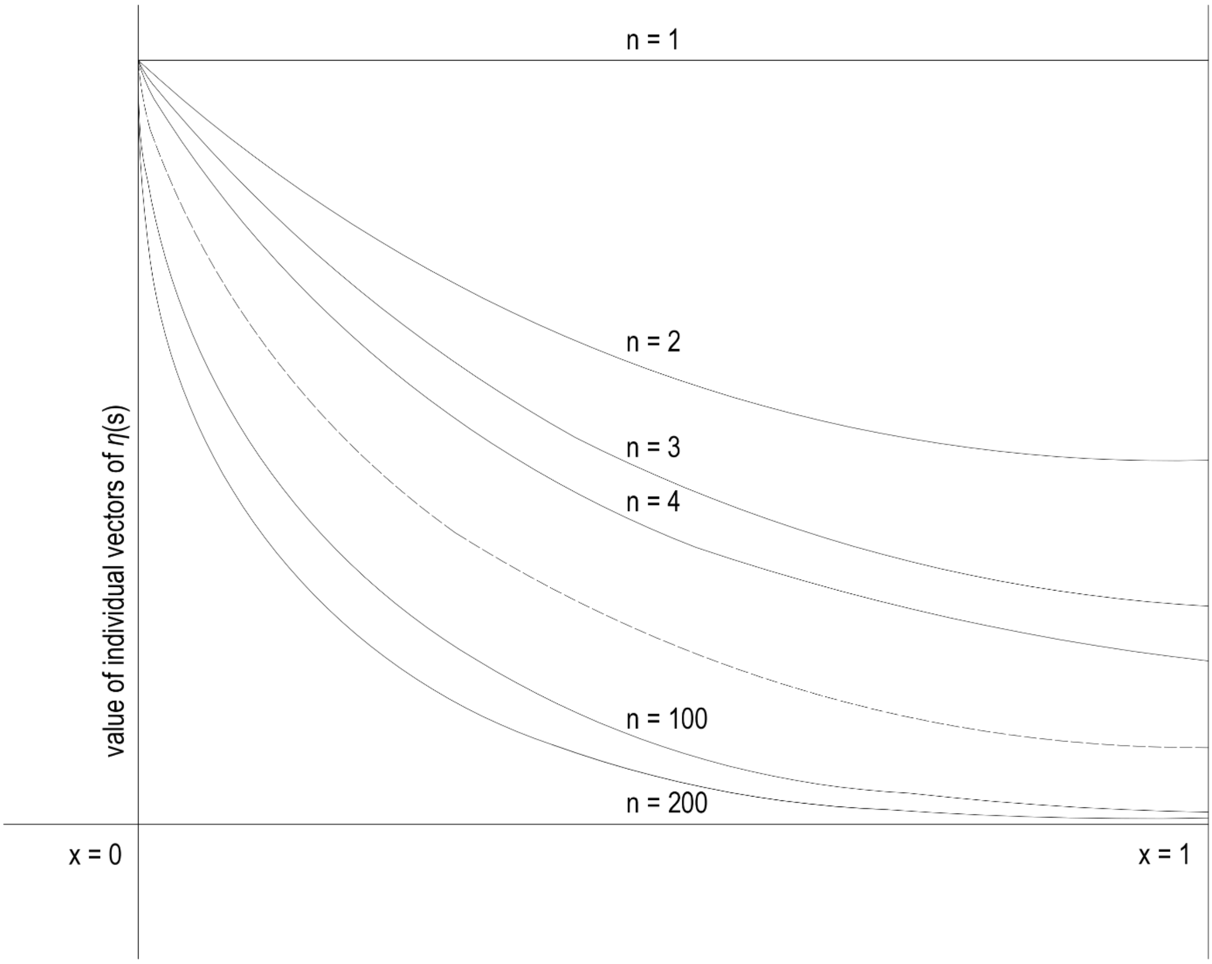


Components of reduced η(s) at high M, ie. after applying Aaiky Operator



In order to understand how this works, we need to study how the individual vectors representing the terms of the Dirichlet Eta series behave individually and in the group.

The following diagram (not accurate to the scale) shows how the 'value' of the individual vectors fall from value=1 to a lower value, for different 'n' , as we move from x=0 to x=1.



Except for n=1, they all fall towards value=0, at different rates depending on the value of 'n'. Vectors with higher number 'n' fall in value at higher rate.

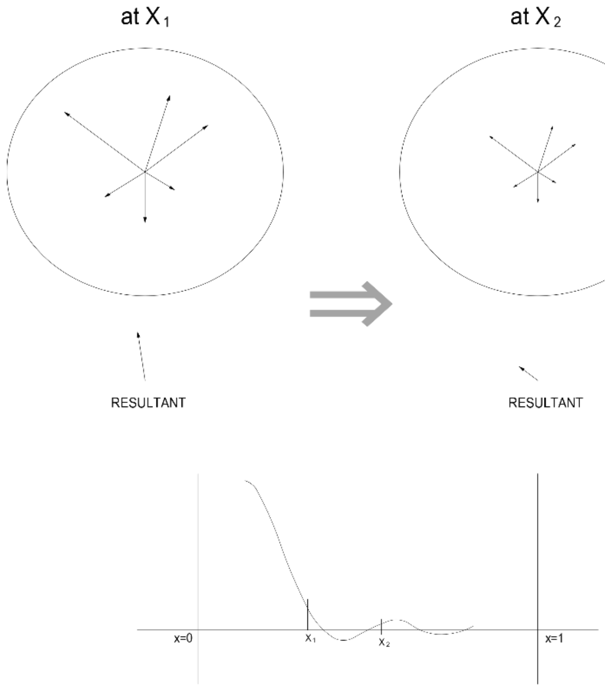
The rate of fall = derivative of the vector:

For vector $T_{\mathrm{n}} = \frac{1}{n^x} \angle y.log(n)$, the rate of fall = $\frac{log(n)}{n^x} \angle y.log(n) \quad \because \frac{d}{dx}\frac{1}{n^x}\angle y.log(n) = -\frac{log(n)}{n^x}\angle y.log(n)$

Also, the rate of fall at any point in proportion to the value of the vector = $(-\frac{d}{dx} T_{\mathrm{n}}) / T_{\mathrm{n}}$ = log(n)

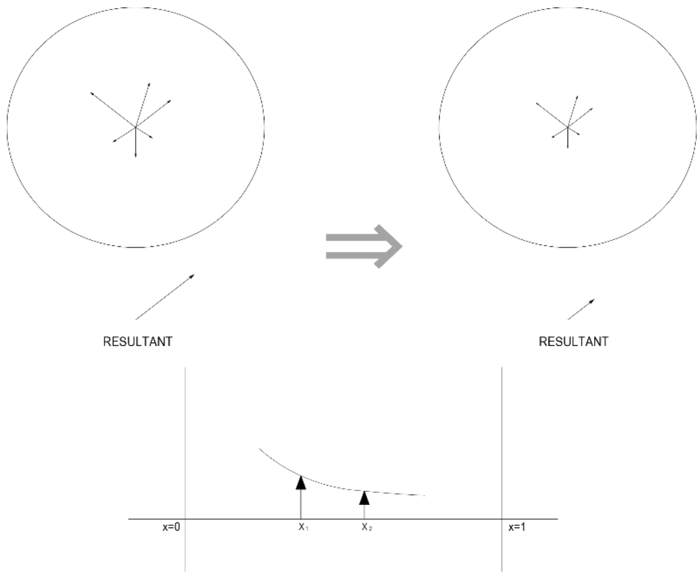
(The author refers to this as per-unit-value fall-rate, in later parts of this paper)

Consider a group of some vectors, to intuitively study their behaviour, at 2 different values of 'x':



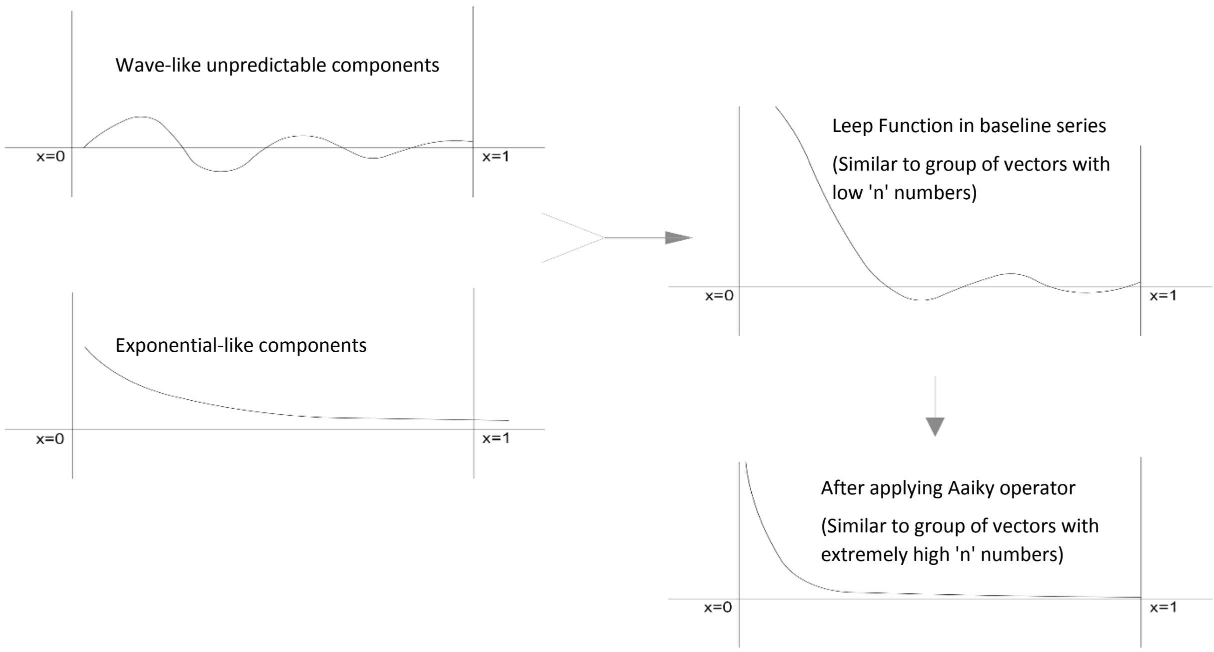
The illustrative (not to the scale) diagram above for group of vectors with low value of 'n' , say n=2, 3, 4, 5, 10, 12, 18,19 etc..

The resultant vector despite having general tendency to fall with increase in 'x', can behave in unpredictable wave-like fashion, because the per-unit-value fall-rate of individual vectors in the group is quite different from one another; log2 for n=2, log5 for n=5, log18 for n=18 etc., i.e. all different from one another.



The illustrative diagram above (not to the scale) for group of vectors with high value of 'n', say n=1000009, 1000013, 1000031 etc..

The resultant vector still share the same general tendency to fall with increase in 'x', and the behaviour is much more predictable (not meaning predictable in value but meaning predictable in terms of the shape of the graph of the function) because of negligible effect of unpredictable wave-like behaviour, because the per unit-value fall-rate of individual vectors in the group is more or less equal with respect to one another (except when 'x' is close to '0'). The per-unit-value fall-rate of individual vectors will be $\approx$ log1000009 for all the vectors in the group.

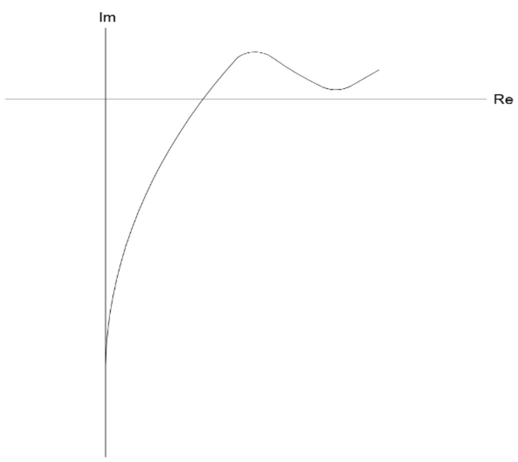


<u>Consider η(x + i 77777) :</u> If we plot the graph for η(s) for 'x' ranging from '0' to '1', we can notice some wave-like behaviour with both the maxima and the minima present in the graph. Then, one can always think of possibility of such a function passing through 'Zero' at more than one location for some other value of 'y' (even if none observed yet).

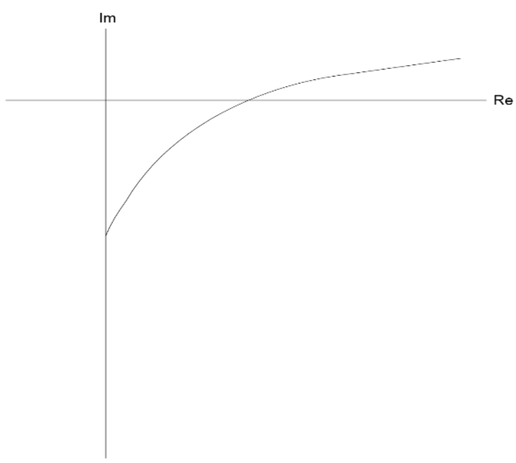
Let's study η(x + i 77777) in two components: $Mewada_{(x+i77777)}$ & $Leep_{(x+i77777,2)}$

As discussed in a previous chapter, the Mewada function is always well behaved and is like a gentle sloping line. Let's look at $Leep_{(x+i77777,2)}$ for 0<x<1. The graph is plotted below. We can see some wave-like nature, despite the general tendency of the function to lose value with increase in 'x'.

Now apply $Aaiky_{(10)}$[ ] operator to $\eta(\mathrm{x} + \mathrm{i}77777)$, and get $Leep_{(x+i77777,10)}$. The graph for $Leep_{(x+i77777,10)}$ is plotted below (not to scale). We can see that even for 'M' as low as just '10', we get rid of all wave-like components from the graph, and it becomes more or less like of an exponential and constant components. Such wave-free graph can't intersect the graph of Mewada function at 2 separate locations. If we consider higher and higher 'M', the Leep function becomes more and more wave-free.



$\mathrm{Leep}_{(x+i77777,2)}$
Baseline



$\mathrm{Leep}_{(x+i77777,10)}$
After applying Aaiky (10) Operator

We can look at any value of 'y' where there is any unpredictable wave-like nature in η(x-iy), and when we start applying Aaiky operator, the wave-like components disappear and the Leep functions becomes more and more wave-free and of exponential nature.

The author has numerically checked/verified this for several inputs. The readers can also check and verify numerically for whatever inputs they like. The author obviously doesn't rely on mountain of empirical evidence as any proof whatsoever, but this gives the readers an intuitive insight as to why the theory actually works.

While the function $Leep_{(x-iy,M)}$ changes with change of chosen value of 'M', the function $'Mewada_{(x-iy)}'$ is independent of 'M'.

For any chosen real value of 'y' , if there is/(are) 'Zero'/('Zeros') of Eta function for 0<x<1, then at each 'Zero', $'Leep_{(x-iy,M)}'$ must be equal to '$-Mewada_{(x-iy)}$', for any and all values of chosen 'M', right from M=2 up to extremely high 'M', and even if we consider M→∞

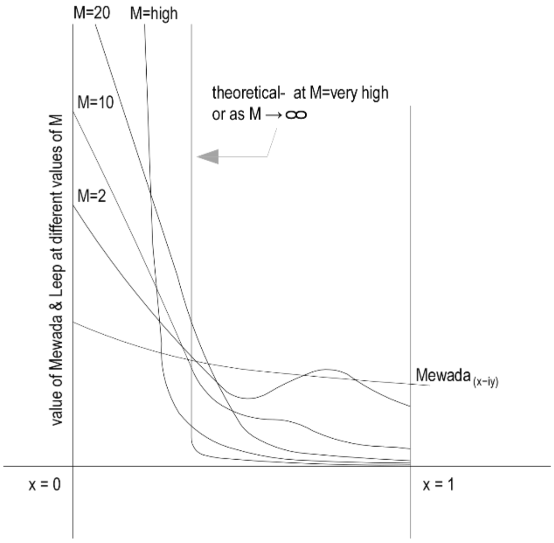
While the above conditions are possible to be met and that is indeed always the case if there is only 1 'Zero' (i.e. at $x = 0.5$) for any given value of 'y', we will prove that the above conditions are absolutely-impossible to be met if there were 'hypothetically' 2 or more 'Zeros' of Eta at any given value of 'y' within 0<x<1.

If you chose any value of 'y' for which there is a single non-critical 'Zero', i.e. at x=0.5, then

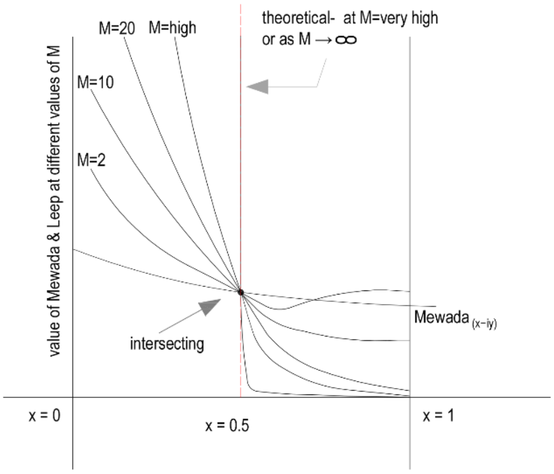
$Leep_{(0.5-iy,M)}$ = $-Mewada_{(0.5-iy)}$ = √2 in value and 'polar vector argument' dependent on 'y' ....for any, each, and every value of 'M' … Even if M → ∞

[NOTE: While the above observations are very simple, this has been computationally verified by the author for various values of 'y', for M=4, M=10, M=20, and any reader/scrutinizer may independently verify the same and/or request the author for details on how such computations may be done. If a reader/scrutinizer wants to independently verify the above by numerical computations method, for various values of 'y' for known non-trivial 'Zeros' of Eta/Zeta functions, and at the reader's own chosen values of 'M', the author recommends to compute up to $'10^6$ x M' number of terms for medium accuracy of convergence, or, up to $'10^9$ x M' or higher number of terms for high accuracy of convergence, as obviously we can't compute upto infinite number terms with computers.]

Graphs of some typical 'y' randomly chosen by the Author, one without a 'Zero', another with a 'Zero'.



If there are no 'Zeros' of η(s) at given value of 'y'



If there is/are 'Zero/s' of η(s) at given value of 'y'

The diagram above represents the empirical observations for some typical know 'Zero' of Dirichlet Eta, just to give the readers an intuitive explanation of how the author's theory actually works. Leep and –Mewada function intersect at 'Zero' of the Eta, for each and every value of 'M'. Similar graphs for Leep Function up to M=20 for several known 'Zeros' and for non-zeros are computationally verified by the author, however for high 'M' and for M→∞ the graphs are only theoretical due to unavailability of necessary computational resources. The interested readers are welcome to verify up to higher values of 'M' if they like and if they have resources, although there is no real need for computational verification up to '∞' as the theory is well founded. (The plots in the diagram above are for 'vector values' only and not the 'vector angle', because they are 2D plots)

If in a hypothetical scenario there were more than 1 'Zero' for a real value of 'y' and 0<x<1, then it is even theoretically absolutely-impossible for conditions to be met at more than 1 location of 'x', when we change value of 'M', each and every time, all the way to 'M → ∞'.

Let's say that in a hypothetical scenario there are 2 'Zeros' of Eta for a given value of 'y', at x=X1 and at x=X2, such that X1=1-X2.

$Leep_{(X1-iy,M)} = -\ Mewada_{(X1-iy)}$ ..for each and every value of 'M' … Even if M → ∞

&

$Leep_{(X2-iy,M)} = -\ Mewada_{(X2-iy)}$ ..for each and every value of 'M' … Even if M → ∞

The author will prove in later sections that these conditions are absolutely impossible to me met at more than 1 location, unless of-course if X1=X2=0.5

$Leep_{(X1-iy,M)}$ and $Leep_{(X2-iy,M)}$ would be incomparable in value/size if very high 'M' is chosen. At extremely high value of 'M' the Leep function drops its value exponentially at astonishingly high rate as 'x' goes from X1 to X2. This makes intersection of Mewada & −Leep functions at 2 different points impossible. (This is proven mathematically in the next chapter.)

At low 'M', i.e. M=2, we have original equation, where it's difficult to deduce any conclusion about change of '$Leep_{(x-iy,M)}$' when x changes from X1 to X2.

Because we don't have direct solution for Eta or Zeta, and values are different for different 'x' and different 'y', it's extremely difficult to prove that we can't have more than one zero at any given value of 'y'

However, the $Aaiky_{(M)}[\,]$ operator helps in making the comparison lot easier, and crystal clear, at higher value of M, without changing the value of the equation η(x+iy) = 0, by getting rid of the most of the terms where 'n'<M

In a nut-shell:

The $Leep_{(x-iy,M)}$ function changes with every change in 'M', and gets exponentially steeper as 'M' increases.

The $Mewada_{(x-iy)}$ function is gentle sloping in 0<x<1 range and is a fixed function that doesn't change with change in 'M'

The 'Zeros' of η(s) act like pivot joints where $Mewada_{(x-iy)}$ & $-Leep_{(x-iy,M)}$ must intersect for each and every 'M'.

The reasons why there can't be more than one 'Zero' are multiple, and any one of these is a sufficient reason:

1. It is impossible for there to be more than 1 pivot joints for 2 graphs where one graph is constant and the other graph keep changing its slope with each change of 'M'.
   (This is the "Logical" reason. And, the author doesn't know how to prove this mathematically. If some reader has useful suggestion to offer to the author for a better theory, they are most welcome)
2. It is absolutely impossible for there to be more than 1 point of intersection, if one graph is a gentle sloping line and the other graph is a pure vertical line as we consider M→∞
   (This is the "Mathematical" reason… And, the author knows how to prove this, and it's done in the next chapter)

Chapter: 7

THE BEHAVIOUR OF THE 'LEEP FUNCTION' FOR HIGH VALUE OF 'M' AND AS M→∞ AND THE PROOF OF THE RIEMANN HYPOTHESIS

This chapter has the proof that :

1) The function $Leep_{(x-iy,M)}$ become almost-exponentially-value-dropping function with negligible wave-like behaviour at high values of 'M' , for X_0<x<1 at any value of 'y'. (X_0: a small +ve number close to '0')

&

2) The function $Leep_{(x-iy,M)}$ becomes pure exponentially-value-dropping function, dropping like a pure vertical line as we consider M→∞, for X_0<x<1 at any value of 'y'.

Because at x=0 the Mewada Function may have a pole for some values of 'y', the Leep function may also have a pole at x=0, so the author does not study the behaviour of Leep Function at 'x' equal to exactly '0'. Also it is well known that Riemann Zeta function has a 'Zero free region' and there are no 'Riemann Zeta Zeros' on x=0 line, we do not need to worry about the behaviour of Leep Function at x=0. We study the behaviour for X_0<x<1 only.

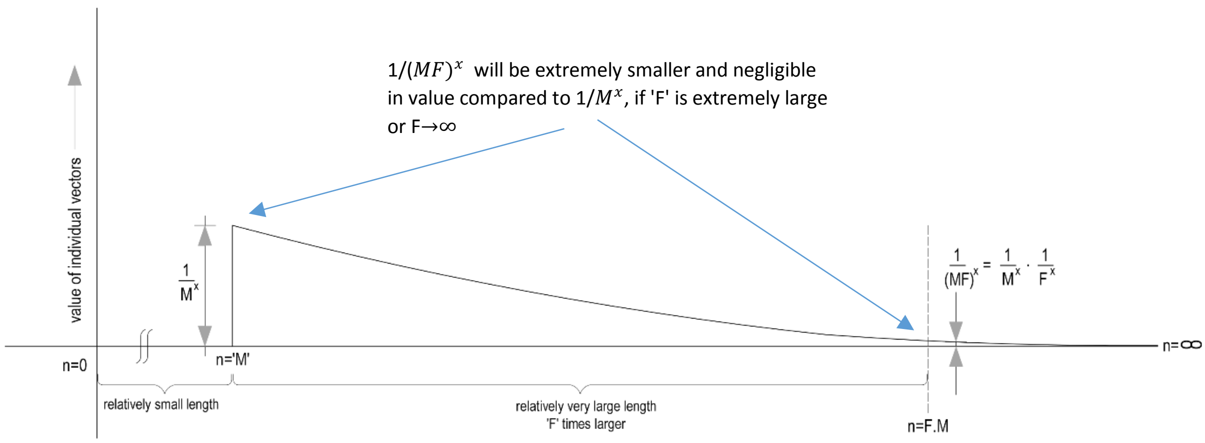
To study the behaviour for x>0, we choose an arbitrary value x = X_0 such that X_0 is very a small positive real number very close to '0' but not exactly '0'. Let's say we chose $X_0 = 10^{-3}$, so we will be studying the behaviour of Leep Function between 0.001 to 1, which is sufficient to demonstrate how the proof works (although there is no theoretical limit on how small X_0 one may choose, as theoretically the proof remains valid as long as $X_0 > 0$, and $X_0 \neq 0$. Theoretically, if one wants to (though there in no need to) one can always chose X_0 to even fall in 'Zero-free-region' close to '0' , no matter how narrow the region may be.

No matter how small, or infinitesimally small, a number one may chose for X_0 close to '0', there can always be a high number 'F' such that $\frac{1}{(F\,.\,M)^{X_0}}$ $i.e. \frac{1}{F^{X_0}\,.\,M^{X_0}}$ is infinitesimely smaller compared to $\frac{1}{M^{X_0}}$, such that partial series up to 'n'=F.M term accurately represents value of the full series with negligible error. (The negligible error will actually become '0' as we consider limits later on)

Let's say we want to include the terms in the series, with vector magnitude ranging all the way from $\frac{1}{(M)^{X_0}}$ up to $\frac{1}{10^{10}\,.\,(M)^{X_0}}$.

Then, for our example we would need to choose F = $(10^{10})^{10^3}$ = 10^{10^4} = 10^{10000}

[If we take Lim F→∞, we will actually cover whole series up to n=∞, and up to the term with value equal to '0'. I.e. with no error. But, in order to study the limits we first need to start with a high but a finite value]



We got the 'Leep Function':

$Leep_{(x-iy,M)} = \frac{1}{(M+P1)^x} \angle y.log(M+P1) + \frac{1}{(M+P2)^x} \angle y.log(M+P2) + \frac{1}{(M+P3)^x} \angle y.log(M+P3)$ + ...

- ... + ... $\pm\frac{1}{(F.M)^x} \angle y.log(F.M)$ +/- discarded tiny terms (zero error if we consider Lim F→∞)

(Signs to be retained as per the original series, + for odd numbered terms, and – for even numbered terms)

....Where 'n'=M+P1 is the 1st term of $Leep_{(x-iy,M)}$, n=M+P2 is the 2nd term, ...and so on up to n=F.M being the term at which $\frac{1}{F^{X_0}}.\frac{1}{M^{X_0}}$ becomes infinitesimly smaller/negligible in comparison with $\frac{1}{M^{X_0}}$.

The value of this series from ('n'=M)th term up to (n=F.M)th term, will be almost equal to the value of series all the way up to (n=∞)th term, with extremely tiny error if we are choosing F=10^{10000}. To get perspective, if we were to examine the Leep function at x=0.5, the term $\frac{1}{(F.M)^x}$ and terms discarded after that would be $\frac{1}{10^{5000}}$ times smaller than the initial terms included in the $Leep_{(x-iy,M)}$ function.

There will be no error or say the error=0, if we take Limit F→∞.

Now, no matter how high a number 'F' one choses, there can always be a higher number 'M' such that : F^F=M

For our example, we are choosing M = F^F= $[10^{10000}]^{10^{10000}}$ = $10^{10^{10004}}$

If 'F' is very high F . F^F is 'F' times bigger than F^F . For our example F. F^F is 10^{10^4} times bigger than F^F.

If we consider Lim F→∞, F . F^F will still be be infinite-times bigger than F^F, even if $F^F$is infinite itself.

To study the behaviour of the Leep function for X_0<x<1, we take its derivative w.r.t. 'x'

$$\frac{d}{dx} Leep_{(x-iy,M)} = \frac{d}{dx}\frac{1}{(M+P1)^x} \angle y.log(M+P1) + \frac{d}{dx}\frac{1}{(M+P2)^x} \angle y.log(M+P2) + \frac{d}{dx}\frac{1}{(M+P3)^x} \angle y.log(M+P3)$$

$$+ ... - \frac{d}{dx}\frac{1}{(F.M)^x} \angle y.log(F.M)$$

$$= \frac{-log(M+P1)}{(M+P1)^x} \angle y.log(M+P1) + \frac{-log(M+P2)}{(M+P2)^x} \angle y.log(M+P2) + \frac{-log(M+P3)}{(M+P3)^x} \angle y.log(M+P3) + ... - ... + ... - ... - \frac{-\log(F*M)}{(F.M)^x} \angle y.log(F.M)$$

$$= -\text{logM} . [\frac{log(M+P1)/logM}{(M+P1)^x} \angle y.log(M+P1) + \frac{log(M+P2)/logM}{(M+P2)^x} \angle y.log(M+P2) +$$

$$\frac{log(M+P3)/logM}{(M+P3)^x} \angle y.log(M+P3) + ... - ... - \frac{log(F.M)/logM}{(F.M)^x} \angle y.log(F.M)]$$

For very high 'F':

$\frac{log(F.F^F)}{logF^F} = \frac{log(F^{F+1})}{logF^F} = \frac{(F+1)\, logF}{F\, .\, logF} \simeq 1$ (because (F+1)/F ≃ 1 for very high 'F')

for our example it would be $\frac{1+10^{10000}}{10^{10000}} \simeq 1$

$so, \frac{log(F^F+P1)}{log(F^F)} \simeq 1$ (because $F^F + P1 \ll F.F^F$) ..and so on...

... because all the $log(F^F + P_k)$ are smaller than $log(F.F^F)$, all $log(F^F + P_k)/ log(F^F) \simeq 1$

… for our example it would be $\frac{(something<1)+10^{10000}}{10^{10000}} \backsimeq 1$

And, Lim F→∞ $\frac{log(F.F^F)}{logF^F}$ = 1

Lim F→ ∞ $\frac{log(F^F+P1)}{log(F^F)}$ = 1 (because $F^F + P1 \ll F.F^F$) ..and so on…

And since M=F^F, $log(F^F)$ = logM, and so:

$$\frac{d}{dx}\ Leep_{(x-iy,M)} \backsimeq -\text{logM}\ .\ [\frac{1}{(M+P1)^x}\ \angle y.log(M+P1) + \frac{1}{(M+P2)^x}\ \angle y.log(M+P2) +$$

$$\frac{1}{(M+P3)^x}\ \angle y.log(M+P3) + \ldots - \text{etc. Up To } \frac{1}{(F.M)^x}\ \angle y.log(F.M)\ ]$$

So,

$\frac{d}{dx}\ Leep_{(x-iy,M)} \backsimeq -$logM . $Leep_{(x-iy,M)}$ …For Vey high 'M' and X_0<x<1

…what this means is that if one choses extremely high value of 'M', all the individual terms of the $Leep_{(x-iy,M)}$ function fall at almost same per-unit-fall-rate with respect to each other as we observe from x= X_0 to x=1. (in our example from x=0.001 to x=1)

And, as we consider M→∞ $\frac{d}{dx}\ Leep_{(x-iy,M)} = -$logM . $Leep_{(x-iy,M)}$

…what this means is that if we consider M→∞, all the terms of the $Leep_{(x-iy,M)}$ function simultaneously fall at exactly the same per-unit-fall-rate (i.e. derivative of a function per unit-value of the function) with respect to each other as we observe from x= X_0 to x=1 (in our example from x=0.001 to x=1).

Since Z = V. b^x is the general solution for $\frac{d}{dx}Z = Z.log(b)$

…where 'V' is some constant or a function independent of 'x'

We get : $Leep_{(x-iy,M)} \backsimeq$ V . $1/M^x$ for extremely high 'M' … for X_0<x<1

And, we also get:

As M→∞ , $Leep_{(x-iy,M)}$ = V . $1/M^x$ …for X_0<x<1

(we don't need to worry about evaluating the value of 'V' as we are only interested in the shape of the graph of the 'Leep Function' at extremely high value of 'M', and as M→∞)

We observe that for "high" 'M', the function $Leep_{(x-iy,M)}$ is almost exponentially-value-dropping function at extremely high rate as 'x' goes from X_0 to 1, irrespective of how small positive value of X_0 you chose close to '0'

And, as M→∞, the function $Leep_{(x-iy,M)}$ is a vertical line, falling from almost ∞ to '0'.

Now, a 'vertical' line for $-\ Leep_{(x-iy,M)}$ function at high value of 'M' (or as M→∞) can only intersect the graph of $Mewada_{(x-iy)}$ function (a gentle sloping graph which doesn't depend on 'M') at maximum of only 1 point!

[because $Leep_{(x-iy,M)}$ + $Mewada_{(x-iy)}$ = 0, for each and every value of 'M', at 'Zeros' of Dirichlet Eta function.]

Thus, $Leep_{(x-iy,M)}$ + $Mewada_{(x-iy)}$ = 0 is possible only at 1 point, and not at more than 1 point.

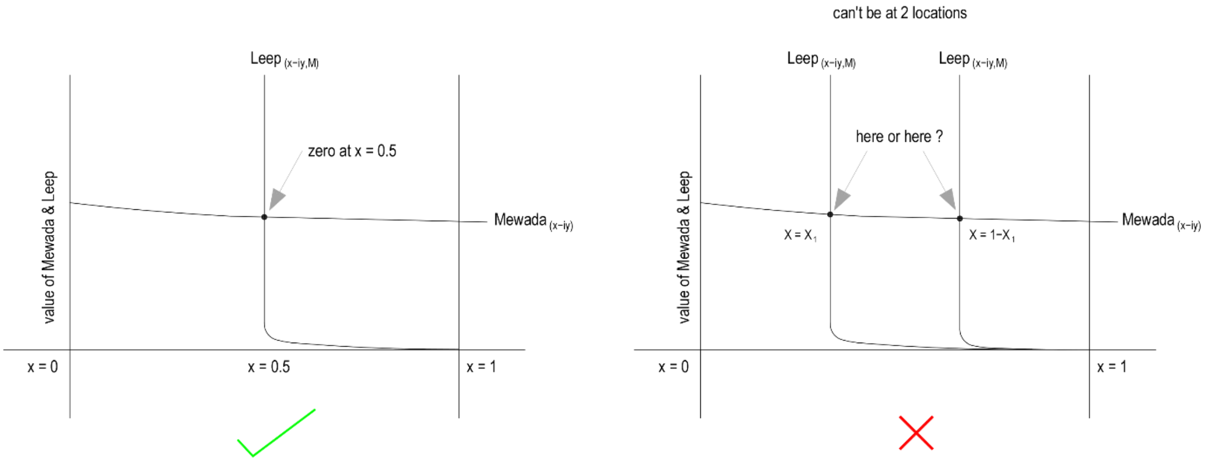
That point has to be at x=0.5 if 0<x<1 (Because of the Riemann Zeta functional equation). *

[* That doesn't have to be the case for Zeros at x=1 & y.log2=2π,4π,6π etc., i.e. at Zeros of Mewada function, because these are unimportant Zeros of Dirichlet Eta but they are not the Zeros of Riemann Zeta so those unimportant zeros of Dirichlet Eta don't need mirror-twins. If there is a Zero of Dirichlet Eta within the critical strip (0<x<1) then there has to be another Zero of Dirichlet Eta at (1-x), i.e. mirror-twin, because of the Riemann Zeta functional equation. This does not apply for x=1 and y.log2=2π,4π,etc. where there are Zeros of Dirichlet Eta but not of the Riemann Zeta. (J.Sondow 2003)

Note1: If in a hypothetical scenario, for some value of 'y' let's say 'V' hypothetically turns out to be a small value or say '0' for a particular value of a high 'M', then $Leep_{(x-iy,M)}$ won't follow 'almost infinity' to 'almost zero' downhill path; however in such a hypothetical scenario there won't be any 'Zero' in the first place (neither at x=0.5 nor at x= 0.5+d & x=0.5-d) of the Eta function for that particular value of 'y', because $Leep_{(x-iy,M)}$ will then never be equal to $-Mewada_{(x-iy)}$ at any value of 'x' when 'x' goes from ' X_0' to 1, as $Leep_{(x-iy,M)}$ will be extremely tiny in value in comparison to the value of the $Mewada_{(x-iy)}$ function.

Note2: Even if a reader/scrutiniser comes up with an arbitrary hypothetical combination of polar vectors each having n>M, which may have nothing to do with Dirichlet Eta function, such that their sum (i.e. $Leep_{(X_1-iy,M)}$) is exactly equal to $-Mewada_{(X_1-iy)}$ for x= X_1, i.e. arbitrarily constructed a hypothetical 'Zero' at ' X_1', then when you move to x= X_2 (i.e. x=1- X_1) the value of $Leep_{(x-iy,M)}$ will drop by a factor $1/M^{X_2-X_1}$, which means $Leep_{(X_2-iy,M)}$ will be 'almost zero' in value, because of extremely high value of 'M' (or when $M \to \infty$), so $|Leep_{(X_2-iy,M)}|$ << $|Mewada_{(X_2-iy)}|$, thus $Leep_{(X_2-iy,M)}$ + $Mewada_{(X_2-iy)}$ ≠ 0, thus there can't be a 'Zero' at $x = X_2$, even in that hypothetical scenario of arbitrarily chosen vectors, let alone in the systematically distributed patterns of vectors of Dirichlet Eta function.

For practical purposes, or for computational verification, at 'M' value as low as 20 or so, the almost-exponentially dropping and almost-wave-free nature of $Leep_{(x-iy,M)}$ function is very obvious. However to prove theoretically with absolute certainty the author has shown that there is no theoretical limit up to which Aaiky operator can be used to make Leep function more and more exponentially-value-dropping type and more and more wave-free, such that when you go for very high M, or consider M→∞, the Leep Function becomes a vertical line passing through a 'Zero', which obviously can only be at x=0.5 (if within 0<x<1 range), because a vertical line cannot pass through 2 'Zeros' at 2 different locations, unless of course if $X_1 = X_2 = 0.5$



Note: The zeros of Dirichlet Eta (but not of Riemann Zeta) can occur at x=1 for y.log2=2π,4π,etc., because they are not the Zeros of Riemann Zeta and don't need mirror-twins for Riemann Functional Equation. This was addressed by J.Sondow (2003) who proved that zeros at x=1 for y.log2=2π,4π,etc. are not the Zeros of Riemann Zeta. They plays no role in Riemann Hypothesis. In those special cases x=1 & y.log2=2π,4π,etc. i.e. at the Zeros of Mewada function, the Mewada and −Leep functions may intersect on real axis at x=1, and so not in 0<x<1 range, resulting in unimportant zeros of Dirichlet Eta but not of Riemann Zeta.

Thus we can conclude that all the non-trivial zeros of Riemann Zeta in the critical strip 0<x<1 are indeed on the critical line x=0.5 as has always been observed.

Chapter: 8

A STEP FURTHER – ALL NON-TRIVIAL ZEROS OF RIEMANN ZETA ARE SIMPLE

Now that we have proven that non-trivial 'Zeros' are only possible at x=0.5, let's check if there can be any double 'Zeros' at x=0.5

Let's go back to how the operator $Aaiky_{(M)}[\,]$ actually operates on a Convergent Eta series η(s) "in general" for 0<x<1, i.e. this time we consider general η(s) and not just at 'Zero' of η(s).

We got the Dirichlet Eta series:

η(s) = $\frac{1}{1^x}\angle y.log1 - \frac{1}{2^x}\angle y.log2 + \frac{1}{3^x}\angle y.log3 - \frac{1}{4^x}\angle y.log4 + \frac{1}{5^x}\angle y.log5 - \frac{1}{6^x}\angle y.log6 + \cdots$

….which is a 'Convergent Series' for x > 0

When we apply the operator $Aaiky_{(4)}[\,]$, what we are actually doing is deducting a sub-series, which is also a convergent series :

i.e. $\frac{1}{3^x}\angle y.log3 - \frac{1}{6^x}\angle y.log6 + \frac{1}{9^x}\angle y.log9 - \frac{1}{12^x}\angle y.log12 + \frac{1}{15^x}\angle y.log15 - \frac{1}{18^x}\angle y.log18 + \cdots$

which is equal to η(s). $\frac{1}{3^x}\angle y.log3$

The resultant series is also a convergent series if we are adding/subtracting 2 convergent series.

So,

$Aaiky_{(4)}$ [η(s)] = η(s) − (s) . $\frac{1}{3^x}\angle y.log3$

= η(s) . ($1 - \frac{1}{3^x}\angle y.log3$)

Similarly if we were to apply the operator $Aaiky_{(6)}[\,]$ what we would get is a resultant converging series:

$Aaiky_{(4)}$ [η(s)] = η(s) − (s) . $\frac{1}{3^x}\angle y.log3 -$ η(s) . $\frac{1}{5^x}\angle y.log5$ + η(s) . $\frac{1}{15^x}\angle y.log15$

= η(s) . ($1 - \frac{1}{3^x}\angle y.log3 - \frac{1}{5^x}\angle y.log5 + \frac{1}{15^x}\angle y.log15$)

…refer to Chapter#5 if required to recall how $Aaiky_{(M)}$ operator was originally used for each step

..and so on..

If we apply $Aaiky_{(M)}$[ ] to (s) , as long as η(s) is convergent, we get a convergent series with following value:

$Aaiky_{(M)}$ [η(s)] = (s) . ($1 - \frac{1}{3^x}\angle y.log3 - \frac{1}{5^x}\angle y.log5 - \frac{1}{7^x}\angle y.log7$ – … + …+ $\frac{1}{15^x}\angle y.log15 + \cdots$

$-\frac{1}{105^x}\angle y.log105$ – … + … – … and so on)

…how long the series goes depends on what value of 'M' you chose

Thus

$Aaiky_{(M)}$ [η(s)] = η(s) . (1 – Some convergent series depending on chosen value of 'M')

So

$Aaiky_{(M)}$ [η(s)] = η(s) . $Mystery_{(\mathrm{x-iy},\ M)}$

For the purposes of this paper we do not need to worry about the content of the $Mystery_{(\mathrm{x-iy},M)}$ as long as we know that it is convergent, which is indeed the case at x > 0.

We know that

$Aaiky_{(M)}$ [η(s)] = $Mewada_{(x-iy)}$ + $Leep_{(x-iy,M)}$ […which would be = '0' if η(s) = 0]

η(s) . $Mystery_{(\mathrm{x-iy},\ M)}$ = $Mewada_{(x-iy)}$ + $Leep_{(x-iy,M)}$

…as long as x>0, these are all convergent series

Now we know that for non-critical 'Zeros' of Riemann Zeta, x = 0.5 (refer to the proof in chapter 7)

So

(0.5-iy) . $Mystery_{(0.5-\mathrm{iy},M)}$ = $Mewada_{(0.5-iy)}$ + $Leep_{(0.5-iy,M)}$

…this equation has to be valid at each and every value of chosen 'M'.

For there to be 'Double Zero' at x=0.5, for any given 'y', we know that it's first derivative w.r.t. 'x' has to be '0'

$\frac{d}{dx}\eta$(x-iy) = 0 at x=0.5 ..for 'Double Zero'

So, for 'Double Zero':

$\frac{d}{dx}$ ($Mewada_{(x-iy)}$ + $Leep_{(x-iy,M)}$) = 0 at x=0.5

…which means that the 2 functions $Mewada_{(x-iy)}$ & $-Leep_{(x-iy,M)}$ will have to be tangential at x=0.5 for each and every chosen value of 'M', that is even if M→∞

As discussed in a previous chapter, while it's hard determine/generalise the behaviour of $Leep_{(x-iy,M)}$ function in base line Dirichlet Eta series (i.e. at M=2, meaning that no $Aaiky_{(M)}$ operator applied), we know that as we go for high value of 'M', $Leep_{(x-iy,M)}$ function become an almost exponentially value dropping curve, which is almost a vertical line at very high value of 'M' or when M→∞, whereas, as studied before, the $Mewada_{(x-iy)}$ is gentle sloping (almost flat) line at x=0.5, with value =√2 regardless of what 'y' is chosen.

At high value of 'M' or if required we consider as M→∞, we can see that the 2 functions $Mewada_{(x-iy)}$ & $-Leep_{(x-iy,M)}$ just cannot intersect tangentially. To determine the exact slope of the two graph at the point of intersection requires the value of 'y', however we are not interested in determining the value of the slopes. We are only interested in the fact that at the point of intersection (i.e. at x=0.5 if there are Zeros) the slope of Mewada function is very low in value compared to slope of Leep function (which is almost ∞ as M→∞), for any given 'y', thus the two slopes just cannot be equal at any 'Zero'.

So the following condition cannot be satisfied :

$\frac{d}{dx}$ ($Mewada_{(x-iy)}$ + $Leep_{(x-iy,M)}$) = 0 at x=0.5 , for each and every value of chosen 'M', and that's definitely impossible as M→∞.

Thus, we know that there can not be any 'Double Zero', thereby proving that all Non-Trivial Zeros of Riemann Zeta Function are indeed 'Simple Zeros' as has always been computationally verified, though the computations are limited up to a finite height. The author just proved that even in theory, there can't be any 'Double Zero' even if one checks up to infinite height.

Chapter: 9

GENERALIZATION OF THE PROOF TO SOME OTHER SIMILAR DIRICHLET L-FUNCTIONS WITH MODIFICATIONS

The Mewada proof of Riemann-Hypothesis may be extended to some other similar functions like other Dirichlet L-Functions, though it cannot be extended to each and every type of Zeta functions.

Let's first consider the method to extend the proof of Riemann-Hypothesis for Dirichlet L-Function for the nontrivial character of conductor 3 i.e. $\boldsymbol{L}(\mathbf{s}, x_3)$

> Note: The author chose to extend the proof to this L-Function after reading Dr. J Brian Conrey's 2003 AMS article where he mentioned that there are striking analogies between Riemann Zeta and some L-Functions though the connections were not fully understood.

$L(\mathrm{s}, \chi_3) = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{4^s} - \frac{1}{5^s} + \frac{1}{7^s} - \frac{1}{8^s} + \cdots$

Assign s = x − i y …where 'x' and 'y' are real, and x > 0, we get:

$L(\mathrm{s}, \chi_3) = \frac{1}{1^x} \angle y.log1 - \frac{1}{2^x} \angle y.log2 + \frac{1}{4^x} \angle y.log4 - \frac{1}{5^x} \angle y.log5 + \frac{1}{7^x} \angle y.log7 - \frac{1}{8^x} \angle y.log8 + \cdots$

At "Zeros" , $L(\mathrm{s}, \chi_3) = 0$

$0 = \frac{1}{1^x} \angle y.log1 - \frac{1}{2^x} \angle y.log2 + \frac{1}{4^x} \angle y.log4 - \frac{1}{5^x} \angle y.log5 + \frac{1}{7^x} \angle y.log7 - \frac{1}{8^x} \angle y.log8 + \cdots$

…Base Series LE.0

$0 . \frac{1}{5^x} \angle y.log5 = \frac{1}{5^x} \angle y.log5 \ . \ (\frac{1}{1^x} \angle y.log1 - \frac{1}{2^x} \angle y.log2 + \frac{1}{4^x} \angle y.log4 - \frac{1}{5^x} \angle y.log5 + \frac{1}{7^x} \angle y.log7 - \frac{1}{8^x} \angle y.log8 + \cdots)$

$0 = \frac{1}{5^x} \angle y.log5 - \frac{1}{10^x} \angle y.log10 + \frac{1}{20^x} \angle y.log20 - \frac{1}{25^x} \angle y.log25 + \frac{1}{35^x} \angle y.log35 - \frac{1}{40^x} \angle y.log40 + \cdots$

…Series LE.5

Add this convergent series LE.5 to the base series LE.0 , and we get rid of the terms where 'n' divisible by 5. We get new convergent series LE.0.5

Similarly we can get:

$0 = \frac{1}{7^x} \angle y.log7 - \frac{1}{14^x} \angle y.log14 + \frac{1}{28^x} \angle y.log28 - \frac{1}{35^x} \angle y.log35 + \frac{1}{49^x} \angle y.log49 - \cdots$

…Series LE.7

Also

$0 = \frac{1}{35^x} \angle y.log35 - \frac{1}{70^x} \angle y.log70 + \frac{1}{140^x} \angle y.log140 - \frac{1}{165^x} \angle y.log165 + \frac{1}{245^x} \angle y.log245 - \cdots$

…Series LE.35 …because 3x5=35

Add convergent series LE.35 to LE.0.5 then subtract LE.7 from the result, we get resultant convergent series LE.0.7

…This series will not have any terms where 'n' is divisible by either 5 and/or 7.

Basically we can continue the same operation i.e. apply Aaiky-Operator as discussed in earlier chapters.

Refer to earlier chapters for details on how 'Aaiky' operator can be applied up to any value 'M' and result will be a convergent series with value converging to '0', at "Zeros" of the base line series.

Here, after applying $Aaiky_{(M)}$ [L(s, χ_3)] at "Zero" of L(s, χ_3) we get:

$0 = MewadaL_{(x-iy)} + LeepL_{(x-iy,M)}$

Where $MewadaL_{(x-iy)}$ is the 'Modified Mewada Function' for L(s, χ_3) and it will also be an absolutely convergent series for x>0, just like 'Mewada Function', but with a different value.

$$MewadaL_{(x-iy)} = 1 - \frac{1}{2^x}\angle y.log2 + \frac{1}{4^x}\angle y.log4 - \frac{1}{8^x}\angle y.log8 + \frac{1}{16^x}\angle y.log16 - \frac{1}{32^x}\angle y.log32 + \cdots$$

Just like $Mewada_{(x-iy)}$, the new $MewadaL_{(x-iy)}$ function will also have no poles as long as x>0, and $MewadaL_{(x-iy)}$ function will not change the value with change of 'M' in the Aaiky operator.

Just like $Leep_{(x-iy,M)}$ function, the new $LeepL_{(x-iy,M)}$ function will also be dependent on value of 'M' chosen in Aaiky operator, and $LeepL_{(x-iy,M)}$ function will also be an 'Almost vertical line' for very high value of 'M' or if M→∞, and will intersect $-MewadaL_{(x-iy)}$ at "Zeros" of L(s, χ_3) .

Similar to Dirichlet Eta, we can conclude that there can only be 1 'Zero' of L(s, χ_3) for any given value of 'y' in s=x−iy, because the at high M or as M→∞, because the almost vertical line of $LeepL_{(x-iy,M)}$ can only possibly intersect $-MewadaL_{(x-iy)}$ at 1 point only. An because of 'Functional Equation' requiring 'Zero' at 2 points if x≠0.5, we can conclude that non-trivial 'Zeros' of L(s, χ_3) can only lie on x=0.5 line. Also, just like Dirichlet Eta, we can also conclude that 'Zeros' of L(s, χ_3) are also 'simple zeros'.

Dirichlet L-Series (3 mod 4) : L(s, $x_{4,3}$)

The method to extend the proof of Riemann-Hypothesis for Dirichlet L-Function for the nontrivial character of conductor 4 i.e. L(s, $x_{4,3}$)

$$L(s, \chi_{4,3}) = \frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \frac{1}{9^s} - \frac{1}{11^s} + \frac{1}{13^s} - \frac{1}{15^s} + \frac{1}{17^s} - \frac{1}{19^s} + \frac{1}{21^s} \ldots$$

Assign s = x + i y ...where 'x' and 'y' are real, and x>0, we get:

$$L(s, \chi_{4,3}) = \frac{1}{1^x}\angle y.log1 - \frac{1}{3^x}\angle y.log3 + \frac{1}{5^x}\angle y.log5 - \frac{1}{7^x}\angle y.log7 + \frac{1}{9^x}\angle y.log9 - \frac{1}{11^x}\angle y.log11 + \cdots$$

At "Zeros" , L(s, $\chi_{4,3}$) = 0

$$0 = \frac{1}{1^x}\angle y.log1 - \frac{1}{3^x}\angle y.log3 + \frac{1}{5^x}\angle y.log5 - \frac{1}{7^x}\angle y.log7 + \frac{1}{9^x}\angle y.log9 - \frac{1}{11^x}\angle y.log11 + \cdots$$

...Base Series LE.0

$$0 . \frac{1}{3^x}\angle y.log3 = \frac{1}{3^x}\angle y.log3 \;.\; (\frac{1}{1^x}\angle y.log1 - \frac{1}{3^x}\angle y.log3 + \frac{1}{5^x}\angle y.log5 - \frac{1}{7^x}\angle y.log7 + \frac{1}{9^x}\angle y.log9 - \cdots)$$

$0 = \frac{1}{3^x} \angle y.log3 - \frac{1}{9^x} \angle y.log9 + \frac{1}{15^x} \angle y.log15 - \frac{1}{21^x} \angle y.log7 + \frac{1}{27^x} \angle y.log27 - \ ...$

...Series LE.3

Add this convergent series LE.3 to the base series LE.0 , and we get rid of the terms where 'n' divisible by 3. We get new convergent series LE.0.3

Similarly we get:

$0 = \frac{1}{5^x} \angle y.log5 - \frac{1}{15^x} \angle y.log15 + \frac{1}{25^x} \angle y.log25 - \frac{1}{35^x} \angle y.log35 + \frac{1}{45^x} \angle y.log45 - \cdots$

...Series LE.5

Also

$0 = \frac{1}{15^x} \angle y.log15 - \frac{1}{45^x} \angle y.log45 + \frac{1}{75^x} \angle y.log75 - \frac{1}{105^x} \angle y.log105 + \frac{1}{135^x} \angle y.log135 - \cdots$

...Series LE.15 ...because 3x5=15

Add convergent series LE.15 to LE.0.3 then subtract LE.5 from the result, we get resultant convergent series LE.0.5

...This series will not have any terms where 'n' is divisible by either 3 and/or 5.

Basically we can continue the same operation, i.e. apply Aaiky-Operator as discussed in earlier chapters.

Refer to earlier chapters for details on how 'Aaiky' operator can be applied to any value of 'M' and result will be a convergent series with value converging to '0', at "Zeros" of the base line series.

Here, after applying $Aaiky_{(M)}$ [L(s, $\chi_{4,3}$)] at "Zero" of L(s, $\chi_{4,3}$) we get:

$0 = MewadaM_{(x-iy)} + LeepM_{(x-iy,M)}$

Where $MewadaM_{(x-iy)}$ is the 'Modified Mewada Function' for L(s, $\chi_{4,3}$) and it will also be an absolutely convergent series for x>0, just like 'Mewada Function', but with a different value.

$MewadaM_{(x-iy)} = 1$

Just like $Mewada_{(x-iy)}$, the new $MewadaM_{(x-iy)}$ function will also have no poles for as x>0, and $MewadaM_{(x-iy)}$ function will not change the value with change of 'M' in the Aaiky operator.

Just like $Leep_{(x-iy,M)}$ function, the new $LeepM_{(x-iy,M)}$ function will also be dependent on value of 'M' chosen in Aaiky operator, and $LeepM_{(x-iy,M)}$ function will also be an 'Almost vertical line' for very high value of 'M' or if M→∞, and will intersect $-MewadaM_{(x-iy)}$ at "Zeros" of L(s, $\chi_{4,3}$) .

Similar to Dirichlet Eta, we can conclude that there can only be 1 'Zero' of L(s, $\chi_{4,3}$) for any given value of 'y' in s=x-iy, because the at high M or as M→∞, because the almost vertical line of $LeepM_{(x-iy,M)}$ can only possibly intersect $-MewadaM_{(x-iy)}$ at 1 point only. And because of 'Functional Equation' requiring 'Zero' at 2 points if x≠0.5, we can conclude that non-trivial 'Zeros' of L(s, $\chi_{4,3}$) can only lie on x=0.5 line. Also, just like Dirichlet Eta, we can also conclude that 'Zeros' of L(s, $\chi_{4,3}$) are also 'simple zeros'.

Dirichlet L-Series (4 mod 5) : L(s, $x_{5,4}$)

The method to extend the proof of Riemann-Hypothesis for Dirichlet L-Function for the nontrivial character of conductor 5 i.e. L(s, $x_{5,4}$)

$L(s, \chi_{5,4}) = \frac{1}{1^s} - \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{6^s} - \frac{1}{7^s} - \frac{1}{8^s} + \frac{1}{9^s} + \frac{1}{11^s} - \frac{1}{12^s} - \frac{1}{13^s} + \frac{1}{14^s} + \frac{1}{16^s} - \cdots$

Assign s = x − i y ...where 'x' and 'y' are real, and x>0, we get:

$L(s, \chi_{5,4}) = \frac{1}{1^x} \angle y.log1 - \frac{1}{2^x} \angle y.log2 - \frac{1}{3^x} \angle y.log3 + \frac{1}{4^x} \angle y.log4 + \frac{1}{6^x} \angle y.log6 - \frac{1}{7^x} \angle y.log7 - \cdots$

At "Zeros" , $L(s, \chi_{5,4}) = 0$

$0 = \frac{1}{1^x} \angle y.log1 - \frac{1}{2^x} \angle y.log2 - \frac{1}{3^x} \angle y.log3 + \frac{1}{4^x} \angle y.log4 + \frac{1}{6^x} \angle y.log6 - \frac{1}{7^x} \angle y.log7 - \cdots$

...Base Series LE.0

$0 . \frac{1}{3^x} \angle y.log3 = \frac{1}{3^x} \angle y.log3 \; . \left(\frac{1}{1^x} \angle y.log1 - \frac{1}{2^x} \angle y.log2 - \frac{1}{3^x} \angle y.log3 + \frac{1}{4^x} \angle y.log4 + \frac{1}{6^x} \angle y.log6 - \cdots \right)$

$0 = \frac{1}{3^x} \angle y.log3 - \frac{1}{6^x} \angle y.log6 - \frac{1}{9^x} \angle y.log9 + \frac{1}{12^x} \angle y.log12 + \frac{1}{18^x} \angle y.log18 - \ldots$

...Series LE.3

Add this convergent series LE.3 to the base series LE.0 , and we get rid of the terms where 'n' divisible by 3. We get new convergent series LE.0.3 which shall have no terms where 3|n

Similarly we get:

$0 = \frac{1}{7^x} \angle y.log5 - \frac{1}{14^x} \angle y.log14 - \frac{1}{21^x} \angle y.log21 + \frac{1}{28^x} \angle y.log28 + \frac{1}{42^x} \angle y.log42 - \cdots$

...Series LE.7

Also

$0 = \frac{1}{21^x} \angle y.log21 - \frac{1}{42^x} \angle y.log42 - \frac{1}{63^x} \angle y.log63 + \frac{1}{84^x} \angle y.log84 + \frac{1}{126^x} \angle y.log126 - \cdots$

...Series LE.21 ...because 3x7=21

Subtract convergent series LE.21 from LE.0.3 then add LE.7 to the result, we get resultant convergent series LE.0.7 ...This series will not have any terms where 'n' is divisible by either 3 and/or 7.

Basically we can continue the same operation, i.e. apply Aaiky-Operator as discussed in earlier chapters.

Refer to earlier chapters for details on how 'Aaiky' operator can be applied to any value of 'M' and result will be a convergent series with value converging to '0', at "Zeros" of the base line series.

Here, after applying $Aaiky_{(M)}$ [$L(s, \chi_{5,4})$] at "Zero" of $L(s, \chi_{5,4})$ we get:

$0 = MewadaN_{(x-iy)} + LeepN_{(x-iy,M)}$

Where $MewadaN_{(x-iy)}$ is the 'Modified Mewada Function' for L(s, $\chi_{5,4}$) and it will also be an absolutely convergent series for x>0, just like 'Mewada Function', but with a different value.

$$MewadaN_{(x-iy)} = 1 - \frac{1}{2^x}\angle y.log2 + \frac{1}{4^x}\angle y.log4 - \frac{1}{8^x}\angle y.log8 + \frac{1}{16^x}\angle y.log16 - \frac{1}{32^x}\angle y.log32 + \cdots$$

Just like $Mewada_{(x-iy)}$, the new $MewadaN_{(x-iy)}$ function will also have no poles for as x>0, and $MewadaN_{(x-iy)}$ function will not change the value with change of 'M' in the Aaiky operator.

Just like $Leep_{(x-iy,M)}$ function, the new $LeepN_{(x-iy,M)}$ function will also be dependent on value of 'M' chosen in Aaiky operator, and $LeepM_{(x-iy,M)}$ function will also be an 'Almost vertical line' for very high value of 'M' or if M→∞, and will intersect $-MewadaN_{(x-iy)}$ at "Zeros" of L(s, $\chi_{5,4}$) .

Similar to Dirichlet Eta, we can conclude that there can only be 1 'Zero' of L(s, $\chi_{5,4}$) for any given value of 'y' in s=x−iy, because the at high M or as M→∞, because the almost vertical line of $LeepN_{(x-iy,M)}$ can only possibly intersect $-MewadaN_{(x-iy)}$ at 1 point only. And because of 'Functional Equation' requiring 'Zero' at 2 points if x≠0.5, we can conclude that non-trivial 'Zeros' of L(s, $\chi_{5,4}$) can only lie on x=0.5 line. Also, just like Dirichlet Eta, we can also conclude that 'Zeros' of L(s, $\chi_{5,4}$) are also 'simple zeros'.

General Observation for Dirichlet L-Function family:

Thus we can see that if we take any Dirichlet L-Function, at "Zero" we are able to derive pattern of sub-serieses, which are parts of the original series but containing those terms where p|n where 'p' is an odd prime number such that if you multiply the sub-series by factor $\frac{1}{p^x}\angle y.log(p)$, it results in the original series, thus the sub-series also has to converge to '0' at any "Zero" of the original series.

In this paper the Author gave several examples of how the Mewada theory for Riemann Hypothesis can be extended in general to several Dirichlet L-Functions. There is no need to work it out for each individual Dirichlet L-Function as they are all of similar nature, and all those series are manipulable in similar fashion.

Thus the operator $'Aaiky_{(M)}$ [ ] ' may be generally applied to any chosen Dirichlet L-Function at "Zeros", resulting in a convergent series of type: 0 = Mewada + Leep . The exact Mewada Function and Leep Function depends on the type of Dirichlet L-Series chosen, however in any case the Mewada function doesn't not alter with change in value of 'M' of the Aaiky Operator, while as Leep Function change with change of 'M', and as proven earlier, at high 'M' or as M→∞, Leep Function is almost a vertical line which can intersect –Mewada Function at maximum 1 possible value of 'x', for any given 'y' in any Dirichlet L-Function of s=x−iy.

Thus we can conclude that a Generalized Riemann Hypothesis, applying to Riemann Zeta (Dirichlet Eta) and other Dirichlet L-Function, holds true.

Note: The author's theory can be extended to only those functions where Aaiky Operator may be suitably applied (i.e. to all Dirichlet L-Functions for e.g.). So, the Author's theory cannot be extended to a complete set of all the Zeta-like functions even if they may have similar functional equations.

Chapter: 10

INAPPLICABLITY OF AAIKY FUNCTION & MEWADA THEORY TO HURWITZ ZETA (EXCEPT 2 SPECIAL CASES) OR TO THE EPSTEIN ZETA FUNCTION

Consider the Hurwitz Zeta function:

$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$ …for $0<a\leq 1$

In special cases of Hurwitz Zeta, i.e. if a=1 or if a=0.5, the Hurwitz Zeta reduce to Riemann Zeta Function (if a=1) or a simple function of 's' multiplied by Riemann Zeta Function(if a=0.5). So the Aaiky operator and the Mewada theory can be applied. We can actually observe there are mini-serieses within main series such that the mini-serieses converge to zero when main series converge to zero, so we may apply Aaiky operator for these 2 special cases of Hurwitz Zeta function.

However, if $0<a<1$ and $a\neq 0.5$, the terms of the series do not form a repeating patterns of mini-serieses that are versions of original series, unlike in case of Dirichlet Eta Function or other Dirichlet L-Functions. So the Aaiky operator cannot be applied, and so the Mewada theory cannot be applied.

For e.g. let's consider a=.314

$\zeta(s,.314) = \sum_{n=0}^{\infty} \frac{1}{(n+0.314)^s} = \frac{1}{(0.314)^s} + \frac{1}{(1.314)^s} + \frac{1}{(2.314)^s} + \frac{1}{(3.314)^s} + \frac{1}{(4.314)^s} + \dots$

…we can't see any mini-series within the series, that are simple-function-multiples of the original series, unlike in cases of Dirichlet Eta or Dirichlet L-Functions. So, at 'Zeros' of ζ(s,.314) , we can't apply Aaiky operator to get rid of terms up to certain value of 'n' without changing the value of the series. Thus Mewada theory cannot be applied.

So if any Hurwitz Zeta function has 'Zeros' off the critical line for Re(s)>0 (as proved by Davenport, Heilbronn and Cassels), then it does not conflict with Mewada theory, as Aaiky operator has no applicability to such series.

Consider the Epstein zeta function:

The Epstein zeta function $\zeta_Q(s)$ (Epstein 1903) for a positive definite integral quadratic form

$Q(m, n) = cm^2 + bmn + an^2$ is defined by

$$\zeta_Q(s) = \sum_{(m,n)\neq(0,0)} \frac{1}{Q(m,n)^s}.$$

Davenport and Heilbronn, and also Voronin, proved the existence of zeros of Epstein zeta functions off the critical line when the class number of the quadratic form is bigger than 1

We can see that we cannot apply Aaiky operator to such series as there are no repeating patterns of mini-serieses that are versions of the main series (contained within the main series) whose value converges to '0', when the value of the main series converges to '0'. Basically we can't apply Aaiky operator to reduce the series into a new series to get rid of terms up to a certain value of 'n' without changing the value of the series at 'Zero'. As such, Mewada theory cannot be applied to such series, and hence Mewada theory does not conflict with existence of any 'Zeros' off the critical line for these functions.

THE CONCLUSION:

It has been proven with absolute certainty that Dirichlet Eta (and hence of Riemann Zeta) can only have maximum one 'zero' for any real value 'y' and $0<x<1$. Riemann functional equation require that if there is a 'Zero' at 'x' in critical strip ($0<x<1$, and $x\neq0.5$) then there has to be another "Zero" at '1-x', so if there is only one "Zero" within the critical strip $0<x<1$ then it has to be at x=0.5. Thus the Riemann Hypothesis is confirmed with absolute certainty, and may be referred to as Riemann-Mewada Theorem in future.

Also, it has been proven that all the non-trivial 'Zeros' of the Dirichlet Eta & Riemann Zeta are 'Simple Zeros'.

The proof can be extended to other suitable functions like other Dirichlet L-Functions, meaning that the generalized Riemann Hypothesis is true.

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