Properties of Tensors

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Abstract

The article explores certain properties of tensors with reference to the rank two covariant tensor. The metric tensor plays a crucial role in the discussion. An interesting fact emerges that in order to avoid certain controversies the metric tensor and in general the rank two tensor will necessarily be null tensors.

Introduction

In this article we explore certain interesting properties of tensors with reference to the rank two covariant tensor. The metric tensor plays a crucial role in the discussion. It emerges that the metric tensor and in general the rank two tensor will necessarily be null tensors.

The Metric Tensor

We start with a discussion on the symmetric nature of the metric tensor[1].

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (1)$$

$$\Rightarrow ds^2 = g_{\nu\mu} dx^\nu dx^\mu \ (2)$$

$$\Rightarrow ds^2 = \frac{1}{2}\left(g_{\mu\nu} + g_{\nu\mu}\right) dx^\mu dx^\nu \ (3)$$

Maintaining ds^2 constant we may replace $g_{\mu\nu}$by $\left(g_{\mu\nu} + g_{\nu\mu}\right)/2$which is symmetric .It is possible to have the metric tensor as symmetric [though other alternatives might exist][2]

Again (1) implies

$$m ds^2 = m g_{\mu\nu} dx^\mu dx^\nu \ (4)$$

Equation (2) implies

$$n ds^2 = n g_{\nu\mu} dx^\nu dx^\mu \ (5)$$

Therefore

$$(m+n) ds^2 = (m g_{\mu\nu} + n g_{\nu\mu}) dx^\mu dx^\nu$$

$$ds^2 = \frac{mg_{\mu\nu} + ng_{\nu\mu}}{2} dx^\mu dx^\nu \quad (6)$$

The expression $\frac{mg_{\mu\nu}+ng_{\nu\mu}}{2}$ is not symmetric with respect to the interchange of μ and ν unless $g_{\mu\nu} = g_{\nu\mu}$

That means the metric coefficients are not unique for any given ds^2

[Invariance of ds^2 in respect of transformations has not been used anywhere]

For a confirmation we consider a tensor a rank two covariant tensor[3] $A_{\mu\nu}$ which is not symmetric and form the expression

$$d\chi^2 = A_{\mu\nu} dx^\nu dx^\mu \quad (7)$$

$$\Rightarrow d\chi^2 = A_{\nu\mu} dx^\nu dx^\mu \ (8)$$

$$\Rightarrow d\chi^2 = \frac{1}{2}\left(A_{\mu\nu} + A_{\nu\mu}\right) dx^\mu dx^\nu \ (9)$$

From (9) we cannot conclude that $A_{\mu\nu}$ is symmetric since we took $A_{\mu\nu}$ as a tensor which is not symmetric.

Maintaining $d\chi^2$ constant we may replace $A_{\mu\nu}$by $\left(A_{\mu\nu} + A_{\nu\mu}\right)/2$ which is symmetric

Again (7) implies

$$md\chi^2 = mA_{\mu\nu} dx^\mu dx^\nu \quad (10)$$

Equation (8) implies

$$nd\chi^2 = nA_{\nu\mu} dx^\nu dx^\mu \ (11)$$

Therefore

$$(m+n)d\chi^2 = (mA_{\mu\nu} + nA_{\nu\mu}) dx^\mu dx^\nu \quad (12)$$

$$d\chi^2 = \frac{mA_{\mu\nu} + nA_{\nu\mu}}{2} dx^\mu dx^{\ \nu} \ (13)$$

An Arbitrary Tensor being Null

Next we consider/recall equations(1) and (7)

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$d\chi^2 = A_{\mu\nu} dx^\nu dx^\mu$$

at a given space time point and a given pair of dx^{μ} and dx^{ν}

$$\frac{d\chi^2}{ds^2} = \frac{A_{\mu\nu}dx^{\mu}dx^{\nu}}{g_{\mu\nu}dx^{\nu}dx^{\mu}}$$

$$\frac{d\chi^2}{ds^2} = f(x, y, z, t, dx^{\mu}, dx^{\nu})(14)$$

$$\frac{A_{\mu\nu}dx^{\mu}dx^{\nu}}{g_{\mu\nu}dx^{\mu}dx^{\nu}} = f$$

Intentionally we make f dimensionless by considering a multiplicative factor of suitable k units in the denominator on the left side. Then we make the value of k unity

We choose dx^{μ} and dx^{ν} so that $dx^0 = dx^1 = dx^2 = dx^3$;

$$\frac{A_{\mu\nu}dx^{\mu}dx^{\nu}}{g_{\nu\mu}dx^{\mu}dx^{\nu}} = \frac{\sum_{\mu\nu} A_{\mu\nu}}{\sum_{\mu\nu} g_{\nu\mu}}$$

$$\frac{\sum_{\mu\nu} A_{\mu\nu}}{\sum_{\mu\nu} g_{\nu\mu}} = F(x, y, z, t)$$

$$\sum_{\mu\nu} A_{\mu\nu} = F(x, y, z, t) \sum_{\mu\nu} g_{\nu\mu} \quad (15)$$

$$\sum_{\mu\nu} A_{\mu\nu} - F(x, y, z, t) \sum_{\mu\nu} g_{\nu\mu} \quad (16)$$

The above relation will not change even if we do not impose $dx^0 = dx^1 = dx^2 = dx^3$. It will be independent of the values of dx^{μ} and dx^{ν}

$$A_{\mu\nu}dx^{\mu}dx^{\nu} = f dx^{\mu}dx^{\nu}$$

$$(A_{\mu\nu} - f(x, y, z, t, dx^{\mu}, dx^{\nu})g_{\nu\mu})dx^{\mu}dx^{\nu} = 0 \quad (17)$$

$$(A_{\mu\nu} - f(x, y, z, t, dx^{\mu}, dx^{\nu})g_{\nu\mu})dx^{\mu}dx^{\nu} + K\left[\sum_{\mu\nu} A_{\mu\nu} - F(x, y, z, t) \sum_{\mu\nu} g_{\nu\mu}\right] = 0$$

K isa number. It could be a complex number with a non zero imaginary part since $\sum_{\mu\nu} A_{\mu\nu} - F(x, y, z, t) \sum_{\mu\nu} g_{\nu\mu} = 0$[Equation (16)]

$$A_{\mu\nu}[dx^{\mu}dx^{\nu} + K] + g_{\nu\mu}(KF - f)dx^{\mu}dx^{\nu} = 0 \quad (18)$$

But dx^{μ} and dx^{ν} are be arbitrary [and independent of $A_{\mu\nu}$ and $g_{\nu\mu}$]mplying $A_{\mu\nu} = 0$ and $g_{\nu\mu} = 0$

In fact if we partial differentiate with respect to A_{km} we obtain

$$[dx^k dx^m \; + K] + g_{\nu\mu} dx^\mu dx^\nu \left[K \frac{\partial F}{\partial A_{km}} - \frac{\partial f}{\partial A_{km}}\right] = 0 \; (19)$$

By varying K vigorously we have to maintain (17). That will not be possible unless $A_{\mu\nu} = 0$ and $g_{\nu\mu} = 0$

$$K \frac{\partial F}{\partial A_{km}} - \frac{\partial f}{\partial A_{km}} = \frac{K + dx^k dx^m}{g_{\nu\mu} dx^\mu dx^\nu} = \frac{K + dx^k dx^m}{ds^2}$$

$$K \left[\frac{\partial F}{\partial A_{km}} - \frac{1}{g_{\nu\mu}}\right] = \frac{\partial f}{\partial A_{km}} + \frac{dx^k dx^m}{g_{\nu\mu} dx^\mu dx^\nu} \; (20)$$

If K is taken to be a complex number with a non zero imaginary part there would be a mismatch.

Also by varying K vigorously we might upset equation (20)

The way out would be to consider $A_{\mu\nu} = 0$ and $g_{\nu\mu} = 0$.

Symmetric Tensors

Considering the defining property of a rank two covariant tensor tensor

$$\bar{A}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} A_{\alpha\beta} \; (21)$$

if $A_{\alpha\beta}$ is symmetric then $\bar{A}_{\mu\nu}$ is also symmetric.It would not be possible to transform a tensor which is not symmetric to one which is symmetric no matter what be the transformation elements. If we could do so then the transformation would produce a symmetric tensor leading to a contradiction [unless the tensor is a null tensor]. Alternatively we may fix up the values of the transformation elements $\frac{\partial x^\alpha}{\partial \bar{x}^\mu}$ at a point and consider a set of sixteen linear equations [in 4-dimensional space] where the $\bar{A}_{\mu\nu}$ on the left side of

$$\bar{A}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} A_{\alpha\beta}$$

are symmetric by our choice. Solving these linear equations,the $A_{\alpha\beta}$ on the right have to be symmetric no matter what values we fix up for the transformation elements. The contradiction disappears if the tensor considered is the null tensor.

Antisymmetric Tensors

$$\bar{A}_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial \bar{x}^{\alpha}} \frac{\partial x^{\nu}}{\partial \bar{x}^{\beta}} A_{\mu\nu}$$

$$\bar{A}_{12} = \frac{\partial x^0}{\partial \bar{x}^1}\frac{\partial x^0}{\partial \bar{x}^2} A_{00} + \frac{\partial x^0}{\partial \bar{x}^1}\frac{\partial x^1}{\partial \bar{x}^2} A_{01} + \frac{\partial x^1}{\partial \bar{x}^1}\frac{\partial x^0}{\partial \bar{x}^2} A_{10} + \cdots (21)$$

$$\bar{A}_{21} = \frac{\partial x^0}{\partial \bar{x}^2}\frac{\partial x^0}{\partial \bar{x}^1} A_{00} + \frac{\partial x^0}{\partial \bar{x}^2}\frac{\partial x^1}{\partial \bar{x}^1} A_{01} + \frac{\partial x^1}{\partial \bar{x}^2}\frac{\partial x^0}{\partial \bar{x}^1} A_{10} + \cdots (22)$$

$$\bar{A}_{12} = -\bar{A}_{21} \ (23)$$

Equation (C) holds even if $\bar{A}_{12} = -\bar{A}_{21} = 0$ (24)

We adjust the transformations $\frac{\partial x^i}{\partial \bar{x}^j}$ so that $\bar{A}_{12}$ becomes zero in some frame of reference . Therefore it should be zero in all frames of reference

The Inner Story

We recall equation (21)

$$\bar{A}_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} A_{\alpha\beta}$$

By solving the equations represented by (21) we obtain

$$A_{\alpha\beta} = f_{\alpha\beta}\left(\frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}}, \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}}, \bar{A}_{\mu\nu}\right) \not\equiv \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} A_{\alpha\beta} \ (25)$$

unless we consider specific types of transformations like the Lorentz Transformations

$$x' = \gamma(x - vt) = \gamma x - \gamma vt \ \ (26.1)$$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) = -\gamma\frac{v}{c^2}x + \gamma t \ (26.2)$$

$$vt' = \gamma\left(t - \frac{v}{c^2}x\right) = -\gamma\frac{v^2}{c^2}x + v\gamma t \ \ (27)$$

Adding (26.1) and (27) we obtain

$$x' + vt' = \gamma\left(1 - \frac{v^2}{c^2}\right)x = \frac{1}{\gamma}x$$

$$x = \gamma(x' - (-v)t') \ (28)$$

$$\frac{v}{c^2}x' = \gamma\frac{v}{c^2}x - \gamma\frac{v^2}{c^2}t \ (29)$$

Adding (26.2) and (29)

$$t' + \frac{v}{c^2}x' = \gamma t - \gamma\frac{v^2}{c^2}t = \gamma\left(1 - \frac{v^2}{c^2}\right)t = \frac{1}{\gamma}t$$

$$t = \gamma\left(t' + \tfrac{v}{c^2}x'\right) = \gamma\left(t' - \left(\tfrac{-v}{c^2}\right)x'\right) \quad (30)$$

The direct and inverse transformations do have the same form. If we look at the electromagnetic tensor it is antisymmetric in all frames reference[inertial frames] and we have the Lorentz transformations in the picture and not arbitrary transformations.If arbitrary transformations [non singular transformations] are considered the only resolution for achieving compatibility would be to consider the tensor as the null tensor.In general the direct and inverse transformations are not compatible: we have a difficulty as indicated by (25)

Conclusions

As claimed the metric tensor and in general the rank two tensor turn out to be null tensors. The symmetric and the antisymmetric tensors have been separately investigated.

References

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