Group Geometric Algebras and the Standard Model

Carl Brannen

Abstract. We generalize the massless Dirac equation and Weyl equation to give particles with internal symmetry of the Standard Model fermions. The 2x2 or 4x4 complex matrices are matrix rings R and can be generalized by a group G to a group algebra $G[R]$. For a group of size N, this creates N times as many Pauli spin or gamma matrices and generalizes the wave equations. We consider point group symmetries for G and show that using the full octahedral group and the 2x2 complex matrices gives a group algebra which generalizes the Weyl equation to the Standard Model fermions plus a dark matter particle. An alternative is to use the chiral octahedral group and the 4x4 complex matrices to generalize the massless Dirac equation. The method leaves a dark matter particle and its antiparticle. We describe the symmetry of dark matter.

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We will write the Weyl equation for a single massless fermion as

$$\sigma^\mu \partial_\mu \psi = I_2 \frac{\partial \psi}{\partial t} \pm \sigma_x \frac{\partial \psi}{\partial x} \pm \sigma_y \frac{\partial \psi}{\partial y} \pm \sigma_z \frac{\partial \psi}{\partial z} = 0, \tag{0.1}$$

where the plus signs are for the left handed Standard Model fermions and the minus signs are for the right handed fermions. In the above, the Pauli algebra has a basis of four 2x2 complex matrices $\{\sigma^\mu\} = \{I_2, \sigma_x, \sigma_y, \sigma_z\}$ each of which has two nonzero complex elements. And ψ is a 1x2 vector so the Weyl equation consists of two coupled differential equations with four terms in each.

Changing the signs of the Pauli spin matrices, as seen above in the left and right handed Weyl equations, can be thought of as a symmetry operation. The symmetry group has two elements, it is the point group symmetry that Hestenes and Holt[1] call $\overline{22}$. We will label the point group symmetries we

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use with their notation and to avoid confusion with numerals will prepend them with "Geo" as in Geo$\overline{22}$. The groups explored in this paper are the full octahedral group Geo43 and some of its subgroups. An element of Geo43 can be described by its effect on the Pauli spin matrices. In general, the three matrices can be permuted and can be independently permuted. Together, these give the $3! \times 2^3 = 48$ elements of Geo43.

In the Weyl representation of the gamma matrices, the Dirac equation consists of two Weyl equations, one left handed the other right handed, which are coupled by the mass term. Setting the mass to zero gives the massless Dirac equation and the two Weyl equations are uncoupled. If we let $g \in$ Geo$\overline{22}$, we can write these two uncoupled equations as

$$\sigma^{\mu g}\partial_\mu\ \Psi^g = 0, \tag{0.2}$$

where a sum over μ is implied: $\sigma^{\mu g}$ is I_2, $\pm\sigma_x$, $\pm\sigma_y$ and $\pm\sigma_z$, and the signs are according to the element g in Geo$\overline{22}$.

In the above equation we assumed that the Ψ^g depends on the group element g the same way as the g of the generalized Pauli spin matrices $\sigma^{\mu g}$. An alternative interpretation, the one adopted by this paper, is that the g of the Pauli spin matrices and the h of the wave function Ψ^h are mixed by the point group symmetry operation so that $\sigma^g...\Psi^h$ is a product whose point group symmetry membership is gh. Then the wave equation which corresponds to the group element g is

$$\sigma^{\mu g/h}\partial_\mu\ \Psi^h = 0, \tag{0.3}$$

where a sum over h in the point group symmetry is implied and $(g/h)\ h = g$. The result is a set of coupled Weyl equations whose number is given by the size of the point group. For Geo43, there are 48 group elements so the resulting 96 differential equations each has $4 \times 48 = 192$ terms for a total of $96 \times 192 = 18,432$ terms. The subject of this paper will largely be the uncoupling of these equations into 48 Weyl equations, each with $4 \times 2 = 8$ terms. The Standard Model fermion symmetry arises from this decoupling.

It may be useful to explicitly show how elements of Geo43 modify the Weyl equation. We can use X, Y and Z to signify the three Pauli matrices and give a permutation by listing these three letters. For example, the identity is XYZ and a swap of σ_y and σ_z will be designated XZY. And we can specify which Pauli spin matrices are negated by putting a bar over them. So $X\overline{ZY}$ is a right angle rotation about the x axis. This rotation modifies the left-handed Weyl operator to

$$I_2\frac{\partial}{\partial t} + \sigma_x\frac{\partial}{\partial x} - \sigma_z\frac{\partial}{\partial y} - \sigma_y\frac{\partial}{\partial z}. \tag{0.4}$$

In the above, the partial derivatives no longer match the Pauli spin matrices. We can fix that by having the Pauli spin matrices depend on the group element $X\overline{ZY}$ as in

$$I_2^{X\overline{ZY}}\frac{\partial}{\partial t} + \sigma_x^{X\overline{ZY}}\frac{\partial}{\partial x} + \sigma_y^{X\overline{ZY}}\frac{\partial}{\partial y} + \sigma_z^{X\overline{ZY}}\frac{\partial}{\partial z}, \tag{0.5}$$

where $\sigma_y^{X\overline{ZY}} = -\sigma_z$, etc.

This paper makes the assumption that different particles use physically different Pauli algebra bases and that their wave states are different physical degrees of freedom. For example, writing the left handed electron "eL", and left handed neutrino "νL", their Weyl equations are:

$$\sigma_{eL}^{\mu}\partial_\mu\psi^{eL} = 0, \quad \text{and} \quad \sigma_{\nu L}^{\mu}\partial_\mu\psi^{\nu L} = 0. \tag{0.6}$$

Each of these Weyl equations are two coupled differential equations with the four terms shown above in Equation 0.1. These two Weyl equations are uncoupled. In this paper we will be writing generalized Weyl equations with as many as 96 coupled differential equations and then algebraically manipulating them into an equivalent set of uncoupled Weyl equations. We assume that these uncoupled Weyl equations correspond to different particles.

We will illustrate this generalization in three steps. The first step will use the "Geo$\overline{22}$" point group with two elements, the identity and the inversion, and is the subject of Section 1.0. We write the generalized Weyl equations which are four coupled partial differential equations, and decouple them into two Weyl equations corresponding to two particles. The decoupling is equivalent to putting the group algebra $\overline{22}[\sigma^\mu]$ into block diagonal form. A short cut to diagonalizing a group algebra is to use the finite group's character table. In general, an Abelian group of size n has a character table of size $n \times n$, and an Abelian group generated by a single element has a character table whose entries are the same as the discrete Fourier transform. In the following section we generalize the character tables of non Abelian symmetries to also be of size $n \times n$. This is a generalization of the discrete Fourier transform to non Abelian symmetries.

The second step will use the point group Geo3 that corresponds to the six permutations on the three Pauli spin matrices and is discussed in Section 2.0. The group's generalized Weyl equation has twelve coupled differential equations with 24 terms each. We will use the character table for Geo3 to assist in rewriting the equations. The group is is small enough that writing down the decoupled Weyl equations is manageable. Since Geo3 is not Abelian, block diagonalizing $3[\sigma^\mu]$ leaves a 2x2 block on the diagonal. While the group size of Geo3 is 6, the character table has only 3 columns and 3 rows but we show how to expand it to a 6×6 table that can be interpreted as a generalization of the discrete Fourier transform to the non Abelian Geo3 symmetry. The expanded character table defines an internal symmetry that is an SU(2) analog of the color SU(3) internal symmetry of quarks. Note that the matrices of the Weyl and Dirac wave equations transform the way that density matrices transform rather than state vectors. This is significant because our generalized discrete Fourier transform is a state vector type transform on an object that transforms as a density matrix which is more general than a density matrix transform of a density matrix object. We discuss the thermodynamics of internal symmetries.

In Section 3.0 we discuss the full octahedral group Geo43. Manipulating the 18,432 terms would be exhausting so we use the character table methods shown in the previous section to read off the result of uncoupling. We find four Standard Model first generation leptons, i.e. electron, positron, neutrino and anti-neutrino, four Standard model first generation quarks and a set of Weyl equations with internal SU(2) symmetry left over for dark matter and anti-dark matter. Thinking of our transformation as a discrete Fourier transform, we can reverse the process and act on the electric charge, weak hypercharge and weak isospin to find what elements of $43[\sigma^\mu]$ transform to these operators. The result is compatible with the observation that Fourier transforms are useful for simplifying complicated structures in physics.

In Section 4.0 we discuss the dark matter doublet found in the previous section and the implications for dark matter. The internal SU(2) symmetry is assumed to act similarly to the internal SU(3) color symmetry so we call them "dark quarks" or "duarks". Corresponding to the quark color force boson the gluon, dark quarks are bound by a dark color force boson called the "duon". We assume duarks combine similarly to the usual quarks so we have dark matter composed of "daryons" and "desons". Instead of colors that sum to white, the dark quarks do not interact with photons so their colors must sum to black. Since they are SU(2) instead of SU(3), we need only two dark colors to form a basis set for their state vectors and we call the two dark colors "doom" and "gloom".

The conclusion, Section 5.0 discusses the outstanding lack of this paper, a theory of the gauge bosons. There are some hints as to what is going on. This is followed with a short aknowledgement.

1. $\overline{22}[\sigma^\mu]$, Generalized Weyl Equations and Decoupling

The group Geo$\overline{22}$ is equivalent to S_2, the group of permmutations of two elements. There are only two permutations on two elements. Either they are swapped $S = (12)$ which alters the Pauli spin matrices to $\overline{XYZ}$, or they are not $E = (\ )$ leaving the Pauli spin matrices in their default condition XYZ. The multiplication table is:

$$\begin{array}{c|cc} S_2 & E & S \\ \hline E & E & S \\ E & S & E \end{array} \qquad (1.1)$$

The four basis elements of the Pauli algebra are $\{I_2, \sigma_x, \sigma_y, \sigma_z\}$; The Geo$\overline{22}$ group has two elements so the group algebra $\overline{22}[\sigma^\mu]$ will have two basis elements for each of these, corresponding to the two group elements E and S: $\{I_2^E, \sigma_x^E, \sigma_y^E, \sigma_z^E\ I_2^S, \sigma_x^S, \sigma_y^S, \sigma_z^S\}$. These eight elements are a basis for the $\overline{22}[\sigma^\mu]$ algebra so that an arbitrary element α of the algebra can be defined as

$$\alpha = \alpha_{IE} I_2^E + \alpha_{xE}\sigma_x^E + \alpha_{yE}\sigma_y^E + \alpha_{zE}\sigma_z^E + \alpha_{IS} I_2^S + \alpha_{xS}\sigma_x^S + \alpha_{yS}\sigma_y^S + \alpha_{zS}\sigma_z^S \qquad (1.2)$$

where the α_χ are eight complex numbers. Thus the $\overline{22}[\sigma^\mu]$ vector space is eight dimensional.

Addition is the usual for a complex vector space, as is multiplication by a complex number. We will discuss statistical mechanics of group algebra quantum states in the next section; that will be our only use of multiplication of two group algebra elements so we define it here. Multiplication is term by term with products of basis elements given by the group product and the usual Pauli algebra rules. So for example:

$$\begin{array}{rcl} (1+2i)\sigma_x^E \;\; (2-3i)\sigma_y^S = (2+4i-3i+6)\;\; i\sigma_z^{ES} & = & (-1+8i)\sigma_z^S, \\ I_2^E \;\; \alpha \;\; = \;\; \alpha \;\; I_2^E & = & \alpha, \\ (\sigma_x^S)(\sigma_x^S+\sigma_y^E) & = & I_2^E + i\sigma_z^S, \end{array} \tag{1.3}$$

where we've used the usual Pauli algebra multiplication $\sigma_x\sigma_y = i\sigma_z$, $\sigma_x\sigma_x = I_2$ and the S_2 group multiplication rules $SS = E, SE = ES = S$. These rules reduce any product to the $\overline{22}[\sigma^\mu]$ basis.

The Weyl equation for $\overline{22}[\sigma^\mu]$ is

$$\sigma^{\mu g/h}\partial_\mu\Psi^h = 0, \tag{1.4}$$

where h is summed over Geo$\overline{22} = \{E, S\}$. There are two equations, one for $g = E$, the other for $g = S$. Writing out the sums over h gives:

$$\begin{array}{rcl} \sigma^{\mu EE}\partial_\mu\Psi^E + \sigma^{\mu ES}\partial_\mu\Psi^S & = & 0, \\ \sigma^{\mu SS}\partial_\mu\Psi^S + \sigma^{\mu SE}\partial_\mu\Psi^E & = & 0, \end{array} \tag{1.5}$$

where we've used the fact that $h^{-1} = h$ for this group. The group products $EE = E$, $ES = S$, $SS = E$ and $SE = S$ simplifies the generalized Pauli spin matrices to give

$$\begin{array}{rcl} \sigma^{\mu E}\partial_\mu\Psi^E + \sigma^{\mu S}\partial_\mu\Psi^S & = & 0, \\ \sigma^{\mu E}\partial_\mu\Psi^S + \sigma^{\mu S}\partial_\mu\Psi^E & = & 0, \end{array} \tag{1.6}$$

and we see we have two coupled Weyl equations.

To uncouple these two equations, consider the character table for the S_2 group:

Class:	$\{E\}$	$\{S\}$
Size:	1	1
A	1	$+1$
B	1	-1

(1.7)

The group is Abelian so the two elements are each their own classes. There are two irreducible representations consisting of the sum and difference of the E part and S part. Taking this as a hint for how to decouple the two Weyl equations, we compute the sum and differences of Equation 1.6 and indeed obtain two decoupled Weyl equations:

$$\begin{array}{rcl} (\sigma^{\mu E}+\sigma^{\mu S})/2 \;\; \partial_\mu(\psi^E+\psi^S) & = & 0, \\ (\sigma^{\mu E}-\sigma^{\mu S})/2 \;\; \partial_\mu(\psi^E-\psi^S) & = & 0. \end{array} \tag{1.8}$$

The normalization factor of 1/2 is included so that the new Pauli basis elements $\sigma^{\mu\pm} = (\sigma^{\mu E} \pm \sigma^{\mu S})/2$ satisfy the equation $(\sigma^\mu)^2 = I_2$.

We can also illustrate the method using a matrix representation of the finite group S_2. Using

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{1.9}$$

as a representation, note that the representation satisfies the group multiplication and that they are linearly independent matrix (considered as a vector space) elements. The eight basis elements of the $\overline{22}[\sigma^\mu]$ group algebra are defined by eight 4x4 complex matrices. The basis elements of the Pauli algebra are four 2x2 complex matrices that are linearly independent. In an abuse of notation, we can write these familiar matrices as a single 2x2 complex matrix:

$$\sigma^\mu \quad = \quad \begin{pmatrix} I_2 + \sigma_z & \sigma_x - i\sigma_y \\ \sigma_x + i\sigma_y & I_2 - \sigma_z \end{pmatrix}, \tag{1.10}$$

that defines the four 2x2 complex matrices. The same abuse of notation allows us to write the eight basis elements of $\overline{22}[\sigma^\mu]$ in a single 4x4 matrix:

$$\overline{22}[\sigma^\mu] \quad = \quad \begin{pmatrix} I_2^E + \sigma_Z^E & \sigma_x^E - i\sigma_y^E & I_2^S + \sigma_z^S & \sigma_x^S - i\sigma_y^S \\ \sigma_x^E + i\sigma_y^E & I_2^E - \sigma_z^E & \sigma_x^S + i\sigma_y^S & I_2^S - \sigma_z^S \\ I_2^S + \sigma_z^S & \sigma_x^S - i\sigma_y^S & I_2^E + \sigma_z^E & \sigma_x^E - i\sigma_y^E \\ \sigma_x^S + i\sigma_y^S & I_2^S - \sigma_z^S & \sigma_x^E + i\sigma_y^E & I_2^E - \sigma_z^E \end{pmatrix}. \tag{1.11}$$

Note that in the above, the E and S subscripts follow the S_2 group representation of Equation 1.9 while the Pauli algebra basis appears in the four 2x2 blocks. For example, the above indicates that

$$\sigma_x^S = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{1.12}$$

and similarly for the other seven basis elements of $\overline{22}[\sigma^\mu]$.

The four Pauli basis elements $\{I_2, \sigma_x, \sigma_y, \sigma_z\}$ are Hermitian, one for each real degree of freedom in the 2x2 complex Hermitian matrices. The 4x4 complex Hermitian matrices have sixteen real degrees of freedom but $\overline{22}[\sigma^\mu]$ is only of dimension eight.

Given a matrix U and its inverse U^{-1} we can transform an algebra by $\alpha \to \alpha' = U\alpha U^{-1}$. Choosing

$$U = U^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_2 & I_2 \\ I_2 & -I_2 \end{pmatrix} \tag{1.13}$$

conveniently puts the $\overline{22}[\sigma^\mu]$ group algebra into block diagonal form. This corresponds to a transformation on our representation of S_2. Using the usual abuse of notation, in an even more confusing way, that is, to show two equations in one:

$$U \begin{pmatrix} E & S \\ S & E \end{pmatrix} U^{-1} = \frac{1}{2} \begin{pmatrix} E + S & 0 \\ 0 & E - S \end{pmatrix}. \tag{1.14}$$

Applied to the $\overline{22}[\sigma^\mu]$ basis, $(E \pm S)/2$ corresponds to the Pauli spin matrices of Equation 1.9 or

$$(E \pm S)/2 \equiv (\sigma^{\mu E} \pm \sigma^{\mu S})/2, \tag{1.15}$$

so the block diagonalizing U transformation is what is needed to decouple the Weyl equations. In the next two sections we will be decoupling Weyl equations for more complicated finite groups; we will approach those problems as a matter of block diagonalization.

The two coupled Weyl equations of Equation 1.8 differ in the wave function: $\psi^E \pm \psi^S$. This fact can be extracted from the block diagonal form of $\overline{22}[\sigma^\mu]$ given in Equation 1.14 by considering it as the symmetry of a density matrix. The $U\alpha U^{-1}$ transformation of the algebra is also how density matrices are traditionally transformed. This follows from the fact that the Weyl equations are matrix equations and so transform as matrices.

2. $3[\sigma^\mu]$, Nonabelian Group Algebras and Internal Symmetry

The Geo3 point group has six elements. They permute the Pauli spin matrices. The group is equivalent to the permutation group on three elements S_3 and we will use S_3 notation for the group elements. The conversion of S_3 group element names to the Pauli spin matrix octahedral point group are:

$$\begin{array}{rclrclrcl} () & \equiv & XYZ, & (123) & \equiv & YZX, & (132) & \equiv & ZXY, \\ (12) & \equiv & YXZ, & (13) & \equiv & ZYX, & (23) & \equiv & XZY. \end{array} \tag{2.1}$$

The three element permutations (123) and (132) correspond to 120 degree rotations about the $(1,1,1)$ axis. The swaps (12), (13) and (23) are the three mirror planes with perpendiculars $(1,-1,0)$, $(-1,0,1)$ and $(0,1,-1)$ that intersect in the $(1,1,1)$ direction. Since the Pauli algebra has a basis of four elements, The basis for the $3[\sigma^\mu]$ algebra has $4 \times 6 = 24$ elements and we will designate them as $\{I_2^{()}, I_2^{(123)}, ..., \sigma_z^{(23)}\}$.

In the previous section we were able to completely diagonalize the Geo$\overline{22}$ part of the $\overline{22}[\sigma^\mu]$ algebra. This was possible only because Geo$\overline{22}$ is Abelian. This section discusses Geo3 which is non Abelian so we will be unable to completely diagonalize the Geo3 part. As with the previous section, the Pauli algebra is non Abelian so that part of the group algebra cannot be diagonalized.

The mathematics literature on group algebras frequently does not specify the "field" or "ring" and instead consider the general subject of a group algebra over an unspecified field that satisfies a few easy requirements. Here we are using the Pauli algebra as the ring. A convenient choice for the field is the complex number. The group algebra over the field of complex numbers is discussed at length in Hammermesh's book "Group Theory and its Applications to Physical Problems" Section 3-17 [2] where he uses the group algebra as a way of defining all the irreducible representations of a finite symmetry. Here we will be doing the same diagonalization but with the objective of uncoupling the generalized Weyl equations.

The diagonalization method given by Hammermesh is identical to how one block diagonalizes a group algebra over the Pauli algebra and we refer the reader to his book. The overall result is that each line of the character table of the finite group corresponds to a diagonal block. The size of the block is defined by the character of the identity ( ). Our Geo$\overline{22}$ example of the previous section was Abelian so all the representations have character 1 for the identity and the block diagonalized group algebra had only 1x1 blocks.

The character table for Geo3 has three irreducible representations, two, A and B, with character 1 for the identity, and the third C with character 2 for the identity:

Class:	{( )}	{(123)}	{(12)}
Size:	1	2	3
A	1	+1	+1
B	1	+1	−1
C	2	−1	0

(2.2)

This corresponds to two 1x1 diagonal blocks for the A and B irreps, and one 2x2 diagonal block for the C irrep. The size of the Geo3 group is 6, so that the sizes of the three classes sum to $1 + 2 + 3 = 6$. Diagonalizing preserves the degrees of freedom so our diagonalization amounts to writing 6 as a sum of the squares of the sizes of the diagonal blocks:

$$6 = 1^2 + 1^2 + 2^2. \tag{2.3}$$

This shows how the 6 dimensions of $3[\sigma^\mu]$ as a vector space over the Pauli algebra appear in block diagonalized form. The previous example $\overline{22}[\sigma^\mu]$ had two elements in the finite group so its sum of squares was $2 = 1^2 + 1^2$.

Since the size of Geo3 is six, the $\overline{22}[\sigma^\mu]$ algebra has enough degrees of freedom for six Weyl equations. Two equations are given by the A and B irreps. We can read them off of the A and B horizontal lines of the character table. The A irrep has character 1 for all the classes so all 6 group elements contribute equally and its Pauli spin algebra basis is given by the four $\overline{22}[\sigma^\mu]$ elements:

$$\sigma^{A\mu} = (\sigma^{\mu(\,)} + \sigma^{\mu(123)} + \sigma^{\mu(132)} + \sigma^{\mu(12)} + \sigma^{\mu(13)} + \sigma^{\mu(23)})/6. \tag{2.4}$$

The division by 6 is a normalization so that these elements square to their 2x2 identity $\sigma^{A0} = I_2^A$. The B irrep has -1 for the $\{(12)\}$ class so its Pauli spin algebra basis is

$$\sigma^{B\mu} = (\sigma^{\mu(\,)} + \sigma^{\mu(123)} + \sigma^{\mu(132)} - \sigma^{\mu(12)} - \sigma^{\mu(13)} - \sigma^{\mu(23)})/6. \tag{2.5}$$

Since the A and B Pauli spin algebra bases are in different diagonal blocks, they must annihilate each other. For example $\sigma^{A0}\,\sigma^{B3} = I_2^A\,\sigma_z^B = 0$. This is in contrast to the original basis where all the products are nonzero.

The 2x2 diagonal block will give four Weyl equations. It would be natural to label them according to their position in the 2x2 block. With this method, the four uncoupled Weyl equations would carry superscripts of

$11, 12, 21, 22$ and would fit into the 2x2 block as follows:

$$\sigma^{C\mu} = \begin{pmatrix} \sigma^{C11\mu} & \sigma^{C12\mu} \\ \sigma^{C21\mu} & \sigma^{C22\mu} \end{pmatrix}. \quad (2.6)$$

However, we'll be considering SU(2) transformations on these equations (which correspond to SU(2) transformations on the internal symmetries of the corresponding particles) and our transformations will be simpler if we manipulate these four uncoupled Weyl equations into four new uncoupled Weyl equations according to

$$\begin{pmatrix} \sigma^{C11\mu} & \sigma^{C12\mu} \\ \sigma^{C21\mu} & \sigma^{C22\mu} \end{pmatrix} = \begin{pmatrix} \sigma^{C1\mu} + \sigma^{Cz\mu} & \sigma^{Cx\mu} + i\sigma^{Cy\mu} \\ \sigma^{Cx\mu} - i\sigma^{Cy\mu} & \sigma^{C1\mu} - \sigma^{Cz\mu} \end{pmatrix}. \quad (2.7)$$

So that, for example, $\sigma^{C1\mu} = (\sigma^{C11\mu} + \sigma^{C22\mu})/2$ and $\sigma^{Cz\mu} = (\sigma^{C11\mu} - \sigma^{C22\mu})/2$.

The four equations $\{\sigma^{C1\mu}, \sigma^{Cx\mu}, \sigma^{Cy\mu}, \sigma^{Cz\mu}\}$ can be manipulated by a transformation

$$\sigma^{C\mu} \ \rightarrow \ \sigma^{C'\mu} \ = \ U\,\sigma^{C\mu}\,U^{-1} \quad (2.8)$$

where U is in SU(2), to another equivalent set of four decoupled Weyl equations.

As an example of this internal SU(2) symmetry of the uncoupled Weyl equations, given an angle $\theta/2$, note that $(\theta/2)\sigma_x$ is in su(2) so that $U = \exp(i(\theta/2)\sigma_x)$ is in SU(2). This U leaves $\sigma^{C1\mu}$ and $\sigma^{Cx\mu}$ unchanged but rotates the other two according to

$$\begin{aligned} \sigma^{C'y\mu} &= \cos(\theta)\sigma^{Cy\mu} + \sin(\theta)\sigma^{Cz\mu}, \\ \sigma^{C'z\mu} &= \cos(\theta)\sigma^{Cz\mu} - \sin(\theta)\sigma^{Cy\mu}, \end{aligned} \quad (2.9)$$

so that, as expected, U rotates the y and z components of $\sigma^{C\mu}$ by an angle θ. This simplifies our understanding of these four Weyl equations as an internal SU(2) symmetry so rather than using the natural matrix description of the four equations shown in Equation 2.6, we will use the Pauli algebra description of these four degrees of freedom which we can abbreviate as:

$$\sigma^{C\nu\mu} = \begin{pmatrix} \sigma^{C1\mu} + \sigma^{Cz\mu}, \sigma^{Cx\mu} - i\sigma^{Cy\mu} \\ \sigma^{Cx\mu} + i\sigma^{Cy\mu}, \sigma^{C1\mu} - \sigma^{Cz\mu} \end{pmatrix} \quad (2.10)$$

where $\nu =\in \{1, x, y, z\}$ is the index for the Pauli algebra of the internal SU(2) symmetry, while μ is the usual index for the Pauli spin-1/2.

Our next task is to write out $\sigma^{C\nu\mu}$ in terms of the algebra. The first, $\sigma^{C1\mu}$ is easiest; similar to $\sigma^{A\mu}$ and $\sigma^{B\mu}$ defined in equations (2.4) and (2.5), it's given by the third line of the character table "$C \mid 2 \ -1\ 0$" and is:

$$\sigma^{C1\mu} = 2(2\sigma^{\mu(\,)} - \sigma^{\mu(123)} - \sigma^{\mu(132)})/6. \quad (2.11)$$

The overall multiplication by 2 is needed for normalization and comes from the fact that this irrep has a character of 2 for the identity.

To find the remaining three degrees of freedom, i.e. $\{\sigma^{x\mu}, \sigma^{y\mu}, \sigma^{z\mu}\}$, let us first reexamine the character table Equation (2.2). The three columns correspond to the three classes. These three degrees of freedom are rewritten as the three irreducbile representations $\{A, B, C\}$. The three classes include all six of the group elements so we can rewrite this table from 3x3 to 6x3 if we replace the classes with their elements. For example, the second class {(123)} has two elements (123) and (132). The 6x3 character table is:

Elements:	( )	(123)	(132)	(12)	(13)	(23)
A	1	+1	+1	+1	+1	+1
B	1	+1	+1	−1	−1	−1
C	2	−1	−1	0	0	0

. (2.12)

In terms of rewriting the degrees of freedom of the algebra, we are missing three lines that would fit under the irreducible representations. These three will be orthogonal to the three irreps and will correspond to multiples of $\sigma^{Cx\mu}$, $\sigma^{Cy\mu}$ and $\sigma^{Cz\mu}$. From the table we can see that the missing three degrees of freedom can be written as (123) – (132), (12) – (13) and (12) – (23). But these are not normalized and do not satisfy the commutation relations of the Pauli spin matrices. However, since they span the Pauli spin matrices, we can write them using the Pauli spin matrices as the bases for a complex vector space:

$$(123) - (132) = \alpha\sigma_x + \beta\sigma_y + \gamma\sigma_z, \tag{2.13}$$

where α, β and γ are three complex numbers. Squaring both sides gives

$$\begin{array}{rcl} [(123) - (132)]^2 &=& [\alpha\sigma_x + \beta\sigma_y + \gamma\sigma_z]^2, \\ (132) + (123) - 2(\,) &=& (\alpha^2 + \beta^2 + \gamma^2)\sigma_1 = (\alpha^2 + \beta^2 + \gamma^2)\sigma^{C1\mu}. \end{array} \tag{2.14}$$

But the left hand side is the negative of the C irrep line. And since $\alpha^2 + \beta^2 + \gamma^2$ is just a complex number, we can take its square root and use it as a normalization factor on (123) – (132) to give us $\sigma^{Cx\mu}$. After doing this, we can square the next degree of freedom (12) – (13) to give the normalization for $\sigma^{Cy\mu}$. We then define $\sigma^{Cz\mu}$ by $i\sigma_z = \sigma_x\sigma_y$. Taking all this into account, and including the normalization factors in the character table converts it into a 6x6 table that shows how the degrees of freedom of the algebra are written in terms of irreducible representations and their internal symmetries:

Elements:	( )	(123)	(132)	(12)	(13)	(23)
$\sigma^{A\mu}$	1/6	+1/6	+1/6	+1/6	+1/6	+1/6
$\sigma^{B\mu}$	1/6	+1/6	+1/6	−1/6	−1/6	−1/6
$\sigma^{C1\mu}$	2/3	−1/3	−1/3	0	0	0
$\sigma^{Cx\mu}$	0	$i\sqrt{1/3}$	$-i\sqrt{1/3}$	0	0	0
$\sigma^{Cy\mu}$	0	0	0	$\sqrt{1/3}$	$-\sqrt{1/3}$	0
$\sigma^{Cz\mu}$	0	0	0	1/3	1/3	−2/3

. (2.15)

This is a general result that does not depend on the choice of field used in this paper so the μ is unneeded. The squared magnitude of each line is the same as the coefficient for the identity, i.e 1/6 for A and B and 2/3 for the four components of C. Each line of this table defines four equations depending on the choice of μ. For example, the fourth equation of the last line is $\sigma^{Czz} = \sigma^{(12)z}/3 + \sigma^{(13)z}/3 - 2\sigma^{(23)z}/3$.

Our new table upgrades the character table to a transformation on all the basis elements of the group algebra, and therefore, it is also a transformation on the group algebra itself. For Abelian groups of the form $G = \{a^k|0 \leq k < n\}$ for some $a \in G$, the character table is of size nxn and is a discrete Fourier transform. Thus our new table is a generalization of the discrete Fourier transform to a non Abelian symmetry group.

In 2010 this author published a paper that rewrote the Koide equation for the masses of the charged leptons as the result of Feynman paths taken over spin in the x, y and z directions. [3] Given a path of n steps, the $n+1$ step will either be in the same direction, clockwise or counterclockwise around the $(1,1,1)$ direction. Three of the elements of Geo3 correspond to these choices: $(\;) = XYZ$, $(123) = YZX$ and $(132) = ZXY$. They form the point group Geo$\overline{3}$ which is an Abelian subgroup of Geo3 of size three so they imply a discrete Fourier transform on three objects as was pointed out by Marni Sheppeard in 2009. She asked "Is there a noncommutative transform that extends this analysis to nonclassical underlying spaces?"[4] This paper suggests that there is and that it will help understand the masses and symmetries of the Standard Model fermions. Rewriting the Koide equation allowed it to be extended to the neutrinos and, for the charged leptons, revealed another mass coincidence, a phase factor of 2/9. In 2012 Zenczykowski extended the 2/9 coincidence to the up and down quarks which take 1/3 and 2/3 of the charged lepton value.[5] This paper is another step in fully understanding the Standard Model fermions both symmetry and generation structure.

A Weyl equation is a matrix equation, so when we transform it according to its spin-1/2 SU(2) symmetry, we transform it the way we would a density matrix. That is, it transforms as $\rho' = U\ \rho\ U^{-1}$ rather than $\psi' = U\ \psi$. Our treatment of the internal SU(2) symmetry of the C irreps as shown in Equation (2.8), uses the density matrix type transformation. Density matrices are convenient for calculations in quantum statistical mechanics so we briefly discuss them here.

The Gibbs form for the density matrix of an ensemble is

$$\rho(T) \propto e^{-\frac{H}{T}}, \tag{2.16}$$

where the Hamiltonian H specifies the energy of the possible states, T is the temperature, the Boltzmann constant is unity, and we've left off the normalization factor that gives $\rho(T)$ a trace of 1. Squaring both sides gives

$$\rho^2(T) \propto e^{-\frac{2H}{T}} = e^{-\frac{H}{T/2}}, \tag{2.17}$$

so squaring such a density matrix gives a density matrix that is proportional to one with half the temperature. Repeating the procedure of squaring and dividing by the trace allows one to find a density matrix for a temperature close to zero. If the Hamiltonian depends on the state, this low temperature limit will be the pure density matrix corresponding to the state with the lowest energy.

The high temperature limit for a density matrix is given by letting $T \to \infty$ and is proportional to the unit matrix. The proportionality constant

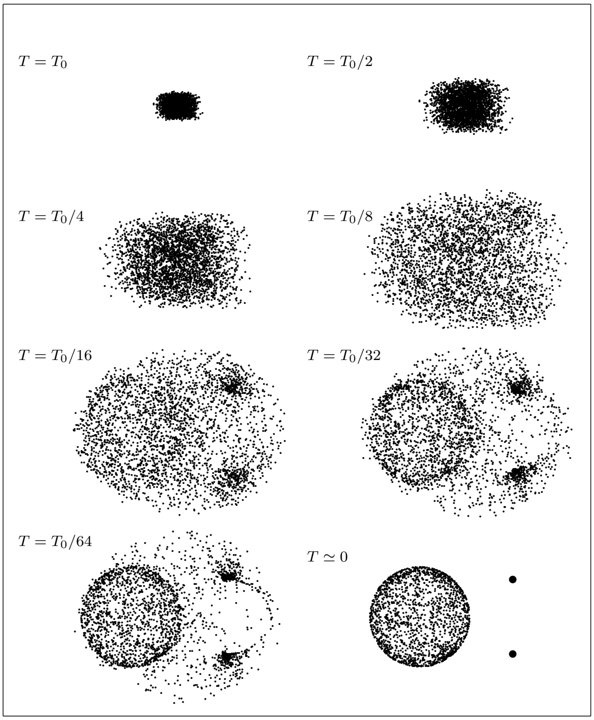

FIGURE 1. Beginning with 3000 random density matrices near the high temperature limit $T = T_0$, we square the density matrix six times to show the states beginning to converge to the two singletons A and B, and the internal SU(2) doublet C. Of the six degrees of freedom in the $3[\sigma^\mu]$ algebra, we choose the x and y axes so as to spread A, B and C apart, and to show the Bloch sphere for C. Continuing the cooling process, the final image $T \simeq 0$ shows the cold temperature limit (pure states).

is the trace so the high temperature limit for $3[\sigma^\mu$ is

$$\rho(T = \infty) = (\ )/6, \tag{2.18}$$

as the trace of the identity of a group algebra is the size of the group (and the other elements of the group algebra have zero trace). Density matrices have to be Hermitian and the same principle applies to a group algebra. The generalization of "Hermiticity" from a matrix to a group algebra is covered in introductory graduate math texts or the reader can work backwards from the Hermiticity requirement for the matrix form of the algebra.

If we choose a small random Hermitian element of the group algebra we can add it to the high temperature limit to get a Hermitian element that is a valid state near the high temperature limit. Cooling this state down to a nearly pure state gives us a random pure state. For $3[\sigma^\mu]$ there are three possible pure states, A, B and C. Figure (1) shows the result of this cooling for a few thousand random high temperature states. The A and B irreps appear as two dots on the right side of the images while the internal SU(2) C state appears as a Bloch sphere on the left. The algebra has six group degrees of freedom and the images are two dimensional so four degrees of freedom are not graphed. The degrees of freedom that are graphed have been selected to separate the A, B and C states and to spread the C states with $\sigma^{Cx\mu}$ in the x-direction and $\sigma^{Cy\mu}$ in the y-direction so the Bloch sphere appears as an x-ray of a sphere. If you assign a graduate student to read and understand this paper, a suitable task is to reproduce this graph for this or other point group symmetries.

In this paper the field we use is the Pauli algebra but the figure does not depend on the field so when writing a computer program to plot the cooling process it is more convenient to use the complex numbers as the field. Then an element of the group algebra $3[\mathbb{C}]$ consists of six complex numbers. Addition is simply vector addition. To multiply algebra elements the computer program uses the Geo3 group multiplication rules. Repeatedly squaring and dividing by the trace creates a path that is then graphed.

In 2014, Steven Weinberg showed that density matrices can have more interesting symmetries than those that are available to state vectors.[6] These last few paragraphs have shown that the group algebra on the Pauli algebra implies a symmetry on the density matrices of $3[\sigma^\mu]$. This is a symmetry of the Weinberg type; a symmetry that is more interesting than those available when considering state vectors only. That is, the 4x4 density matrices that follow the symmetry of Equation 1.11 can be put into block diagonal form:

$$\begin{pmatrix} A & A & & \\ A & A & & \\ & & B & B \\ & & B & B \end{pmatrix} \tag{2.19}$$

where A and B are two uncoupled Weyl equations and so correspond to two density matrices. The remaining entries have been left blank as they are always zero. This is a packing of density matrices into a density matrix. Weinberg's example places state vectors in a density matrix. His example is:

$$\rho = \begin{pmatrix} a_1 & b_3 & b_2^* \\ b_3^* & a_2 & b_1 \\ b_2 & b_q^* & a_3 \end{pmatrix} \tag{2.20}$$

"... where under SU(3), the real diagonal elements a_n transform as singlets, with $a_1 + a_2 + a_3 = 1$, and the triplet (b_1, b_2, b_3) transforms as the representation **3**. This SU(3) transformation of ρ cannot be put into the form $\rho \to U\rho U^\dagger$ required in ordinary quantum mechanics..." [6]

Our transformation of the Weyl equations is of the Weinberg form, that is, the objects being transformed σ^μ, transform by spin SU(2) in the usual density matrix form $\rho' \to U\rho U^{-1}$, but we are transforming it with a single-sided transformation matrix, i.e. a generalization of the discrete Fourier transform. Thus the transformations we use in this paper cannot be accomplished on state vectors. This could be the reason these transformations appear to have previously escaped notice.

3. $43[\sigma^\mu]$ and the Standard Model

In this section we will be discussing a single generation of the Standard Model, i.e. the electron, neutrino, up quark and down quark. In the usual quantum mechanical description of these particles, each is modeled using bispinors and the Dirac equation. Bispinors have four complex components. For example the electron bispinor basis can be chosen to be the four states "spin-up electron", "spin-down electron", "spin-up positron" and "spin-down positron". An alternative natural to the Weyl basis would be "spin-up left", "spin-down left", "spin-up right" and "spin-down right". We cannot define a basis of elements that mix handedness with charge, i.e. "spin-up left electron" because charge and handedness do not commute.

We will avoid the mass interaction in this section. In the massless limit (relativistic or high energy) the Dirac equation splits into two uncoupled Weyl equations, left handed and right handed. Each of these equations is a mix of particle and antiparticle. The other way of eliminating mass from the Dirac equation is to take the infinite mass limit (non relativistic or low energy). In this limit the Dirac equation also splits into two Weyl type equations, but one is for particles the other for antiparticles. The $43[\sigma^\mu]$ algebra includes both left handed and right handed coupled Weyl equations, but the irreducible representations will mix these leaving particles and antiparticles. So our uncoupled Weyl equations will have a basis of "spin-up electron", "spin-down electron", "spin-up positron" and "spin-down positron". This makes it easy to find the operator for electric charge. By looking at the left and right handed parts of the algebra, we can pick out the left and right handed parts of the particles and therefore we can derive operators for weak hypercharge and weak isospin.

An alternative to our calculation would be to write a generalized massless Dirac equation using the handed octahedral point group Geo$\overline{43}$, which has 24 elements and is the subgroup consisting of Geo43's proper rotations. Then the algebra would be $\overline{43}[\gamma^\mu]$. The reason we've chosen $43[\sigma^\mu]$ instead is that it makes the projection operators for handedness part of the point group symmetry. With $\overline{43}[\gamma^\mu]$, the handedness projection operators are $(1 \pm \gamma^5)/2$ which would mean our particle identities would be split between the point group symmetry and the gamma matrices.

Half of the 24 elements of the Geo43 point group are proper rotations and leave the Pauli spin matrices left handed. The other 24 elements change the handedness of the Pauli spin matrices to right handed. The irreducible representations exist in two types. Half have the same signs for the left-handed and right handed elements, the other have opposite signs. The usual convention is that particles take the representations with the same signs while the antiparticles use the representations with opposite signs. In addition to the difference in handedness, the elements of Geo43 can also be classified as “even” and “odd”. The proper rotations which can be obtained as an even number of right angle rotations, and the improper rotations which are a proper even rotation times the inversion i are the “even” elements, the rest are “odd”. For the left handed particles, weak hypercharge depends on left handed even elements while weak isospin depends on left handed odd elements. This is natural in that the Pauli spin matrices are odd for the spatial components σ_x, σ_y and σ_z while the identity 1_2 is even.

It’s clear that the lepton irreps will correspond to 1×1 blocks on the diagonal while the quarks will be 3×3. With these assumptions, it remains to distinguish between the two leptons, i.e. electron and neutrino, and between the two quarks. Until we have a more complete theory that includes the gauge bosons, we will make an arbitrary choice for these. Labeling the irreps of the character table with the particles and antiparticles we have two irreps left over that we assign to dark matter D and its antiparticle $\overline{D}$:

O	E	$3C_2$	$8C_3$	$6C_2$	$6C_4$	i	$3\sigma_h$	$8S_6$	$6S_4$	$6\sigma_d$
Hand:	L	L	L	L	L	R	R	R	R	R
Ev/Od:	E	E	E	O	O	E	E	E	O	O
Size:	1	3	8	6	6	1	3	8	6	6
σ^μ :	XYZ	$X\overline{YZ}$	YZX	$YX\overline{Z}$	$XZ\overline{Y}$	$\overline{XYZ}$	$\overline{X}YZ$	$\overline{YZX}$	$\overline{YX}Z$	$\overline{XZ}Y$
ν	1	1	1	1	1	1	1	1	1	1
$\overline{\nu}$	1	1	1	1	1	−1	−1	−1	−1	−1
e	1	1	1	−1	−1	1	1	1	−1	−1
$\overline{e}$	1	1	1	−1	−1	−1	−1	−1	1	1
d	3	−1	0	1	−1	3	−1	0	1	−1
$\overline{d}$	3	−1	0	1	−1	−3	1	0	−1	1
u	3	−1	0	−1	1	3	−1	0	−1	1
$\overline{u}$	3	−1	0	−1	1	−3	1	0	1	−1
D	2	2	−1	0	0	2	2	−1	0	0
$\overline{D}$	2	2	−1	0	0	−2	−2	1	0	0

(3.1)

In the above table, the five columns on the left are the left handed or proper rotations, and the five columns on the right are the right handed or improper rotations. Each of these five columns are split into three even and two odd columns.

We can write the operator for electric charge Q in terms of the classes of the Geo43 group. That is, Q is a diagonal operator in $43[\sigma^\mu]$ that commutes with all the irreps and so can be written as a sum

$$Q = \Sigma_j q_j I_j \tag{3.2}$$

where q_j is the electric charge of the jth particle, and I_j is the identity for the corresponding block on the diagonal. The blocks on the diagonal correspond to the irreps of the original character table and so each I_j can be written as a sum of products of the group classes. These can be read off of the character table as shown in the previous section. As always occurs with character tables, the number of irreps is the same as the number of classes (columns) so there is only one choice for the electric charge operator when writing it as an element of $43[\sigma^\mu]$.

Since particles and anti particles have opposite electric charge, examining the table shows that the Q operator can be composed only of right handed elements. The left (right) handed particles will be composed only of left (right) handed elements so their weak isospin and weak hypercharge will depend only on left (right) handed group elements. Weak hypercharge for right handed particles is the same as electric charge which we already know is made only of right handed elements. This reduces the algebra needed to define the charges to be a 5x5 transformation instead of 10x10. Writing the left and right handed elements of the particles in the same columns we have:

$L:$	E	$3C_2$	$8C_3$	$6C_2$	$6C_4$			
$R:$	i	$3\sigma_h$	$8S_6$	$6S_4$	$6\sigma_d$			
Hand:	L/R	L/R	L/R	L/R	L/R			
Ev/Od:	E	E	E	O	O		Charge	
Size:	1	3	8	6	6	$Y/2\ +$	$I_3\ =$	Q
ν	1	1	1	1	1	$\frac{-1}{2}/0$	$\frac{1}{2}/0$	0
e	1	1	1	-1	-1	$\frac{-1}{2}/-1$	$\frac{-1}{2}/0$	-1
d	3	-1	0	1	-1	$\frac{1}{6}/\frac{-1}{3}$	$\frac{-1}{2}/0$	$\frac{-1}{3}$
u	3	-1	0	-1	1	$\frac{1}{6}/\frac{2}{3}$	$\frac{1}{2}/0$	$\frac{2}{3}$
D	2	2	-1	0	0	$0/0$	$0/0$	0.

(3.3)

When the classes in the above table are expanded to the elements the five columns will increase to $1+3+8+6+6 = 24$ and the operators are obtained by dividing by 24. We will do the algebra without this division and insert it at the end. Also, in the above table there is a subtlety on the scaling. The projection operators for the quark identities 1_3 is three times the amount shown. This would require us to multiply those rows by three, but these projection operators apply to three colors of quarks while the experimentally measured charges are for a single quark so we also need to divide by three.

Right weak isospin is always zero, and charge is the same as right weak hypercharge leaving us three equations to solve: left and right weak hypercharge, and left weak isospin. Left weak hypercharge depends only on the even left components giving three equations in three unknowns:

$$\begin{aligned} 1\,Y/2^{E} + 1\,Y/2^{3C2} + 1\,Y/2^{8C3} &= -1/2, \\ 3\,Y/2^{E} - 1\,Y/2^{3C2} + 0\,Y/2^{8C3} &= \ \ 1/6, \\ 2\,Y/2^{E} + 2\,Y/2^{3C2} - 1\,Y/2^{8C3} &= \ \ \ \ 0. \end{aligned} \tag{3.4}$$

Left weak isospin depends only on the odd left components so there are two equations in two unknowns:

$$\begin{aligned} 1\ I_3^{6C2} + 1\ I_3^{6C4} &= \ \ 1/2, \\ 1\ I_3^{6C2} - 1\ I_3^{6C4} &= -1/2. \end{aligned} \tag{3.5}$$

Electric charge and right weak hypercharge use all five right components giving five equations:

$$\begin{aligned} 1\,Y/2^{i} + 1\,Y/2^{3\sigma h} + 1\,Y/2^{8S6} + 1\,Y/2^{6S4} + 1\,Y/2^{6\sigma_d} &= 0, \\ 1\,Y/2^{i} + 1\,Y/2^{3\sigma h} + 1\,Y/2^{8S6} - 1\,Y/2^{6S4} - 1\,Y/2^{6\sigma_d} &= -1, \\ 3\,Y/2^{i} - 1\,Y/2^{3\sigma h} + 0\,Y/2^{8S6} + 1\,Y/2^{6S4} - 1\,Y/2^{6\sigma_d} &= -1/3, \\ 3\,Y/2^{i} - 1\,Y/2^{3\sigma h} + 0\,Y/2^{8S6} - 1\,Y/2^{6S4} + 1\,Y/2^{6\sigma_d} &= 2/3, \\ 2\,Y/2^{i} + 2\,Y/2^{3\sigma h} - 1\,Y/2^{8S6} + 0\,Y/2^{6S4} + 0\,Y/2^{6\sigma_d} &= 0. \end{aligned} \tag{3.6}$$

Solving these linear equations for the operator coefficients defines the operators as

$$\begin{array}{rclllllll} I_3 & = & 0 & 0 & 0 & 0 & +12(6C_4), & \text{Left} \\ Y/2 & = & 0 & -4(3C_2) & -8(8C_2) & 0 & 0 & \text{Left} \\ & & 0 & -4(3\sigma_h) & -8(8S_6) & 0 & +12(6\sigma_d), & \text{Right} \\ Q & = & 0 & -4(3\sigma_h) & -8(8S_6) & 0 & +12(6\sigma_d), & \text{Right} \end{array} \tag{3.7}$$

where the terms have been arranged according to their symmetry and there left and right elements are labeled to the right side. The coefficients are for the class; we could have divided by the size of each class and obtained charges per group element of $-4/3$ for $(3C_2)$ and $(3\sigma_h)$, -1 for $(8C_2)$ and $(8S_6)$, and $+2$ for $(6C_4)$ and $(6\sigma_d)$.

In mapping the irreps to the particles we had freedom in that we could independently swap the leptons and swap the quarks. This would change some of the signs in the -4 and -8 columns as well as swapping $(6C_4)$ for $(6C_2)$ and $(6S_4)$ for $(6\sigma_d)$.

Since charge does not commute with handedness, the equation $Y/2 + I_3 = Q$ requires an assumption not obvious in the above: it follows when one uses Q as the charge of a Dirac bispinor so that the particle portion has charge Q while the antiparticle portion has charge $-Q$.

For low temperatures, a heat bath is typically conceived as an object that can exchange electric field gauge bosons (i.e. heat photons) with the system under study. However, when we model such a system with density matrices the gauge bosons do not appear explicitly; their effect is implicit. The gauge boson for the weak force is the $W^{\pm}$. As the temperature rises above the amount needed to create $W^{\pm}$ gauge bosons, it becomes possible for the heat bath to convert between electrons and neutrinos, and to convert between up quarks and down quarks. In the temperature flow shown in Figure (1), we can see that the cooling states clump together into curving arcs. These occur because certain degrees of freedom cool faster than others. We can imagine that these correspond to a cascade of symmetry breaking from the high temperature limit where all states are approximately identical. Then the weak symmetry breaking is the coolest of these.

4. Dark Matter Nomenclature

The extra "dark matter" irreps, D and $\overline{D}$ in the $43[\sigma^\mu]$ character table of Equation (3.1) have zero electric charge, weak hypercharge and weak isospin and so do not participate in the electric or weak forces. In the Standard Model, the color force is determined by quark SU(3) internal symmetries. In this model, the quark SU(3) operators are internal to the quark irreps in that they correspond to 8 of the $3^2 = 9$ degrees of freedom in the 3x3 quark blocks one obtains when diagonalizing the $43[\sigma^\mu]$ grouop algebra. The remaining degree of freedom are the 3×3 unit matrices which are defined as three times the quark irreps of the character table. These quark SU(3) color degrees of freedom are restricted to the corresponding quark 3×3 blocks so they annihilate between different quarks. Examining the character table we see that the quarks differ in the signs of their odd elements, $(6C_2)$, $(6C_4)$, $(6S_4)$ and $(6\sigma_d)$ therefore the gluons must be associated with changes in these elements. But the dark matter irreps are zero in those columns so they cannot participate in the strong force. Thus the dark matter irreps are indeed dark.

As with the leptons and quarks, dark matter comes in particle D and antiparticle $\overline{D}$ form. These differ in sign between the left and right handed parts of the particles as do the leptons and quarks so we expect that dark matter will also have mass.

The dark matter irreps have an internal SU(2) symmetry that is otherwise similar to the quark's internal SU(3) symmetry. By analogy with the quarks, we will label the dark matter SU(2) internal symmetry as colors. As the dark matter particles that correspond to quarks, we will call them "dark quarks" or "duarks". Since these have only an SU(2) symmetry, there are only two dark colors needed as a basis for dark color. Where the colors for quarks are expressive of their participation in electromagnetic photon interactions, the duarks are dark so in contrast to the quark colors of red, green and blue, we will use dark colors of "doom" and "gloom". These are also appropriate for this paper which is being written under the looming threat of the Covid19 pandemic.

We assume that there is a force boson similar to the gluon between them, call it the "duon", that will follow an SU(2) triplet symmetry. Where the eight gluons are often described using a basis of the eight Gell-Mann matrices, the three duons can use the three Pauli matrices to define the dark matter triplet:

$$\begin{array}{c} (d\bar{g} + g\bar{d})/\sqrt{2}, \\ -i(d\bar{g} - g\bar{d})/\sqrt{2}, \\ (d\bar{d} - g\bar{g})/\sqrt{2} \end{array} \tag{4.1}$$

where d and g stand for the doom and gloom dark colors.

A quark and an antiquark can combine to form a meson; we expect that a duark and an antiduark similarly combine to form a "deson". While the mesons decay by electroweak processes, the desons do not participate so we may suppose that they are stable. Similarly, three quarks can combine to form a baryon so two duarks of different dark colors, combine to form a "daryon".

5. Conclusion

We've shown that the relationship between particle and antiparticle in the Dirac equation, i.e. changing the signs of the Pauli spin matrices so as to convert them from being left handed to right handed, can be generalized by right angle rotations and the inversion, that permute the Pauli spin matrices and change their signs. The resulting generalized Pauli spin matrices can be used to define a set of Weyl equations that are coupled. Uncoupling these Weyl equations by algebraic manipulation leads to a set of uncoupled Weyl equations that have the symmetry of the Standard Model fermions.

The method shown here is incomplete in that it does not include a description of the gauge bosons or the mass interaction or Higgs boson. The strong force corresponds to changes to the internal symmetry of the quarks. In this model, these internal symmetries are exact; they are the result of the mathematical freedom to choose the off diagonal elements when block diagonalizing the group algebra. This suggests that the gluons correspond to changes in the quarks that preserve energy exactly. Hence the gluons are massless.

The weak force corresponds to sign changes in the left handed odd elements of the group algebra, i.e. $(6C_2)$ and $(6C_4)$. These changes do not correspond to a Lie algebra type symmetry. That is, with the strong force, we can change the color of a quark by infinitesimal changes in the group algebra. But there are no infinitesimal stable particle states between an up quark and a down quark. On the other hand, the fact that the mass of the weak gauge bosons is so small compared to the Planck mass that we can conclude that sign changes to the group algebra has only very small changes to the energy of a particle.

The mass interaction converts between left and right handed particles. These differ by the signs of the right handed group elements. This implies that the Higgs mass, like the $W^{\pm}$ mass, is very small compared to the Planck mass as the energy of the particles differ little on change of sign. And as with the weak force, the conversion between left and right handed particles is not an infinitesimal or Lie algebra conversion.

The photon does not convert between particles in this model. It might convert an electron from spin-up to spin-down but these are related by a Lie algebra (in the Pauli spin algebra part of the group algebra). The fact that the photon and gluon are both massless appears to be related to the fact that they can modify particles by an infinitesimal Lie algebra type transformation and these transformations are exact. For the strong force the transformation is exact as it is due to an arbitrary choice of representation of the off diagonal elements when block diagonalizing the coupled Weyl equations. The photon transformation is exact as space has no preferred directions and the Pauli spin matrices are an exact SU(2) symmetry.

Dark matter appears to experience only the mass interaction between left and right handed particles, and the proposed dark color interaction that will change a particle's dark color in their internal SU(2) symmetry. As with the quark's internal SU(3) symmetry, the symmetry is exact so we expect the dark quark gauge boson, the duon, to be massless.

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Carl Brannen
210 NW Anthony, Apt 1
Pullman, Washington 99163
United States
e-mail: `carl.brannen@wsu.edu`