Division by Zero

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Abstract

This paper presents a proposed framework for interpreting multiplication, division, and selection as forms of a common transformation between pairs of quantities. Within this framework, a number is treated in its natural form as a ratio between a value and a base measure. The paper argues that reducing such ratios to denominator-one representations discards information about the underlying transformation and motivates a re-examination of proportional equivalence and division by zero. A two-dimensional value–measure representation is introduced to visualize ratios and sets of n-fold relations. The framework is then applied to the functions $f(x) = 1/x$ and $f(x) = \mathrm{tg}(x)$, comparing their conventional real-number graphs with the author's proposed rational-number representation. The central proposal is that expressions with zero measure can be retained as distinct ratio states even though they do not project onto the conventional real-number axis. The paper concludes by discussing consequences of this interpretation for numerical representation and transformation.

Keywords: division by zero; transformation; numerical ratio; measure; value; rational-number representation.

Contents

The natural form of a number is its value relative to a certain measure.

Leszek Mazurek

The worst consequences come from errors in assumptions.

Leszek Mazurek

1. Introduction

Since time immemorial, the prohibition on division by zero has posed a serious problem for all thinking people who have found it difficult to accept this restriction, especially because intuitively it seems very artificial and unjustified. In this work I have undertaken the challenge of analysing this problem very carefully and trying to understand where it comes from, why it creates such difficulty and limitation, and what should be done in order to solve it. The results obtained not only solve this problem, but also cast an entirely new light on the understanding of numbers and operations on numbers and will probably have a number of consequences in other areas of algebra, mathematics, and related fields such as physics, chemistry, or astronomy. It is astonishing that this problem has remained without proper understanding for such a long time.

2. Understanding multiplication and division of numbers

In order to understand division of numbers correctly, and division by zero in particular, we must begin by understanding multiplication itself. In its general form multiplication can be represented as

$$a \cdot b = c. \tag{1}$$

When performing multiplication, we solve the problem in which a and b are given and we seek such a c that is equal to a times b (the sum of b taken a times). This operation can be represented as

$$a \cdot b = ? \tag{2}$$

or graphically as follows.

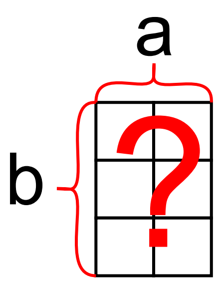


Fig. 1. The problem solved by multiplication

In the example shown in Fig. 1,

$$a = 2, \qquad b = 3, \tag{3}$$

so

$$c = a \cdot b, \tag{4}$$

$$c = 2 \cdot 3, \tag{5}$$

$$c = 3 + 3, \tag{6}$$

$$c = 6. \tag{7}$$

How, then, if we keep to such very basic symbolic notation, will the operation of division look? In general form division can be represented as

$$\frac{c}{a} = b. \tag{8}$$

Thus the task we want to solve consists in having a given c, dividing it into a equally numerous groups, and asking how many elements are in each group:

$$\frac{c}{a} = ? \tag{9}$$

The operation of division presented above causes mathematicians considerable difficulty because it has one, but significant, restriction. In the case of multiplication we have no major problem multiplying by zero, because any number multiplied by zero (any number of zeros added together) gives zero. In the case of division, however, we do not allow the denominator of a fraction to represent division into zero groups. This goes beyond our ability to interpret such a phenomenon on the basis of the rules of logic and our mathematical reasoning.

Looking at the general form of multiplication (1),

$$a \cdot b = c, \tag{10}$$

and at the multiplication problem defined in (2),

$$a \cdot b = ?, \tag{11}$$

division can also be written in a slightly modified form:

$$a \cdot ? = c. \tag{12}$$

This time, therefore, the problem to be solved can be described as follows. We are given a; what must be done with it to obtain c? In somewhat different words: a is available, we want to obtain c; what should we do?

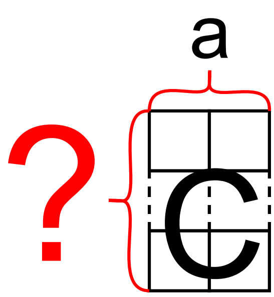


Fig. 2. The problem solved by division

Continuing the example with the numbers 2, 3, and 6, the divisibility problem would look as follows. We are given 2 and want to obtain 6. What should be done? Answer: take it threefold (3 times).

The question posed above appears to be the more primary problem, yet we usually try to solve it by turning the matter "upside down", attempting to calculate the sought value by dividing c into a equal parts.

Let us carefully analyse another example and try to divide 1 by 2. Let us therefore examine carefully what the symbol

$$\frac{1}{2} \tag{13}$$

means.

In the traditional understanding, it means that we divide one into two equal sets. As a result of this operation we obtain two sets, each containing "one half". At first glance this seems somewhat strange, because by dividing 1 by 2 we in fact obtain two occurrences (two sets) of one half, rather than simply one half. Listening carefully to the sentence "one divided by two is one half", we come to the conclusion that it is not consistent with what we observe. It would be more correct to say that "one divided into two equal parts (into two equal sets) gives one half in each of them". It is worth noting that in order to divide into two sets in this case, what is being divided is initially represented by one set. Let us illustrate this classical understanding of division in a drawing.

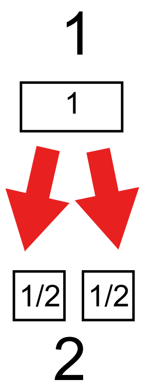


Fig. 3. Division $1 : 2$

The inverse operation would consist in summing, combining these separated sets, as shown below, with the result being 1:

$$\frac{1}{2} + \frac{1}{2} = 2 \cdot \frac{1}{2} = 1. \tag{14}$$

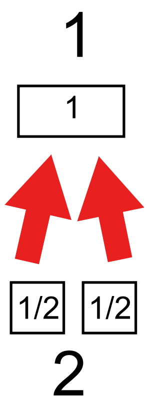


Fig. 4. The inverse of division 1 by 2

In the multiplication operation we combine here two sets of one half, resulting in one, or more precisely one set containing one.

3. Division and multiplication as a transformation

Let us note that in the classical interpretation of a fraction we use top-to-bottom analysis. That is, we start with the numerator and proceed to the denominator. We divide one (the numerator) by two (the denominator). The inverse operation, multiplication by two, can be understood as an operation in the opposite direction: moving from the bottom to the top of the fraction. We collect two (the denominator) and create one (the numerator).

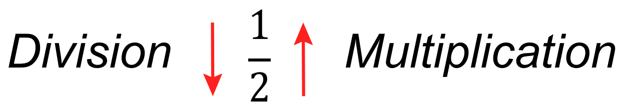


Fig. 5. Multiplication and division as mutually inverse operations

Let us analyse one more case:

$$\frac{3}{5}. \tag{15}$$

This fraction represents the operation of dividing 3 by 5; in other words, we have three and want to divide them into five equal groups. Fig. 6 shows this operation graphically.

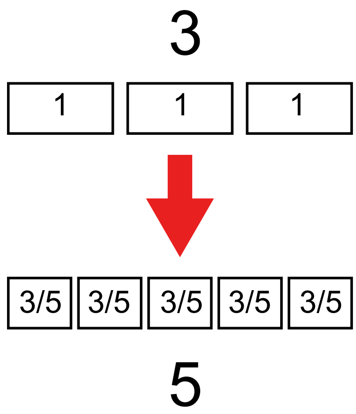


Fig. 6. 3 divided by 5

Mathematically, this can be written, for example, as

$$3 : 5 = \frac{3}{5}. \tag{16}$$

Performing the operation inverse to the above gives multiplication:

$$\frac{3}{5} \cdot 5 = 3. \tag{17}$$

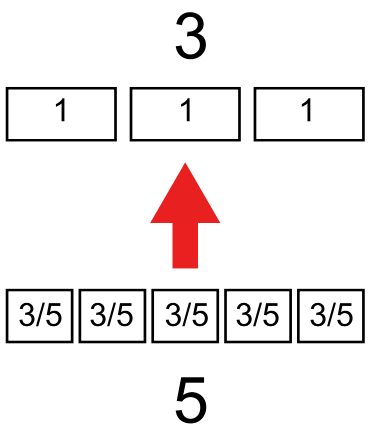


Fig. 7. Multiplication: 5 times $\frac{3}{5}$

By carefully analysing the examples presented above, we can notice that in every case we are in fact dealing with four numbers which together describe a certain kind of conversion (transformation). Thus, for example, division by $\frac{3}{5}$ means the transformation

$$3 \cdot 1 \longrightarrow 5 \cdot \frac{3}{5}. \tag{18}$$

The inverse operation (multiplication) is

$$3 \cdot 1 \longleftarrow 5 \cdot \frac{3}{5}, \tag{19}$$

or, written from left to right,

$$5 \cdot \frac{3}{5} \longrightarrow 3 \cdot 1. \tag{20}$$

Division of 1 by 2, expanded into four numbers forming a transformation, is

$$1 \cdot 1 \longrightarrow 2 \cdot \frac{1}{2}. \tag{21}$$

The inverse operation (multiplication) is

$$1 \cdot 1 \longleftarrow 2 \cdot \frac{1}{2}. \tag{22}$$

The multiplication operation shown at the beginning of this chapter is

$$2 \cdot 3 \longrightarrow 6 \cdot 1. \tag{23}$$

The inverse operation, representing $6/3$ or $6/2$, is

$$2 \cdot 3 \longleftarrow 6 \cdot 1. \tag{24}$$

Let us note that the operation of division causes a transformation which is reversed by the transformation associated with multiplication:

$$3 \cdot 1 \longrightarrow 5 \cdot \frac{3}{5} \longrightarrow 3 \cdot 1. \tag{25}$$

Reading the above notation from left to right, we have 3 (that is, three units), which we separate (transform) into 5 equal parts of $\frac{3}{5}$. Next, we transform them by combining the 5 together (multiplying 5 times), restoring the initial 3 units.

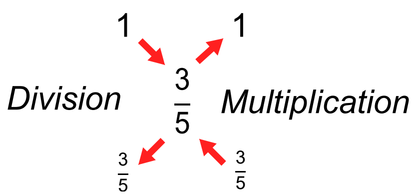


Fig. 8. Division and multiplication as transformation through $\frac{3}{5}$

The next example (Fig. 9) can be read as follows.

Division: We have six units (that is, six times one, six elements of one, six sets of one) and transform them into three equally numerous sets (groups of elements), with two in each set.

Multiplication: We have two elements in each of three equally numerous groups and transform them into six groups of one (for short, six ones can simply be called six).

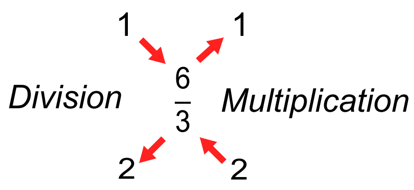


Fig. 9. Division and multiplication as transformation through $\frac{6}{3}$

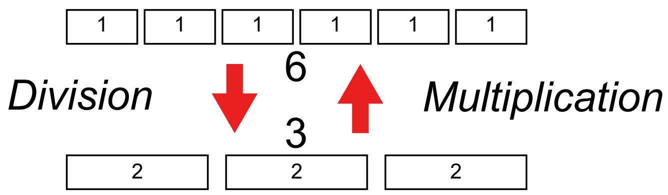


Fig. 10. 6 groups of 1 ↔ 3 groups of 2

For certain reasons, the number 1 usually occurs in operations of this type and is often omitted in formal mathematical notation; however, this is only a special case and need not always be so. The example below shows that this way of understanding multiplication and division works equally well in more general cases.

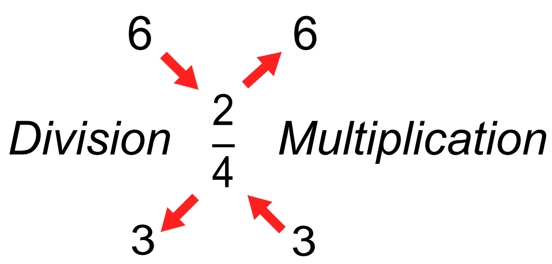


Fig. 11. Division and multiplication as transformation through $\frac{2}{4}$

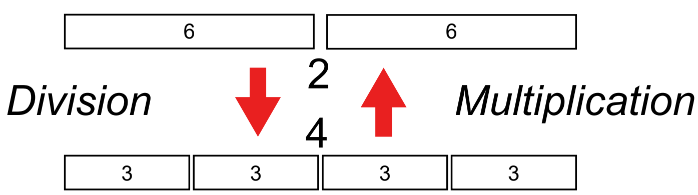


Fig. 12. 2 groups of 6 ↔ 4 groups of 3

This example can be read as follows.

Division: We have two groups of six elements and transform (divide) them into four groups of three elements.

Multiplication: We have four groups of three elements and transform them (combine them, performing the operation inverse to division, i.e. multiply) into two groups of six elements.

As can be seen from the above examples, in both multiplication and division we are in fact performing only a certain transformation, that is, converting two numbers into two other numbers. In the example above, we transformed the pair $(2, 6)$ into the pair $(4, 3)$:

$$(2, 6) \longleftrightarrow (4, 3). \tag{26}$$

In this example, the two numbers 2 and 4 define the transformation; they are the numbers making up the fraction $\frac{2}{4}$ (the magnifying glass in Fig. 13). The other two numbers, 6 and 3, are subject to this transformation in one direction or the other.

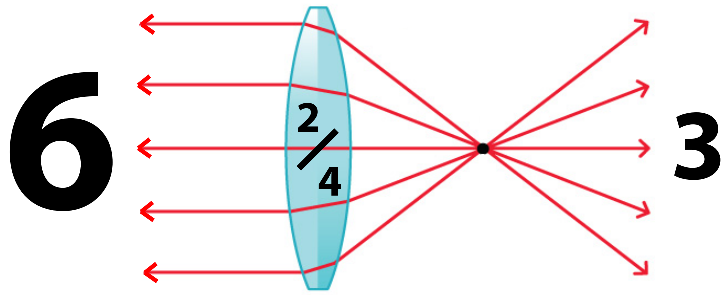


Fig. 13. The fraction $\frac{2}{4}$ as a magnifying glass

In the case of division, the fraction tells us how many groups we transform (two in this case) into how many target groups (four in this case).

In the case of multiplication (the opposite direction), we transform four groups into two target groups.

Having defined the transformation, we perform the conversion. In division we transform 6 (as the cardinality of the initial group) into 3 (the cardinality of the target group). In multiplication (the opposite direction) we transform 3 (the cardinality of the initial group) into 6 (the cardinality of the target group).

Division: 2 groups of 6 elements $\to$ 4 groups of 3 elements.

Multiplication: 2 groups of 6 elements $\leftarrow$ 4 groups of 3 elements (as the operation inverse to division), or equivalently 4 groups of 3 elements $\to$ 2 groups of 6 elements.

It is easy to notice that the ratio defined by the fraction $\frac{2}{4}$ (which defines the transformation) is the same as the ratio between the cardinality of the target group and the cardinality of the initial group in the division, $\frac{3}{6}$:

$$\frac{2}{4} = \frac{3}{6}. \tag{27}$$

$$\frac{\text{number of initial groups}}{\text{number of target groups}} = \frac{\text{cardinality of the target group}}{\text{cardinality of the initial group}}. \tag{28}$$

In the case of division, we transform 6 by the ratio written in the fraction $\frac{2}{4}$, obtaining 3:

$$6 \cdot \frac{2}{4} = 3. \tag{29}$$

Thus, in general,

$$\text{cardinality of initial group} \cdot \frac{\text{number of initial groups}}{\text{number of target groups}} = \text{cardinality of target group}. \tag{30}$$

Let us note that the fraction (in the case discussed, $\frac{2}{4}$) does not unambiguously state the cardinality of the initial group that it is able to transform into the target group. It only states a certain ratio, which we interpret as the numerical ratio between the cardinality of the target group and the cardinality of the initial group. Often we implicitly assume that the cardinality of the elements of the initial group is 1. When we say "two", we understand this as two units which we divide into four parts. What is actually represented, however, is only the number of initial groups — 2 — and the number of target groups — 4 — of this transformation, as well as the ratio between them, $\frac{2}{4}$. Compare the examples below; in each of them we use the same fraction to transform different numbers:

$$6 \cdot \frac{2}{4} = 3. \tag{31}$$

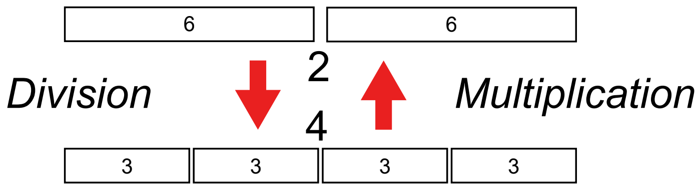


Fig. 14. 2 groups of 6 ↔ 4 groups of 3

$$10 \cdot \frac{2}{4} = 5. \tag{32}$$

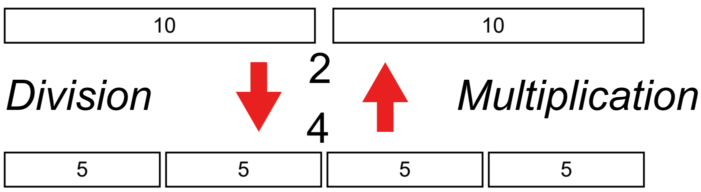


Fig. 15. 2 groups of 10 ↔ 4 groups of 5

$$2 \cdot \frac{2}{4} = 1. \tag{33}$$

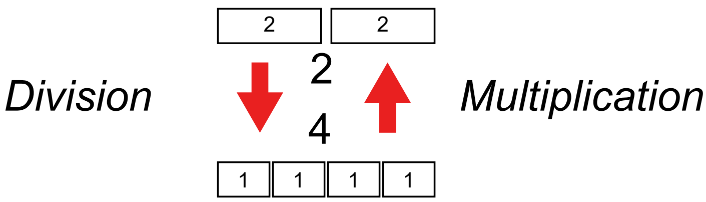


Fig. 16. 2 groups of 2 ↔ 4 groups of 1

In general we can say that the fraction $\frac{2}{4}$ represents:

a) a transformation of two groups into four groups;

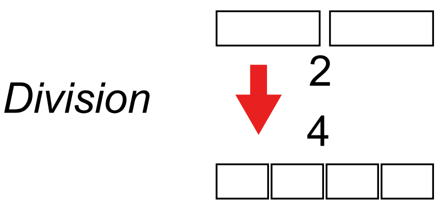


Fig. 17. Transformation of 2 groups into 4 groups

b) the ratio between the cardinality of the target group and the cardinality of the initial group.

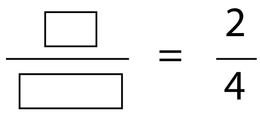


Fig. 18. Ratio between group cardinalities in a transformation through a fraction

But it also represents the inverse operation, i.e. multiplication (read in reverse, from bottom to top, as in Fig. 13):

a) a transformation of four groups into two groups;

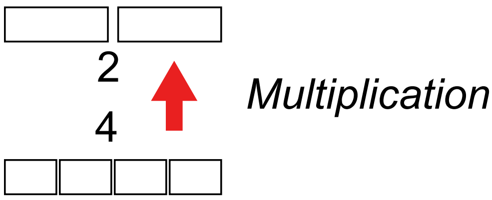


Fig. 19. Inverse transformation: four groups into two groups

b) the numerical ratio between the cardinality of the target group and the cardinality of the initial group. In this case, however, we move from bottom to top, i.e. transform in the reverse direction, so the fraction should be inverted to make this visible.

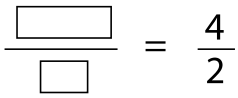


Fig. 20. Ratio between group cardinalities in a transformation through the inverse fraction

4. A fraction as an element transforming itself

If, in the example discussed, we take four as the cardinality of the initial group, then as a result we obtain the cardinality of the target group, namely two:

$$4 \cdot \frac{2}{4} = 2. \tag{34}$$

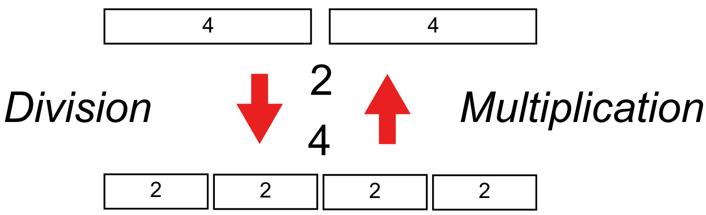


Fig. 21. The fraction $\frac{2}{4}$ as a transformation

$$4 \cdot \frac{2}{4} = 2$$

Fig. 22. The fraction $\frac{2}{4}$ transforms its own denominator 4 into numerator 2

As we can see, in such a case the fraction $\frac{2}{4}$ not only describes the conversion of two groups into four (the red arrow in Fig. 22), but also allows the cardinality of the initial group, 4, to be transformed into the target-group cardinality, 2 (the blue arrow in Fig. 22).

This example shows that a fraction has the property that it can transform its own component elements. In this case $\frac{2}{4}$ transforms $4 \to 2$.

In the general case, the fraction

$$\frac{\text{numerator}}{\text{denominator}} \tag{35}$$

allows the transformation

$$\text{denominator} \longrightarrow \text{numerator}. \tag{36}$$

5. Unification of multiplication and division

The examples presented above clearly show that both multiplication and division can be reduced to one arithmetic operation — **transformation**.

When performing **division**, we begin with a certain number of groups containing certain cardinalities of elements and transform them into another number of groups containing different cardinalities of elements.

When performing **multiplication**, we likewise begin with a certain number of groups containing certain cardinalities of elements and transform them into another number of groups containing different cardinalities of elements.

Both operations describe exactly the same type of operation — transformation. This means that they are not different operations, but one operation. The operation of transformation (multiplication and division) preserves proportion in such a way that when we increase (or decrease) the number of groups during the transformation by a factor of n, we decrease (or increase) the number of elements in a group by the same factor.

For multiplication:

Number of groups reduced by a factor of **n**

$$4 \cdot 2 = 2 \cdot 4$$

Number of elements per group
increased by a factor of **n**

Fig. 23. Reciprocal change in the number of groups and elements per group

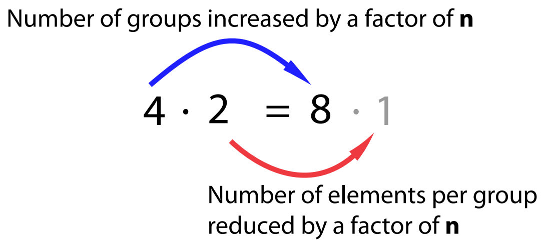


Fig. 24. Reciprocal change in the number of groups and elements per group

For division:

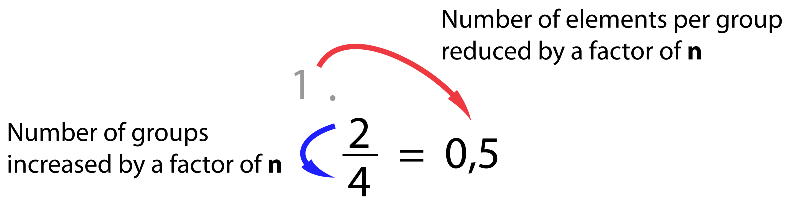


Fig. 25. Reciprocal change in the number of groups and elements per group

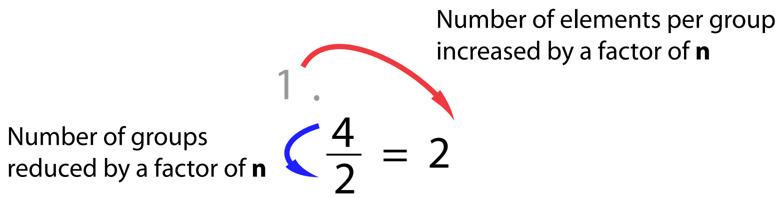


Fig. 26. Reciprocal change in the number of groups and elements per group

The one marked in grey in the example is usually omitted in mathematical notation; however, in order to show the transformations taking place in full, we can make it explicit.

6. Commutativity of multiplication

Treating multiplication as a transformation of groups of elements into a different number of groups of elements makes it possible to illustrate the commutativity of multiplication. Note that with every transformation the proportion

$$\frac{\text{number of initial groups}}{\text{number of target groups}} = \frac{\text{cardinality of target group}}{\text{cardinality of initial group}} \tag{37}$$

is preserved.

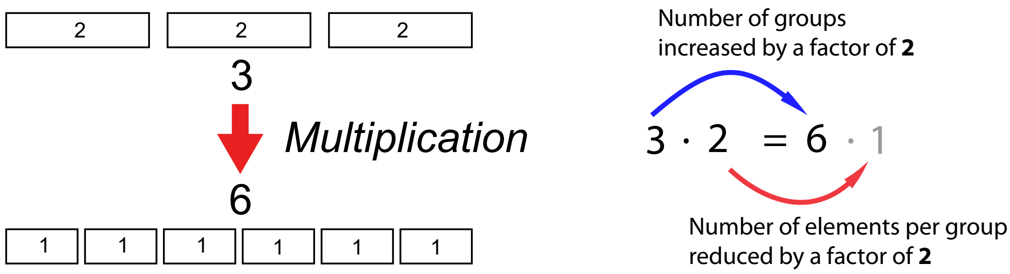


Fig. 27. Multiplication as a transformation

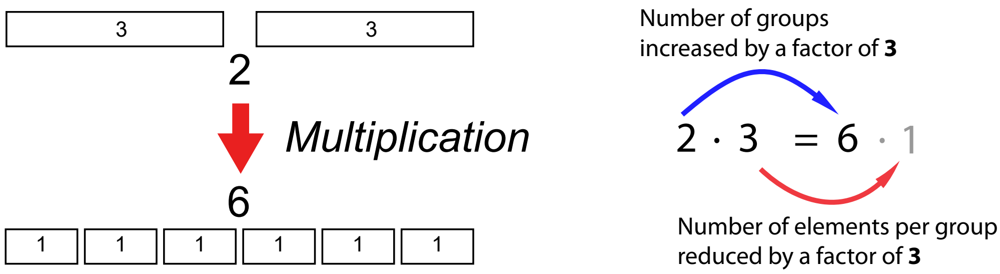


Fig. 28. Multiplication as a transformation

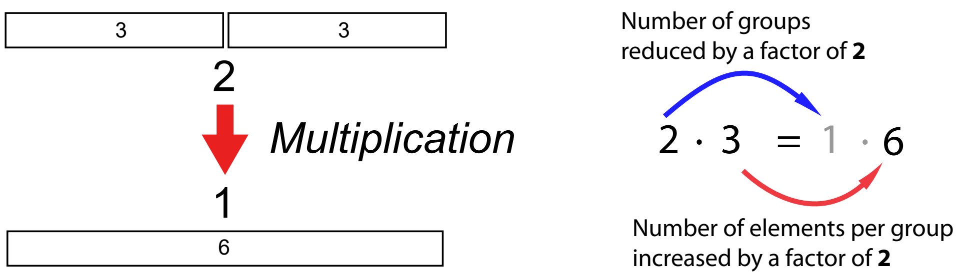


Fig. 29. Multiplication as a transformation

7. Introduction to the selection operation

The operations of division and multiplication presented above preserve the quantity before and after the operation. They seem only to change the "form" of the numbers while preserving their total magnitude at the same time. If we sum the lengths of the rectangles in the examples shown, their total length before and after the operation will be exactly the same. For multiplication and division this is a required property, but we can imagine an operation without such a property.

For example: we have 7 elements and take 5 of them. We can say that we **select** 5 elements out of seven.

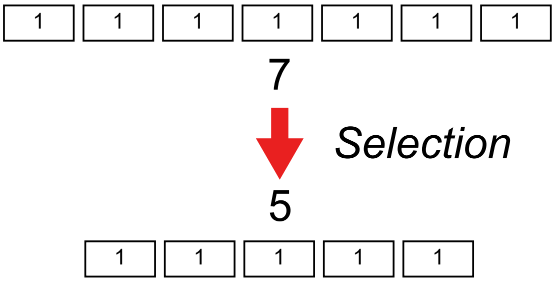


Fig. 30. Selecting 5 elements out of 7

In the case of selection we may have another situation as well. We do not necessarily have to select a certain number of elements; we can also select a certain part of the whole (Figs. 30 and 31).

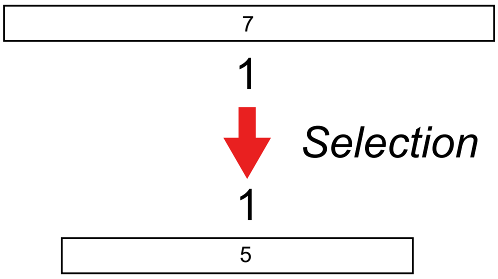


Fig. 31. Selecting a part of a whole

The selection operation differs from multiplication and division in that it does not preserve the total quantity, but changes it during the operation. The total quantity after the operation may be, and usually is, different from that before it is performed.

In the case of selection, in addition to scaling down $7 \to 5$, scaling up $5 \to 7$ is also possible; we will also call such an operation selection. Although it is not possible to select a larger number of elements from a smaller number, we can interpret operations of this type as a desire to select, for example, 7 elements even though only 5 are available.

8. Formal definition of transformation (multiplication, division, selection)

Let us define the following variables:

g_1 – initial number of equally numerous groups in the transformation,
c_1 – cardinality of each initial group,
g_2 – final number of equally numerous groups in the transformation,
c_2 – cardinality of each final group.

We will call a transformation T the conversion of the pair (g_1, c_1) into the pair (g_2, c_2):

$$T(g_1, c_1) \longrightarrow (g_2, c_2). \tag{38}$$

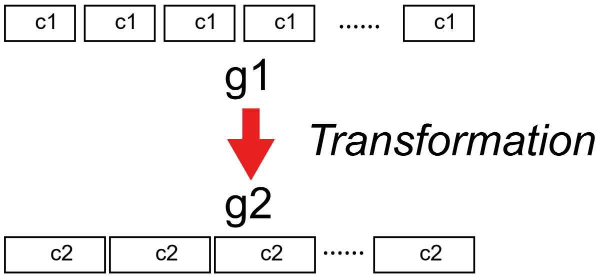


Fig. 32. General form of a transformation

We will call the **transformation coefficient** p (or the **transforming element**) the numerical ratio defined below:

$$p = \frac{g_2}{g_1}, \tag{39}$$

$$g_2 = p \cdot g_1, \tag{40}$$

or

$$p = \frac{c_2}{c_1}, \tag{41}$$

$$c_2 = p \cdot c_1. \tag{42}$$

Note that p transforms one of the elements of the transformation, either the number of groups or their cardinality, depending on the case.

We will call a transformation **full** or **total** if the following condition holds:

$$\forall(g_1, g_2, c_1, c_2) \qquad g_1 \cdot c_1 = g_2 \cdot c_2. \tag{43}$$

Formally: the g_1-fold sum of c_1 equals the g_2-fold sum of c_2.

In a full transformation, the proportion between the numbers of groups and the numbers of elements in the groups is preserved as follows:

$$\frac{g_1}{g_2} = \frac{c_2}{c_1}, \tag{44}$$

$$\frac{\text{number of initial groups}}{\text{number of target groups}} = \frac{\text{cardinality of target group}}{\text{cardinality of initial group}}. \tag{45}$$

We will call a transformation **incomplete** or **non-total** if the condition

$$\forall(g_1, g_2, c_1, c_2) \qquad g_1 \cdot c_1 = g_2 \cdot c_2 \tag{46}$$

does not hold.

We will call a full transformation **multiplication** when $g_2 = 1$:

$$T(g_1, c_1) \longrightarrow (1, c_2), \tag{47}$$

then

$$g_1 \cdot c_1 = 1 \cdot c_2, \tag{48}$$

$$c_2 = c_1 \cdot g_1. \tag{49}$$

The transformation coefficient p is

$$p = \frac{c_2}{c_1}, \tag{50}$$

$$p = \frac{g_1}{1}. \tag{51}$$

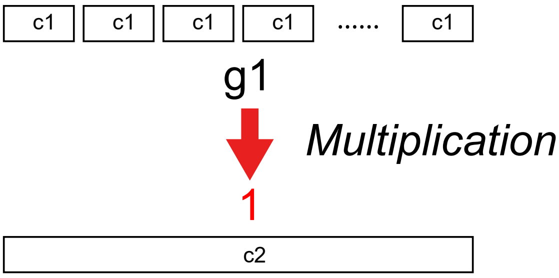


Fig. 33. Multiplication as a full transformation with one target group

We will call a full transformation **multiplication** when $c_2 = 1$:

$$T(g_1, c_1) \longrightarrow (g_2, 1), \tag{52}$$

then

$$g_1 \cdot c_1 = g_2 \cdot 1, \tag{53}$$

$$g_2 = g_1 \cdot c_1. \tag{54}$$

The transformation coefficient p is

$$p = \frac{g_2}{g_1}, \tag{55}$$

$$p = \frac{c_1}{1}. \tag{56}$$

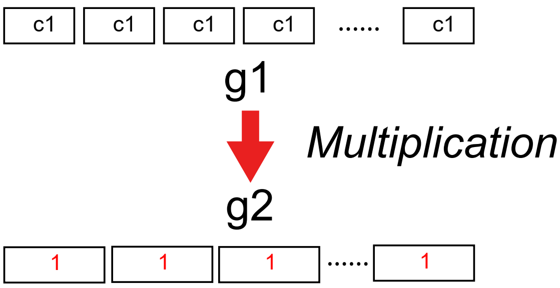


Fig. 34. Multiplication as a full transformation with unit target-group cardinality

We will call a full transformation **division** when $g_1 = 1$:

$$T(1, c_1) \longrightarrow (g_2, c_2), \tag{57}$$

$$1 \cdot c_1 = g_2 \cdot c_2, \tag{58}$$

$$c_1 = g_2 \cdot c_2, \tag{59}$$

$$\frac{c_1}{g_2} = c_2, \tag{60}$$

$$c_1 \cdot \frac{1}{g_2} = c_2, \tag{61}$$

$$\frac{1}{g_2} = \frac{c_2}{c_1}, \tag{62}$$

and the transformation coefficient is

$$p = \frac{1}{g_2}. \tag{63}$$

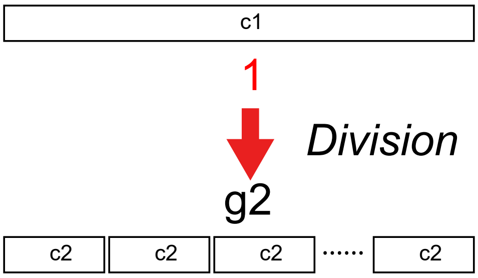


Fig. 35. Division as a full transformation with one initial group

We will call a full transformation **division** when $c_1 = 1$:

$$T(g_1, 1) \longrightarrow (g_2, c_2), \tag{64}$$

$$g_1 \cdot 1 = g_2 \cdot c_2, \tag{65}$$

$$g_1 = g_2 \cdot c_2, \tag{66}$$

$$\frac{g_1}{c_2} = g_2, \tag{67}$$

$$g_1 \cdot \frac{1}{c_2} = g_2, \tag{68}$$

and the transformation coefficient is

$$p = \frac{1}{c_2}. \tag{69}$$

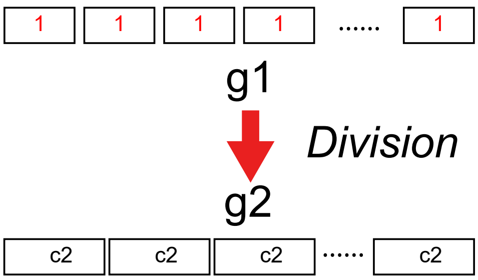


Fig. 36. Division as a full transformation with unit initial-group cardinality

We will call an incomplete transformation **selection** when $g_1 = g_2 = 1$:

$$T(1, c_1) \longrightarrow (1, c_2). \tag{70}$$

Then

$$\frac{c_2}{c_1} = \frac{c_2}{c_1}, \tag{71}$$

$$c_1 \cdot \frac{c_2}{c_1} = c_2, \tag{72}$$

and the transformation coefficient is

$$p = \frac{c_2}{c_1}. \tag{73}$$

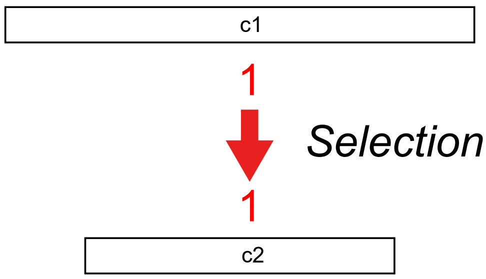


Fig. 37. Selection with one initial and one target group

We will call an incomplete transformation **selection** when $c_1 = c_2 = 1$:

$$T(g_1, 1) \longrightarrow (g_2, 1). \tag{74}$$

Then

$$\frac{g_2}{g_1} = \frac{g_2}{g_1}, \tag{75}$$

$$g_1 \cdot \frac{g_2}{g_1} = g_2, \tag{76}$$

and the transformation coefficient is

$$p = \frac{g_2}{g_1}. \tag{77}$$

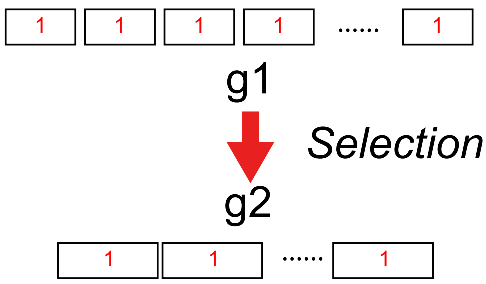


Fig. 38. Selection with unit cardinality in the initial and target groups

Both multiplication and division, as well as selection, are different forms of the transformation defined in (38).

Operation	**Definition of transformation**	**Transformation coefficient (p)**	**Conversion**
Transformation (general form)	$T(g_1, c_1) \to (g_2, c_2)$	$p_1,\ p_2$	$c_1 p_1 = c_2$; $g_1 p_2 = g_2$
Multiplication	$T(g_1, c_1) \to (1, c_2)$	$g_1/1$	$c_1 \cdot (g_1/1) = c_2$
Multiplication	$T(g_1, c_1) \to (g_2, 1)$	$c_1/1$	$g_1 \cdot (c_1/1) = g_2$
Division	$T(1, c_1) \to (g_2, c_2)$	$1/g_2$	$c_1 \cdot (1/g_2) = c_2$
Division	$T(g_1, 1) \to (g_2, c_2)$	$1/c_2$	$g_1 \cdot (1/c_2) = g_2$
Selection	$T(1, c_1) \to (1, c_2)$	c_2/c_1	$c_1 \cdot (c_2/c_1) = c_2$
Selection	$T(g_1, 1) \to (g_2, 1)$	g_2/g_1	$g_1 \cdot (g_2/g_1) = g_2$

Table 1

9. Is $\frac{1}{2}$ equal to $\frac{2}{4}$?

Let us check what we obtain by comparing $\frac{1}{2}$ and $\frac{2}{4}$ understood as transformations.

For $\frac{1}{2}$:

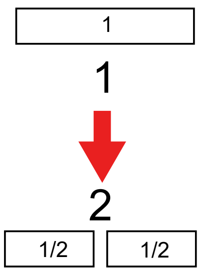
$\frac{1}{2}$	
$\begin{aligned} T(1, c_1) &\longrightarrow (g_2, c_2), \\ T(1, 1) &\longrightarrow (2, c_2), \\ c_1 \cdot \frac{1}{g_2} &= c_2, \\ 1 \cdot \frac{1}{2} &= c_2, \\ c_2 &= \frac{1}{2}, \\ T(1, 1) &\longrightarrow \left(2, \frac{1}{2}\right) \end{aligned}$	 Fig. 39

Table 2

For $\frac{2}{4}$:

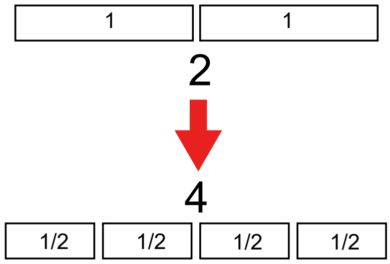
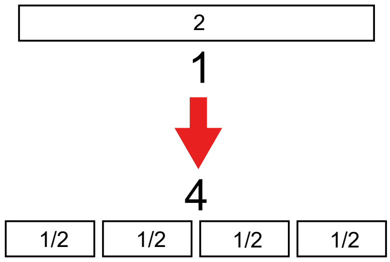
$\frac{2}{4}$	
$\begin{aligned} T(g_1, 1) &\longrightarrow (g_2, c_2), \\ T(2, 1) &\longrightarrow (4, c_2), \\ g_1 \cdot \frac{1}{c_2} &= g_2, \\ 2 \cdot \frac{1}{c_2} &= 4, \\ \frac{1}{c_2} &= \frac{4}{2}, \\ c_2 &= \frac{2}{4} = \frac{1}{2}, \\ T(2, 1) &\longrightarrow \left(4, \frac{1}{2}\right) \end{aligned}$	Fig. 40
Or: $\begin{aligned} T(1, c_1) &\longrightarrow (g_2, c_2), \\ T(1, 2) &\longrightarrow (4, c_2), \\ c_1 \cdot \frac{1}{g_2} &= c_2, \\ 2 \cdot \frac{1}{4} &= c_2, \\ c_2 &= \frac{2}{4} = \frac{1}{2}, \\ T(1, 2) &\longrightarrow \left(4, \frac{1}{2}\right) \end{aligned}$	Fig. 41

Table 3

As we can see, in every case the result of division, c_2, is equal to $\frac{1}{2}$, and in this respect we can say that $\frac{1}{2} = \frac{2}{4}$. But, as is also easy to notice, each of the transformations presented above is different, which indicates the distinctness of $\frac{1}{2}$ and $\frac{2}{4}$. A proposed explanation of this situation will be presented later in this work.

10. Division by zero as a transformation

Let us try to analyse $\frac{1}{0}$ as a transformation:

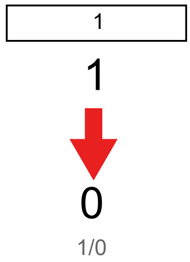
$\begin{aligned} T(1,c_1) &\longrightarrow (g_2,c_2), \\ T(1,1) &\longrightarrow (0,c_2), \\ c_1\cdot\frac{1}{g_2} &= c_2, \\ 1\cdot\frac{1}{0} &= c_2, \\ c_2 &= \frac{1}{0}, \\ T(1,1) &\longrightarrow \left(0,\frac{1}{0}\right) \end{aligned}$	 Fig. 42

Table 4

As can be seen, the result of the transformation $\frac{1}{0}$ is $c_2 = \frac{1}{0}$. What is the meaning of this result? It is something we are not very able to deal with: we cannot divide 1 into zero groups, and an attempt to divide $\frac{1}{0}$ actually leads us again to the symbol $\frac{1}{0}$. The question is whether we necessarily have to do this. Comparing what we obtained with the examples of $\frac{1}{2}$ or $\frac{2}{4}$ above, we can see that we can illustrate the transformation, but we have a problem illustrating the result. Perhaps the meaning of division is **NOT**, as contemporary mathematics understands it, actually the result c_2, but the transformation itself? One could say that we do not really have a problem with the symbol $\frac{1}{0}$; the problem appears when we begin to calculate it, when we want to reduce it to a number of the form 5, 7, or something similar. How to deal with $\frac{1}{0}$ will be explained later in this work.

11. Is $\frac{1}{2}$ equal to $\frac{1}{2}$?

It is not, if we think about two different transformations, and we can think about them as shown by the examples below. Let us begin with classical division, exactly as was already shown above. Treating $\frac{1}{2}$ as a division transformation, we obtain:

$\frac{1}{2}$ as division:

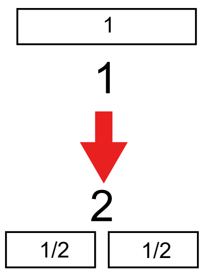
$\frac{1}{2}$ as **division**	
$\begin{aligned} T(1,c_1) &\longrightarrow (g_2,c_2), \\ T(1,1) &\longrightarrow (2,c_2), \\ c_1\cdot\frac{1}{g_2} &= c_2, \\ 1\cdot\frac{1}{2} &= c_2, \\ c_2 &= \frac{1}{2}, \\ T(1,1) &\longrightarrow \left(2,\frac{1}{2}\right) \end{aligned}$	 Fig. 43

Table 5

However, $\frac{1}{2}$ can also be understood as selection.

$\frac{1}{2}$ as selection:

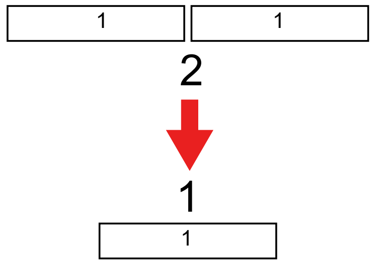
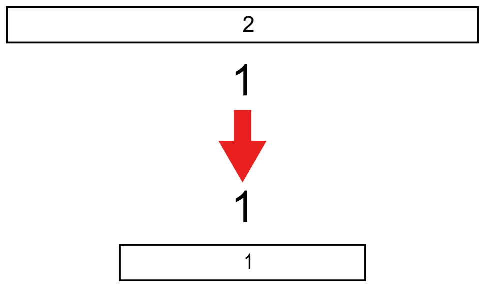
$\frac{1}{2}$ as **selection**	
$T(g_1, 1) \longrightarrow (g_2, 1)$, $T(2, 1) \longrightarrow (1, 1)$, $g_1 \cdot \frac{g_2}{g_1} = g_2$, $2 \cdot \frac{1}{2} = 1$	 Fig. 44
Or: $T(1, c_1) \longrightarrow (1, c_2)$, $T(1, 2) \longrightarrow (1, 1)$, $c_1 \cdot \frac{c_2}{c_1} = c_2$, $2 \cdot \frac{1}{2} = 1$	 Fig. 45

Table 6

Note that the ratio $\frac{1}{2}$ has the property that it transforms two initial groups g_1 into one target group g_2 (Fig. 44), or double initial cardinality c_1 into single final cardinality c_2 (Fig. 45).

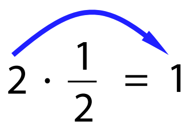


Fig. 46. Transformation by the ratio $\frac{1}{2}$

Is there a way to reconcile such different results and such different possibilities of interpretation between division and selection?

What is the same in both examples presented above is that the transformation coefficient p, the numerical ratio that performs the transformation, is in both cases equal to $\frac{1}{2}$.

$$T(1, c_1) \longrightarrow (g_2, c_2). \tag{78}$$

In division, the initial $c_1 = 1$ is transformed by $p = \frac{1}{2}$ into $c_2 = \frac{1}{2}$.

In selection, the initial $c_1 = 2$ is transformed by $p = \frac{1}{2}$ into $c_2 = 1$. It is the same in the case of transformation of cardinalities (Fig. 45).

$$1 \cdot \frac{1}{2} = \frac{1}{2}$$

Fig. 47. The ratio $\frac{1}{2}$ in division

The difference between division and selection in the examples above is that in division we read the fraction, as it were, from top to bottom. We divide one into two and obtain one half.

$$2 \cdot \frac{1}{2} = 1$$

Fig. 48. The ratio $\frac{1}{2}$ in selection

In selection we read the fraction from bottom to top. From two we select one; we select one half.

$$\frac{1}{2}\downarrow$$

Fig. 49. Reading the fraction from top to bottom (division)

$$\frac{1}{2}\uparrow$$

Fig. 50. Reading the fraction from bottom to top (selection)

In both cases we speak about a half: in the first, "we divide in half"; in the second, "we select half.

What is most important is the ratio, which is the same in every case: the ratio that we obtain each time between the result, the outcome of the transformation, and the beginning, the initial value from which we started the transformation.

$$\frac{\frac{1}{2}}{1} = \frac{1}{2} = \frac{1}{2}$$

Fig. 51. The same ratio between the result and the beginning of a transformation

To unify the understanding of $\frac{1}{2}$, this symbol should be interpreted as the ratio between two "states" of a certain transformation: the final state in the numerator and the initial state in the denominator.

12. What is the meaning of the numerator and denominator of a fraction performing a transformation?

The numerical ratio that is the "processing" element in a transformation converts the initial element into the result, the outcome of the transformation. We can say that the initial element of the transformation is what we start with, that is, something we have, something that exists. Therefore, in the rest of this work we will call it the **measure** (base measure). It denotes the initial state, what is given.

Next, the base measure (the initial element) is converted, transformed through the numerical ratio into the resulting, final element, the outcome of the transformation. This element denotes something we obtain in the transformation, the result of the conversion. It can also denote something we want to obtain, something we expect, something we anticipate as the result of the transformation. In the rest of this work we will call this element the **value** (resulting value), the result of the transformation, or the expected value.

transformation

$$\text{measure} \cdot \frac{\text{value}}{\text{measure}} = \text{value}$$

ratio

Fig. 52. Measure transformed by a numerical ratio into value

Two simple examples follow.

a) Division $\frac{3}{4}$ (dividing three into four).

Division $\frac{3}{4}$ denotes a transformation consisting in transforming three groups of one and dividing them into four groups of $\frac{3}{4}$.

transformation

$$1 \cdot \frac{3}{4} = \frac{3}{4}$$

ratio

Fig. 53. The ratio $\frac{3}{4}$ in division

b) Selection $\frac{3}{4}$ (selecting three out of four).

Selection $\frac{3}{4}$ denotes a transformation consisting in selecting three elements out of four elements. As can be seen, the numerical ratio performing such a transformation is also equal to $\frac{3}{4}$; it is the ratio between the resulting value (three) and the base measure (four).

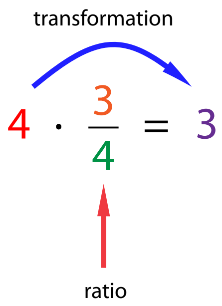


Fig. 54. The ratio $\frac{3}{4}$ in selection

As can be seen in both cases, the same numerical ratio is used, and it is equal to the ratio between the resulting value (the result of the transformation) and the base measure (the beginning of the transformation). In both cases these ratios are equal.

13. Numerical ratio as the natural form of a number

What number is represented by the point below?

Is this 5?

Or perhaps it is 7?

Or perhaps -2?

The number itself, without any reference point, has no meaning by itself. To understand its meaning, we need to know what is being used as the reference — in the familiar number-line convention, in particular, we need to know what is meant by 1.

When speaking about numbers, we always use some reference point, whether it is presented explicitly or only implicitly. By saying 5, what we normally mean is 5 in relation to 1. By saying -2, we likewise mean -2 in relation to 1. The reference does not always have to be 1; it can be another number, including zero. To express a number in this relational sense, we therefore need another number as its reference.

A number is always a certain value relative to a certain "base" measure, so the most natural form of a number is a fraction — a numerical ratio. It tells us what value a number has relative to something, that is, to the measure taken as the base.

The natural form of a number is its value relative to a certain measure.

$$\frac{\text{value}}{\text{measure}}$$

The measure denotes the base, fundamental, available number of elements or magnitude, whereas the value specifies the number of elements or magnitude that we select, that we want to obtain, the magnitude that the number defines or specifies in relation — "in ratio" — to its base measure. Even when we speak about whole numbers 2, 5, 7, and so on, we implicitly always relate them to the measure taken as the base, namely 1. Thus 2 is $\frac{2}{1}$ (twice the number 1), 5 is $\frac{5}{1}$ (five times the number 1), and so on. Even irrational numbers such as $\sqrt{2}$ likewise implicitly refer to the measure 1, and in fact $\sqrt{2}$ means $\frac{\sqrt{2}}{1}$. Although we usually do not show this explicitly, every number indicates a value by referring to a more implicit measure (in the case of non-fractions) or a more explicit one (in the case of fractions). In fractions the measure can be very different and does not always have to be one.

For example, $\frac{1}{2}$ denotes a numerical ratio.

a) In division, it is the ratio between the effect of division and the beginning of division, i.e. what was divided:

$$\frac{\text{effect of division (value)}}{\text{what was divided (measure)}}. \tag{79}$$

We divide one by 2:

$$1 : 2 = \frac{1}{2}. \tag{80}$$

The ratio between the result and the beginning is

$$\frac{\frac{1}{2}}{1} = \frac{1}{2}. \tag{81}$$

b) In selection, it is the ratio between the effect of selection and what we select from:

$$\frac{\text{effect of selection (value)}}{\text{what we select from (measure)}}. \tag{82}$$

In this case we select 1 out of 2:

$$\frac{1}{2}. \tag{83}$$

To illustrate the ideas presented in this work, let us represent numbers by treating them as a fraction, the numerical ratio $\frac{\text{value}}{\text{measure}}$. Every number can be represented in this form by marking the measure on the horizontal axis and the value on the vertical axis. Particular numbers of the form $(\text{value}, \text{measure})$ correspond to points with coordinates $(\text{measure}, \text{value})$. The graph below presents the number $\frac{2}{3}$ (value $= 2$ relative to measure $= 3$), the point with coordinates $(3, 2)$.

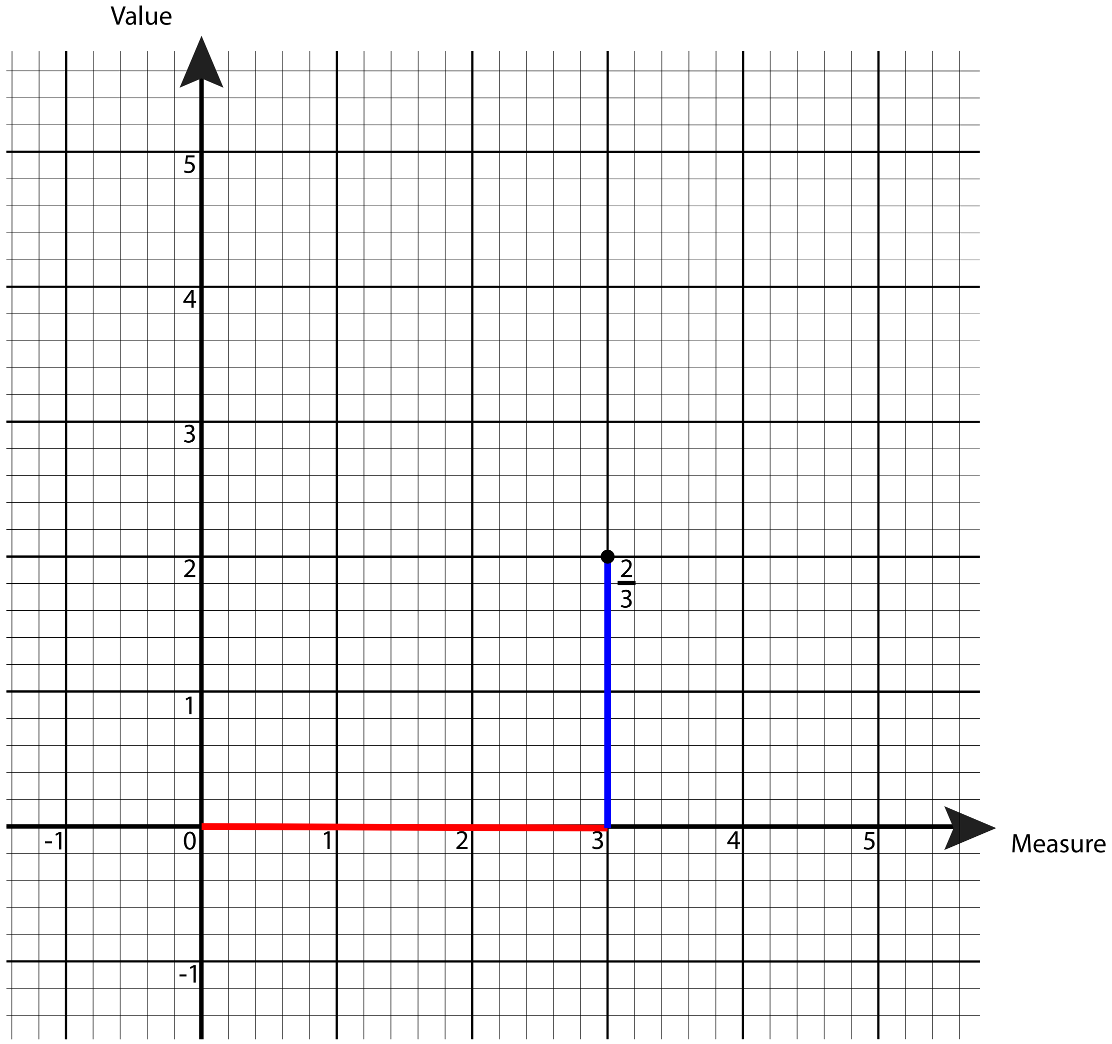


Fig. 55. A number represented as a point (measure, value)

Note: All graphs later in this work, for easier presentation and discussion, usually limit themselves to presenting the ideas only in the range in which both value and measure are greater than or equal to zero. The ideas presented, however, are fully applicable over the entire plane and can also be freely presented in the range below zero.

14. Definition of the set of n-fold relations

Let us call the **set of n-fold relations** (or, more simply, the **n-fold relation**) the set of all numerical ratios equal to $\frac{n}{1}$. For each n-fold relation let us define

$$k_n = \frac{n}{1}.$$

Thus, for example:

- the **half-fold relation** ($\frac{1}{2}$-fold relation) consists of all ratios equal to $\frac{1}{2}$, for example $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{50}{100}, \ldots$;
- the **one-fold relation**: $\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \frac{100}{100}, \ldots$;
- the **two-fold relation**: $\frac{2}{1}, \frac{4}{2}, \frac{6}{3}, \frac{8}{4}, \frac{200}{100}, \ldots$;
- the **three-fold relation**: $\frac{3}{1}, \frac{6}{2}, \frac{9}{3}, \frac{12}{4}, \frac{300}{100}, \ldots$; and so on.

Below is the graph of the half-fold relation, with points satisfying the ratio $\frac{1}{2}$ and the point $k_{1/2}$ marked. The graph shows that fractions (ratios) such as $\frac{1}{2}$ and $\frac{2}{4}$ represent the half-fold relation.

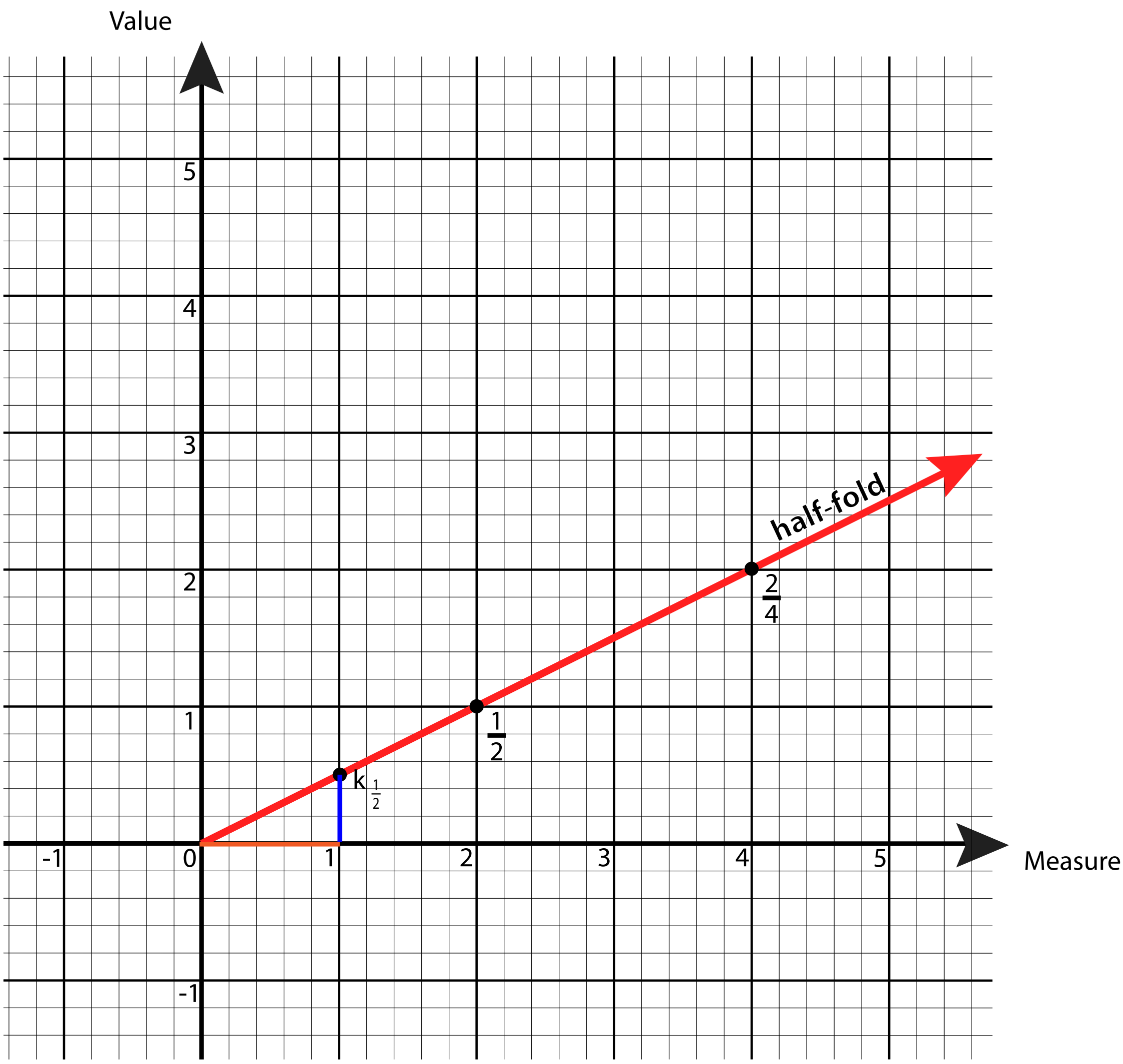


Fig. 56. The half-fold relation and selected ratios belonging to it

The next graph presents several additional multiplicity relations with points marked for each of them.

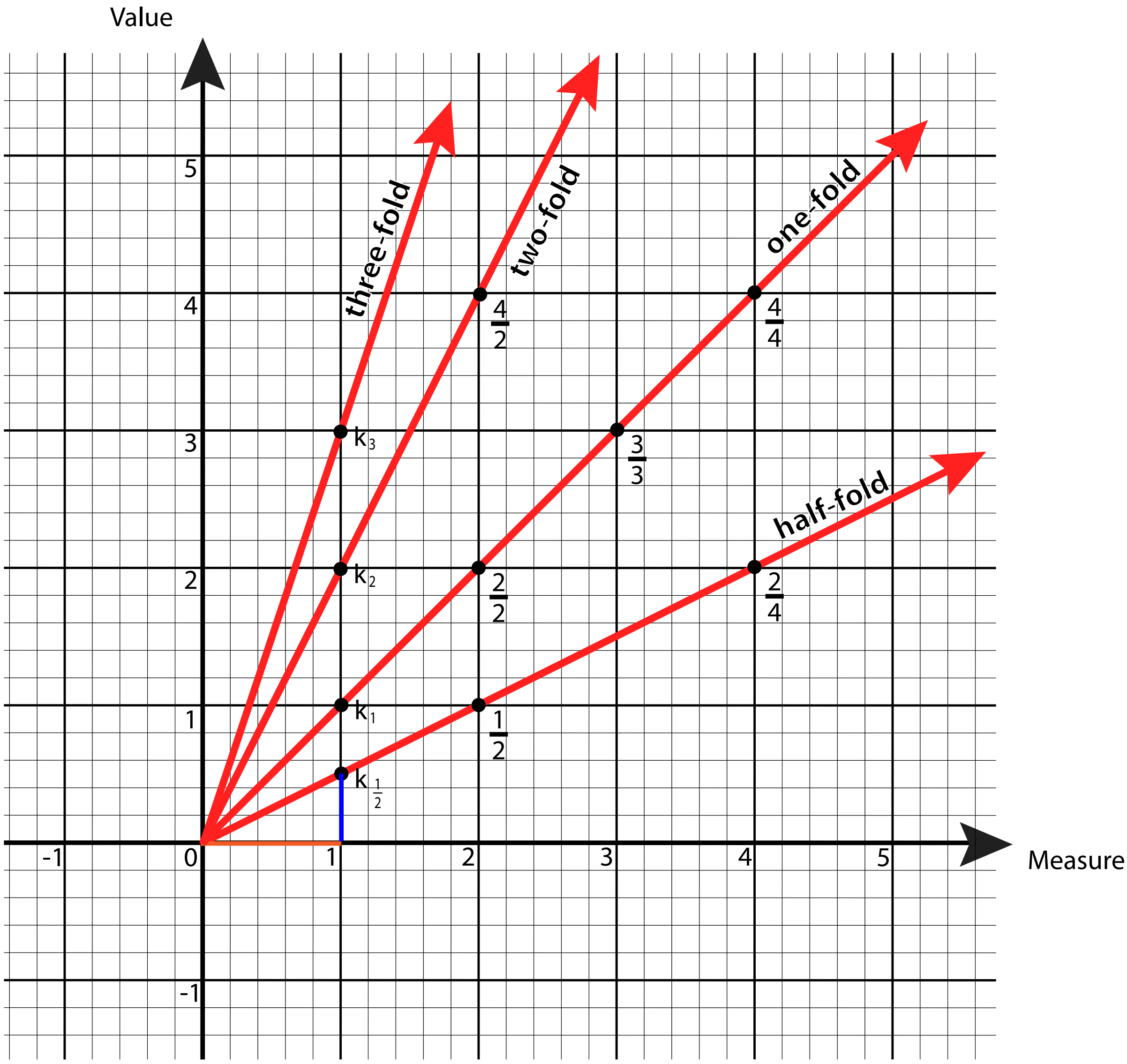


Fig. 57. Several n-fold relations

15. Interpretation of a point on the graph of numerical ratios

Every point on the graph represents a particular number in its true fractional form, that is, as the ratio of its $\frac{\text{value}}{\text{measure}}$. This ratio can also be understood as a transforming element, for example in the division operation or the selection operation.

Thus, in the division operation, the ratio $\frac{2}{3}$ can be represented as a transformation, and likewise in the selection operation.

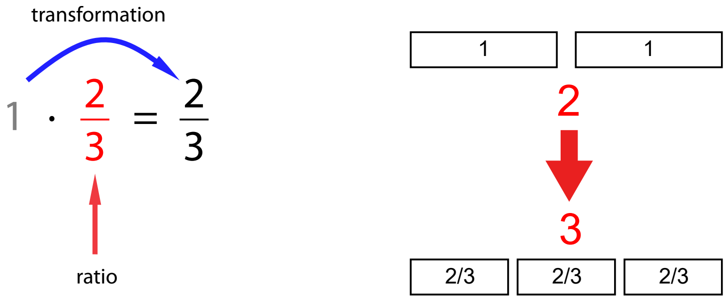


Fig. 58. The ratio $\frac{2}{3}$ used in division

Or in the selection operation:

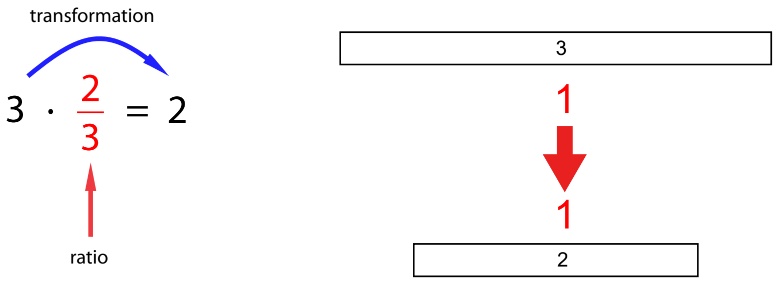


Fig. 59. The ratio $\frac{2}{3}$ used in selection

Note that the ratio between the effect (end) of the transformation and the beginning of the transformation is the same in both cases:

$$\text{division: } \frac{\frac{2}{3}}{1} = \frac{2}{3}, \qquad \text{selection: } \frac{2}{3} = \frac{2}{3}. \tag{84}$$

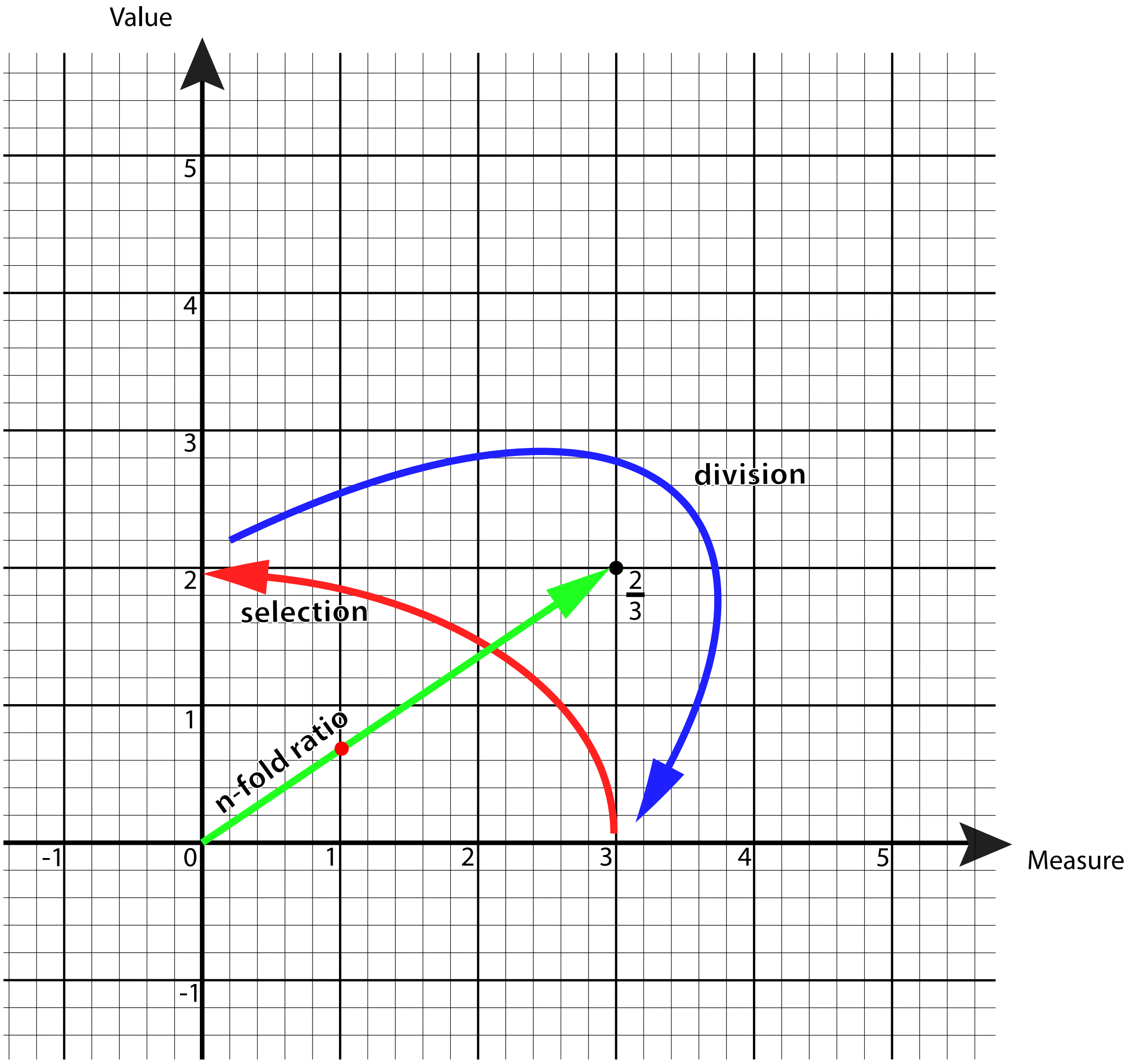


Fig. 60. Four interpretations of a point representing the ratio $\frac{2}{3}$

Looking at the figure above, it is easy to see that $\frac{2}{3}$ can be interpreted in four different ways:

1. as division, in this case 2 divided by 3, and more precisely as the numerical ratio that is the transforming element of the division operation, shown by the blue arrow;

2. as selection, in this case selecting 2 out of 3, and more precisely as the numerical ratio that is the transforming element of the selection operation, shown by the red arrow;

3. as an element of a certain set of n-fold relations, which can be projected onto the point $k_n = \frac{n}{1}$ (the red point). This set is represented by the green axis connecting all points belonging to that set of n-fold relations;

4. as a point, that is, a particular numerical ratio between value and measure, represented by the black point, which is the most correct and accurate representation of the number.

16. The unreality of the real-number axis

The commonly accepted interpretation of numbers and fractions limits their interpretation only to the multiplicity relations associated with them, completely ignoring their diversity and true meaning.

It should be noted that all points $k_n = \frac{n}{1}$ representing individual n-fold relations lie on one line. This line is the real-number axis. By equating all numerical ratios (fractions) belonging to the same multiplicity relation, contemporary mathematics reduces them to points of the form $\frac{n}{1}$. Thus, for example,

$$\begin{aligned}
\cdots = \frac{3}{6} = \frac{2}{4} = \frac{1}{2} = \frac{\frac{1}{2}}{1} = \frac{1}{2},\\
\cdots = \frac{4}{4} = \frac{3}{3} = \frac{2}{2} = \frac{1}{1} = 1,\\
\cdots = \frac{6}{3} = \frac{4}{2} = \frac{2}{1} = 2,\\
\cdots = \frac{9}{3} = \frac{6}{2} = \frac{3}{1} = 3.
\end{aligned}$$

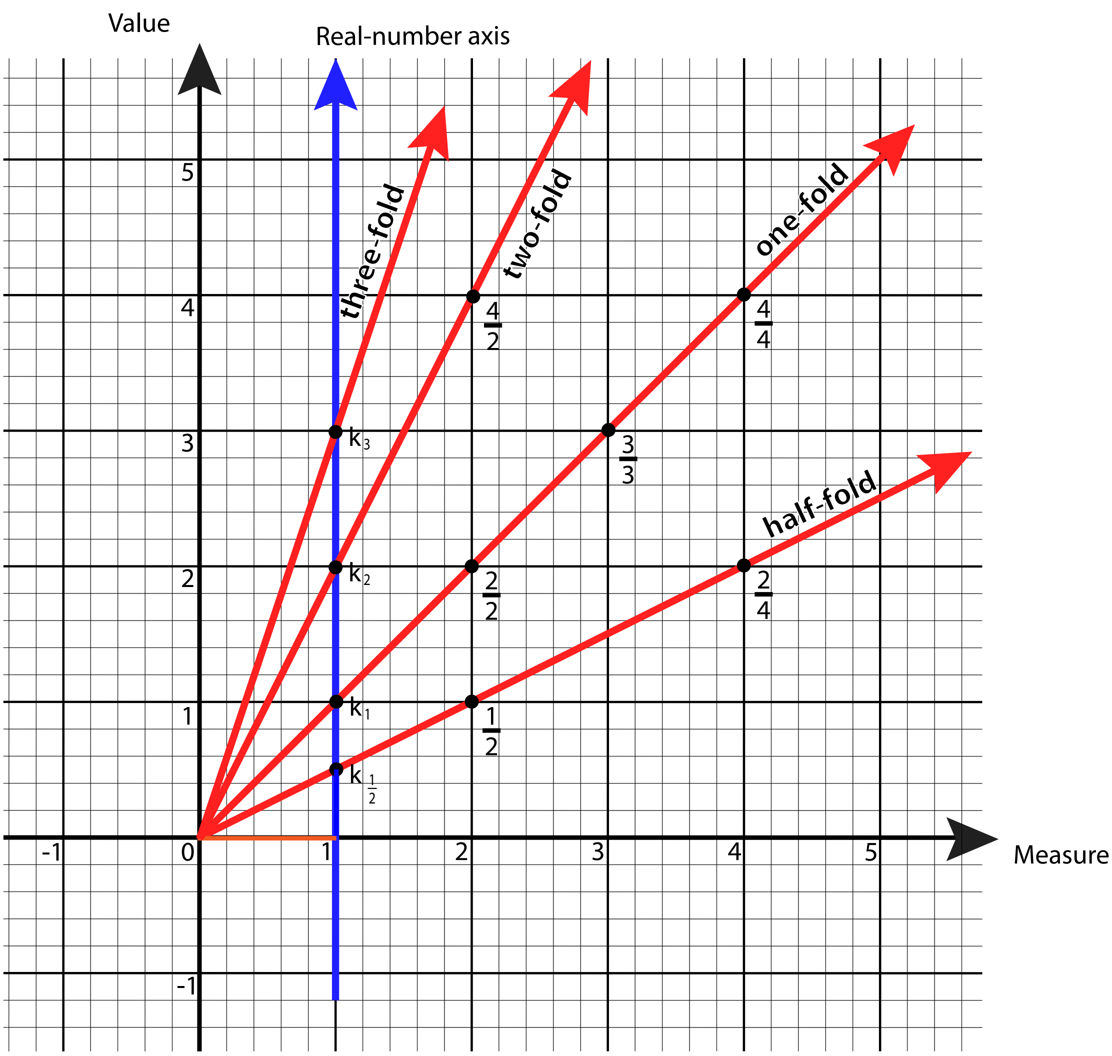


Fig. 61. Projection of multiplicity relations onto the classical real-number axis

In the figure above, the classically understood real-number axis is shown in blue. All points lying on this axis refer to measure $= 1$. This classical understanding causes the 1 in the notation $\frac{x}{1}$ usually to be omitted, and instead of writing $\frac{x}{1}$ we simply write x. It is also easy to notice that what we previously defined as multiplicity relations is nothing other than the projection of individual points onto the real-number axis, made from the perspective of zero. Thus, for example, the point denoting $\frac{2}{1}$ (called simply 2) really denotes value 2 relative to measure 1 and represents the two-fold relation and all fractions associated with it projected onto the real-number axis, such as $\frac{2}{1}, \frac{4}{2}, \frac{6}{3}, \ldots$.

If we do not treat all numbers correctly as fractions of the form $\frac{\text{value}}{\text{measure}}$, as shown in the figure above, we make an unjustified simplification by reducing their meaning only to real numbers of the form $\frac{n}{1}$. Then all fractions lying on one multiplicity relation are reduced to a single point on the real-number axis. Although each of these fractions gives the same result, each also contains additional information: it tells us either about dividing one thing by another, selecting one thing from another, or simply about a ratio — in each case between different numbers, even though their proportions are equal in the result. This means that when we say, for example, that $\frac{1}{2} = \frac{2}{4}$, we omit the fact that in each case we are talking about dividing one number by another; we omit the entire transformation performed in this way and think only about its result.

This situation resembles looking at the world through a pane of glass. It is as if we believed that everything we see through the pane (the real-number axis) is located on the pane, although in fact it lies behind it. For example, a mountain far away may appear to have the same size as a house nearby, just as one fraction can appear to be the same as another because on our "pane" — the real-number axis — both fractions are located at the same point.

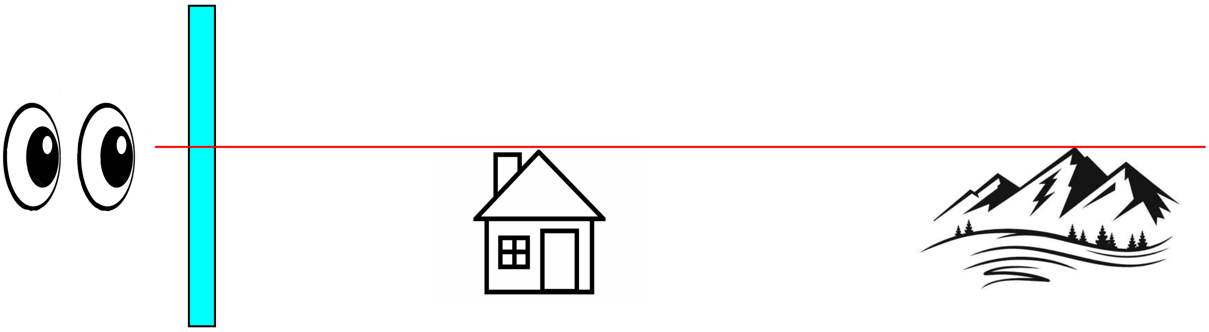


Fig. 62. Perspective analogy for the projection onto the real-number axis

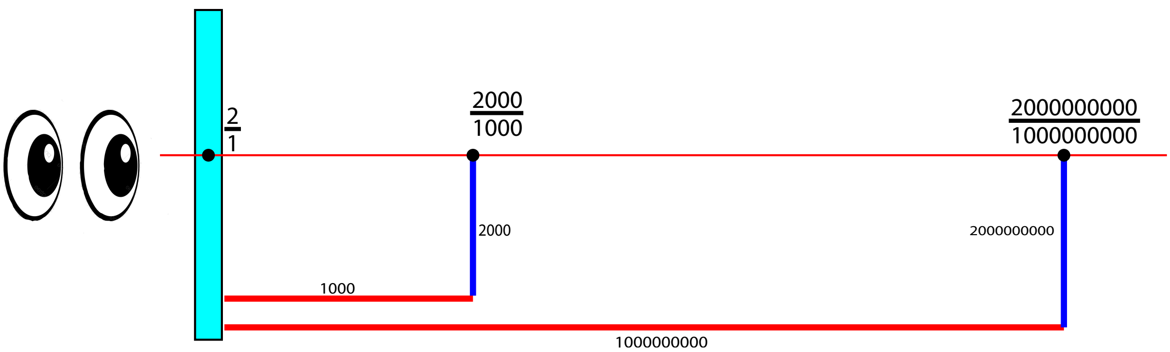


Fig. 63. Perspective construction of the projection

It follows from the reasoning above that the real-number axis is not so "real" after all. It does not "contain" all possible numbers, but only certain representations of them. This is also the reason why we have such great problems understanding "division by zero." All numbers of the form $\frac{x}{0}$ have no representation on the real-number axis (85). The axis determined by all such numbers does not intersect the real-number axis because it is parallel to it, as illustrated later in Fig. 66.

$$\frac{x}{0} \neq \frac{?}{1}. \tag{85}$$

17. Problems with terminology

In view of the explanations proposed, it seems that there is a conflict in terminology between the nomenclature accepted in mathematics and that proposed in this work. The first problem concerns "rational numbers." A definition from Wolfram MathWorld states:

> "A rational number is a number that can be expressed as a fraction p/q, where p and q are integers and $q \neq 0$."

The definition presented is limited to integers for p and q and does not allow q to equal 0. These are artificially imposed restrictions and should be removed. Since the word "ratio" does not mean only the numerical ratio expressed by a fraction, but can also be understood as logical, intelligent, sensible, and rational, it would be good to use it to denote numbers in the broadest, fullest meaning. I propose extending its meaning and at the same time simplifying it by introducing:

Rational number — any number in the form of a fraction $\frac{\textbf{value}}{\textbf{measure}}$.

I see no specific reason why the denominator could not be zero. Nor do I see a reason why the numerator and denominator would have to be integers. The new definition would allow all numbers to be represented by the ratio $\frac{\text{value}}{\text{measure}}$ and to be called rational numbers. Likewise, all points on the graph representing relations between value and measure could then be called rational numbers. Continuing this logic, real numbers should be defined as follows:

Real number — any rational number that has measure $= 1$, represented by the fraction $\frac{\textbf{value}}{1}$.

The consequence of the definitions above would be that the set of rational numbers would be larger than, and would contain as a subset, the set of real numbers.

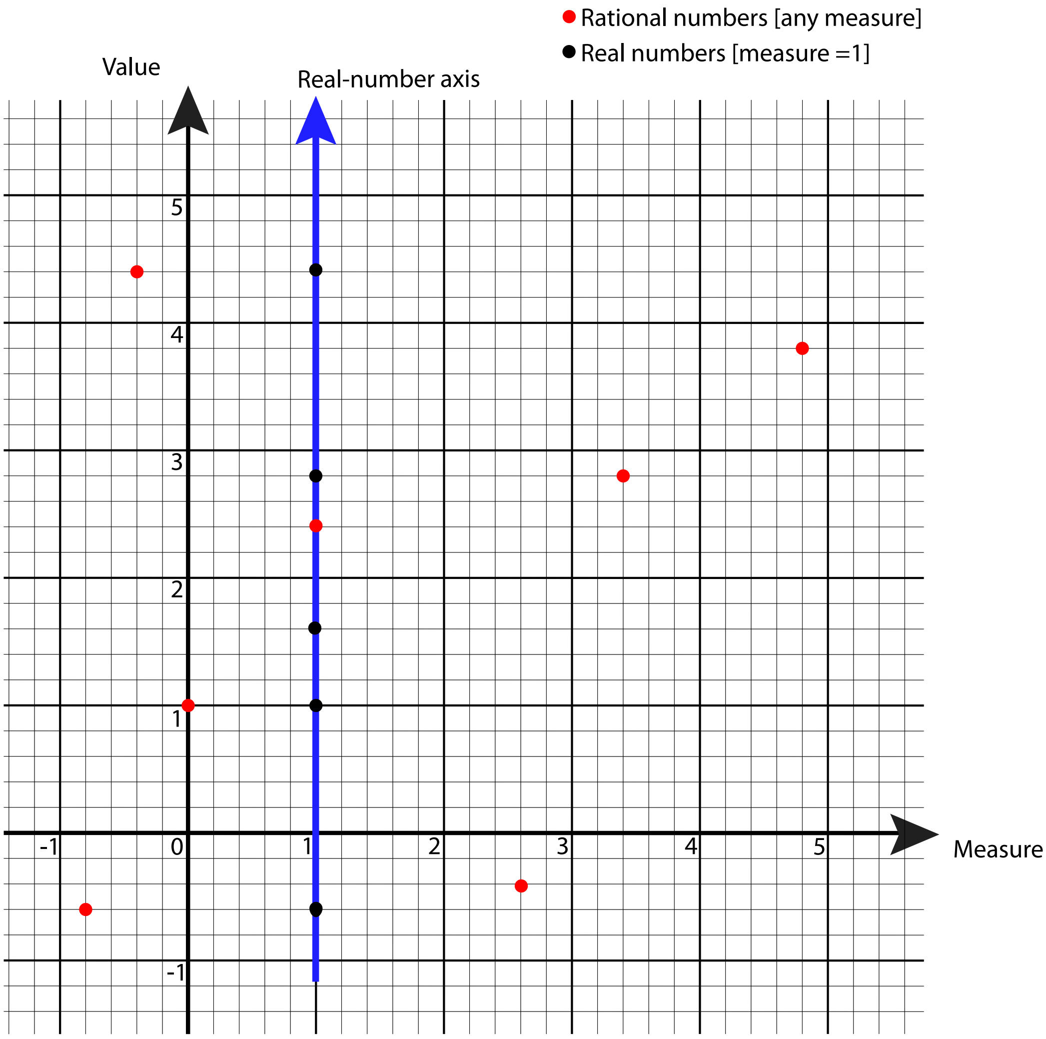


Fig. 64. Proposed relation between the sets of rational and real numbers

18. Problems with numbers

The projection described above of the entire plane of rational numbers, understood as $\frac{\text{value}}{\text{measure}}$ — value relative to a certain "base" measure — onto the standard real-number axis $\frac{\text{value}}{1}$ (Fig. 61), and treating all numbers as always relative to measure 1, has a whole series of consequences.

a) It was incorrectly assumed that rational numbers lying on one "multiplicity line" (for example $\frac{1}{2}$ and $\frac{2}{4}$) are equal, although in fact, understood in their full meaning, they denote completely different things and have only one feature in common: multiplicity, i.e. their projection onto the real-number axis (Fig. 61, (86)). It is enough to think through their true meaning to understand the differences between them.

$$\frac{1}{2} \longrightarrow \frac{\frac{1}{2}}{1} = \frac{\frac{1}{2}}{1} \longleftarrow \frac{2}{4}. \tag{86}$$

b) All fractional numbers of the form $\frac{0}{x}$ (lying on the zero-fold line) were reduced to the single number $\frac{0}{1}$, or more simply 0, which also distorts their true image and true meaning. Trying to understand the meaning of numbers of the form $\frac{0}{x}$, this is easy to grasp because they simply represent a specific ratio: a ratio between two states of a transformation. In the first, initial state, x elements are available, and we simply transform it into another, final state in which there are zero such elements — none at all. It can also mean that we do not want, do not expect, or do not select any of them. But they were initially available, and by saying simply that in the result there are zero of them, we lose this information about the initial state. Even if at the end we obtain zero, at the beginning there were x of them. There is undoubtedly a difference in context if we select zero elements out of, for example, two, and zero out of a million, although the result is the same: we selected zero, i.e. we selected nothing.

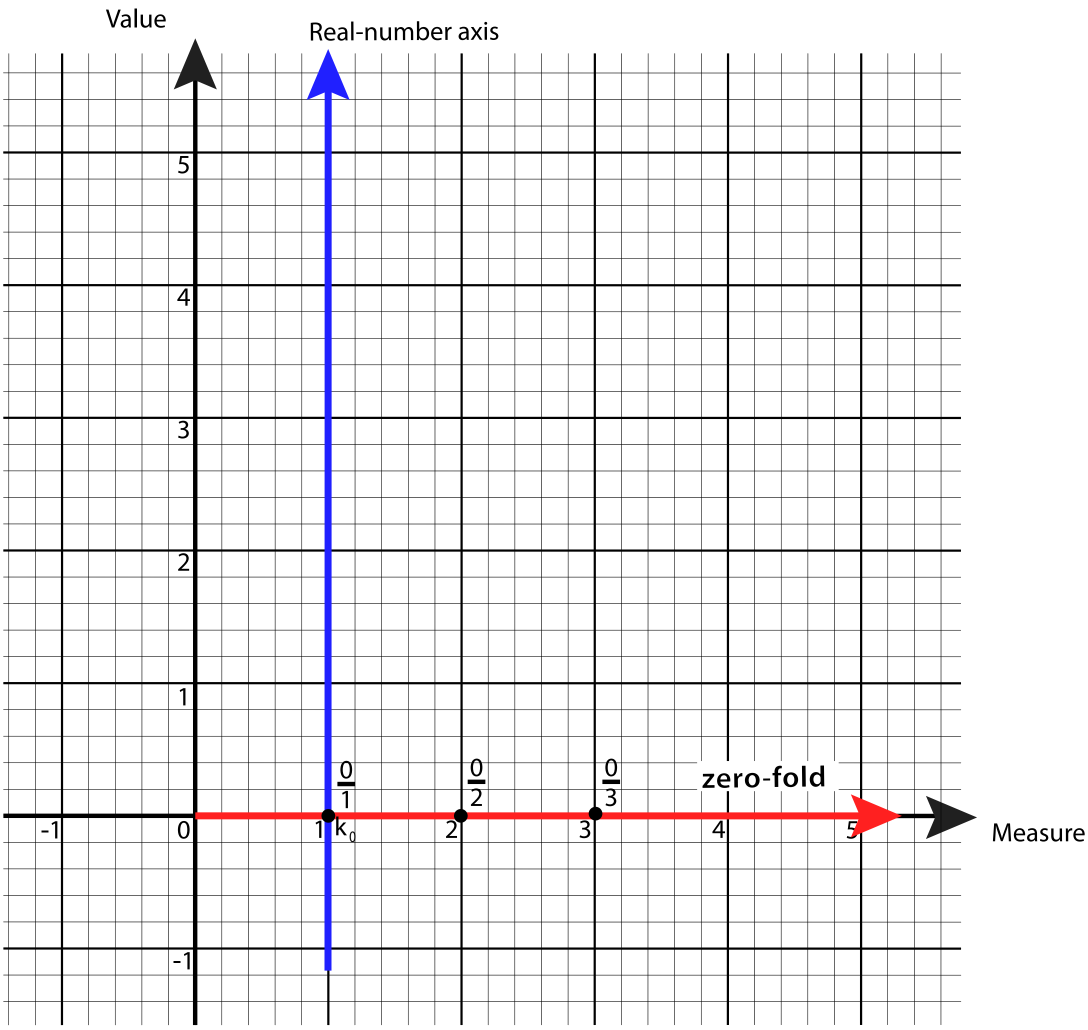


Fig. 65. Distinct ratios of the form $\frac{0}{x}$ projected to zero

c) Division of numbers by zero was artificially prohibited. The reason was an inability to understand that numbers of the form $\frac{x}{0}$ are completely ordinary rational numbers, that is, numbers in their natural form $\frac{\text{value}}{\text{measure}}$, representing value $= x$ relative to base measure $= 0$. The meaning of such numbers is that they are a ratio between two states of a certain transformation: the final state represented by the value (in this case x) and the initial state represented by the measure (in this case 0). It is just as if we wanted to obtain, expected, or wanted to select x elements, while the number of available elements is simply zero. This does not change the fact that we would like to obtain x such elements.

Division by zero was prohibited because the axis formed by all numbers of the form $\frac{x}{0}$ never intersects the real-number axis and could not be projected onto that axis in the way done with all the other rational numbers. Those other numbers were equated within the same multiplicity relation and reduced to the general form $\frac{n}{1}$, their representation on the real-number axis. In the interpretation presented here, there is no problem in indicating a number such as $\frac{1}{0}$, or even in stating that it is something different from $\frac{2}{0}$.

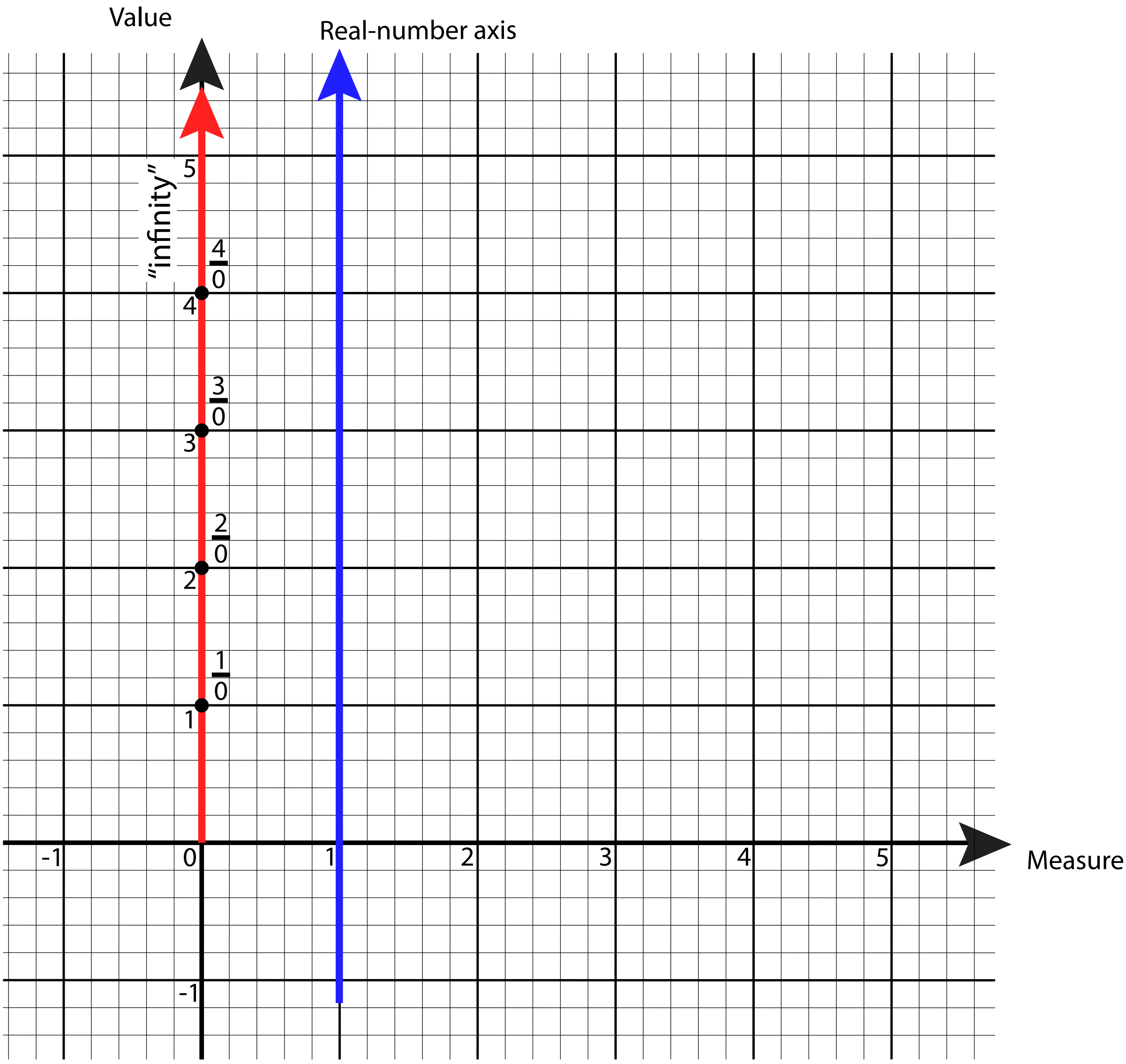


Fig. 66. The line of ratios with measure zero and its relation to the real-number plane

19. Other consequences of perceiving fractional numbers through their multiplicity

Perceiving numbers through the prism of their multiplicity is very deeply rooted in us. Its effect (and perhaps its cause) is the fact that we feel quantitative changes most strongly near the one-fold relation. That is, the difference between a one-fold and a two-fold relation is relatively large and noticeable to us, whereas the difference between a thousand-fold and a thousand-and-one-fold relation is not, although in reality the two differences are equally large.

An everyday example: whether someone has one car or two seems to make a relatively large difference, whereas whether someone has a thousand cars or a thousand and one seems not to make such a difference, although in fact the difference in both cases is equally large — one car. The graph below shows approximately what the problem consists in.

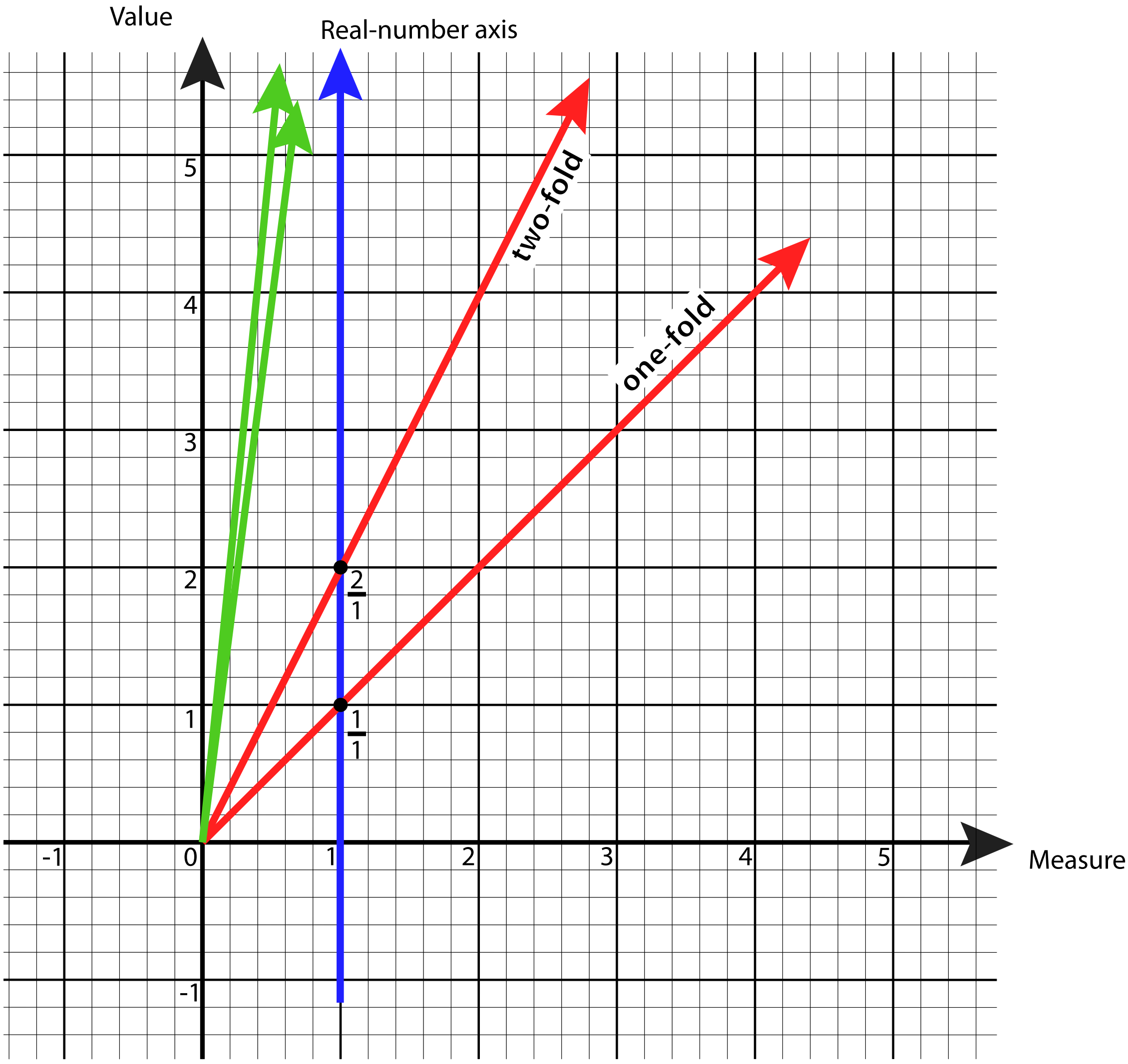


Fig. 67. Perceived differences between multiplicity relations

The red lines show how we perceive the difference in multiplicity between one-fold and two-fold quantities (for example, one car and two cars): the angle between these lines is large. The green lines show how we perceive the difference in multiplicity between a thousand-fold and a thousand-and-one-fold quantity (that is, 1000 cars and 1001 cars): the angle between the green lines is small. In reality, in both cases the difference is identical and equals one (one car).

20. Graphical representation of the function $f(x) = \frac{1}{x}$ for real and rational numbers

The classical representation of the function $f(x) = \frac{1}{x}$, in the form we all know well from school, is presented in the graph in Fig. 68. It has a point of discontinuity for $x = 0$. This representation is limited only to real numbers.

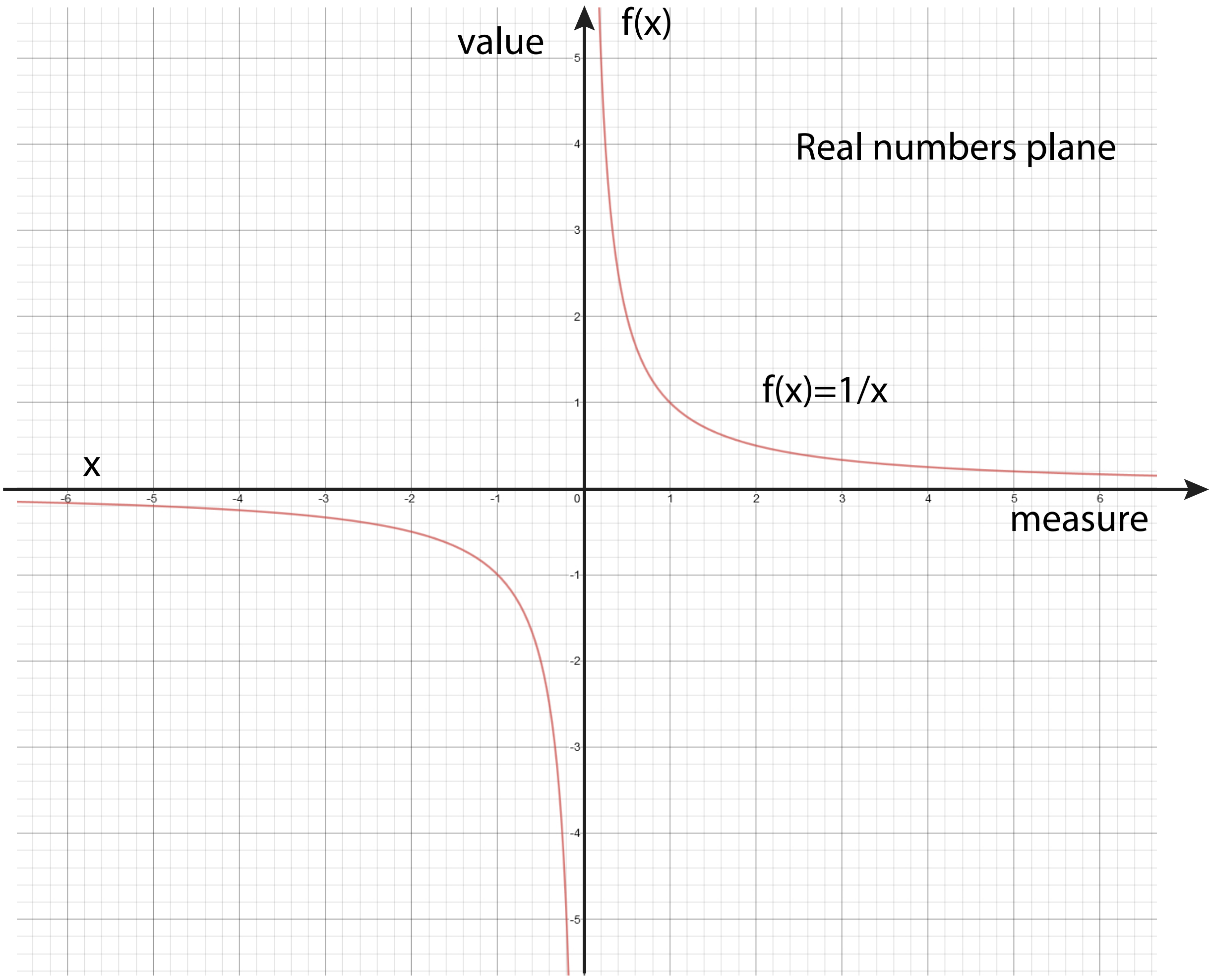


Fig. 68. Classical real-number representation of $f(x) = \frac{1}{x}$

In the following graphs we see a representation of the same function in the domain of rational numbers. This means that the value of the function for each x will be represented by the ratio $\frac{\text{value}}{\text{measure}}$; in this case it will be $\frac{1}{x}$. For better understanding, exactly the same three-dimensional graph is presented from different perspectives. In rational numbers the graph of the function $f(x) = \frac{1}{x}$ is represented by the straight line marked in blue. All the lines in many different colours represent different n-fold relations and map individual points from the graph of our function onto the real plane (measure $= 1$).

Note that the only point that has no representation on the real plane is the point for $x = 0$. The line passing through that point is parallel to the real plane and does not intersect it.

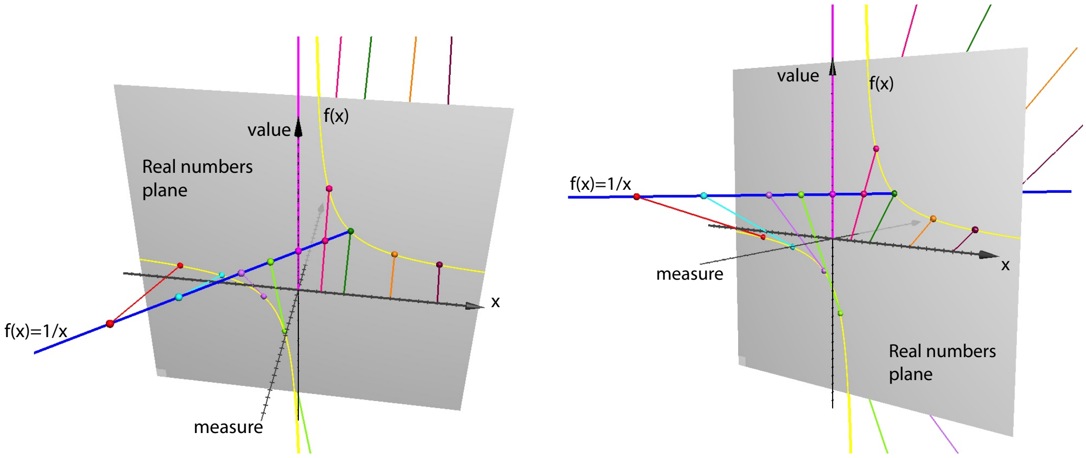


Fig. 69. Rational-number representation of $f(x) = \frac{1}{x}$

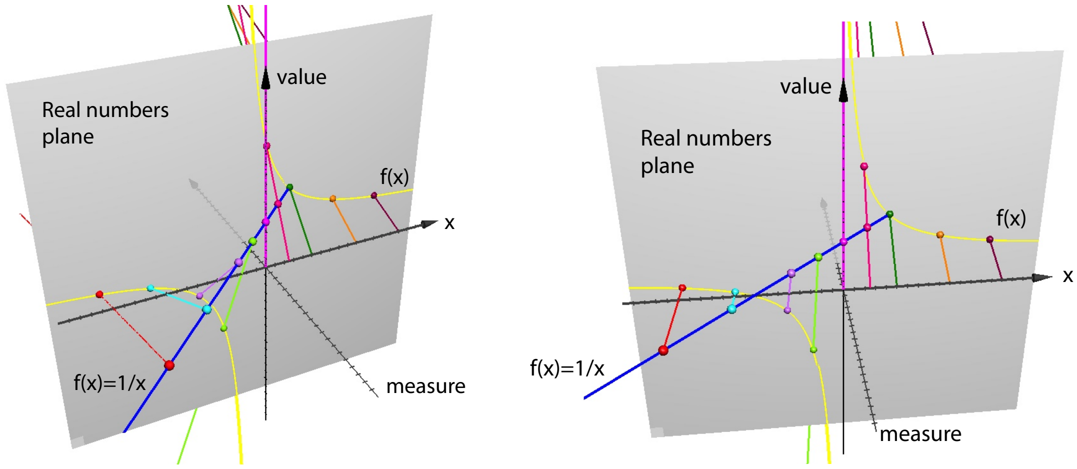


Fig. 70. The same graph from another perspective

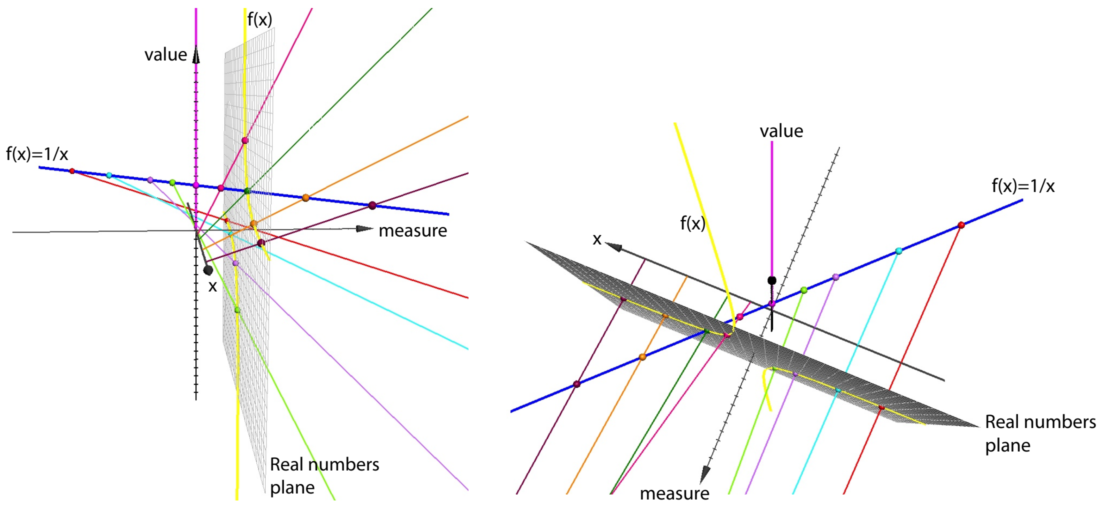


Fig. 71. The same graph from another perspective

From the graphs shown we can conclude that the graph of the function $f(x) = \frac{1}{x}$ is a straight line in the domain of rational numbers, existing for every x. Mapping this function onto real numbers (measure $= 1$) distorts its image into the form known from school, with one point of discontinuity at $x = 0$. The point of discontinuity in our mapping arises because we are unable to map the point for $x = 0$ from our graph onto the real plane.

21. Graphical representation of the function $f(x) = \mathrm{tg}(x)$ in real and rational numbers

A situation similar to that in the previous example occurs for the function $f(x) = \mathrm{tg}(x)$. The traditional graph of this function is presented below. It shows what the function $\mathrm{tg}(x)$ looks like in the domain of real numbers.

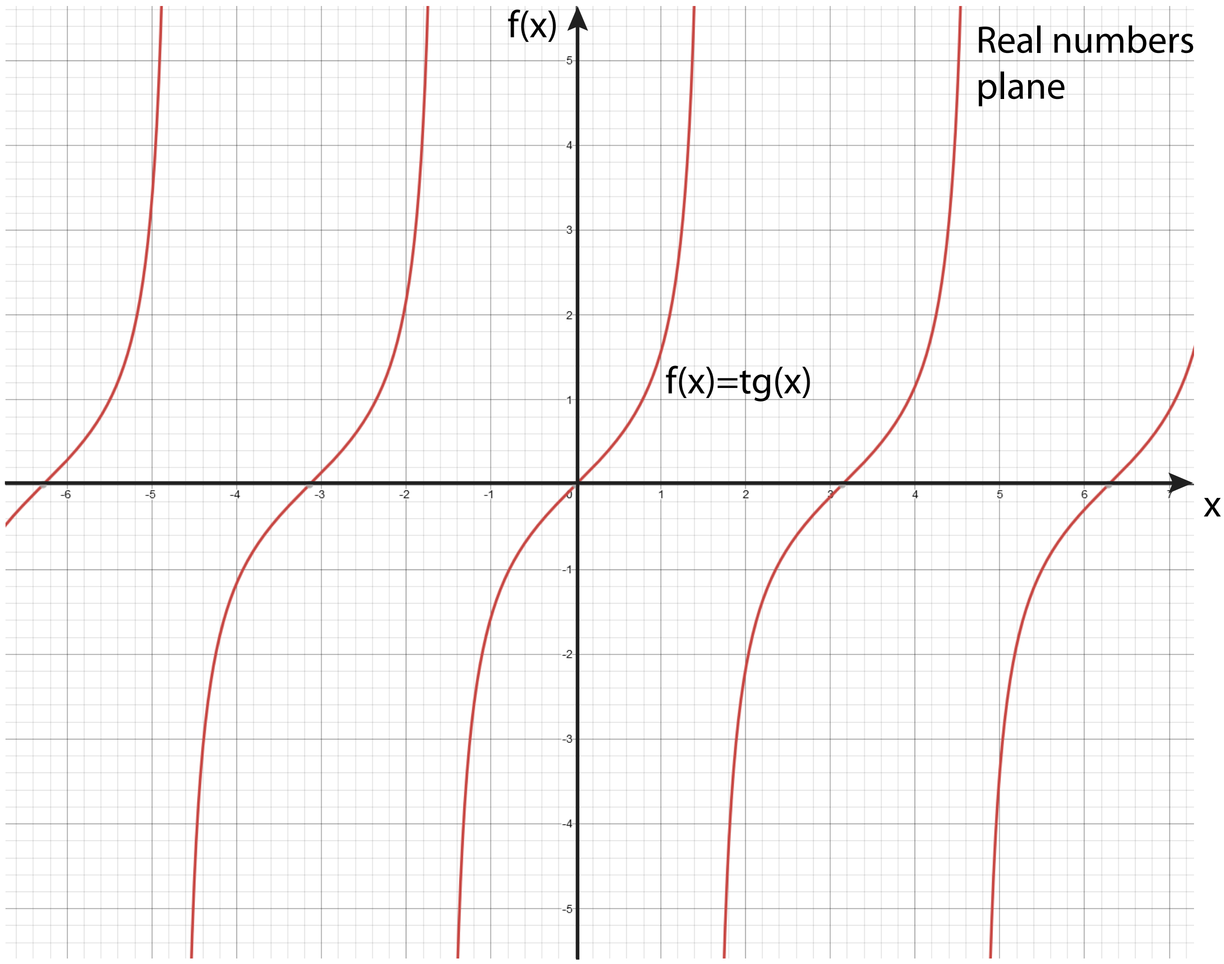


Fig. 72. Traditional real-number graph of $f(x) = \mathrm{tg}(x)$

Let us check what this function looks like in the domain of rational numbers. In the next several graphs its graph is shown from different perspectives. The function $f(x) = \mathrm{tg}(x)$ is represented by the blue spiral. All coloured lines are individual n-fold relations that map points from the graph of our function onto the real plane (measure $= 1$). Here again, for values of x for which this mapping is not possible, points of discontinuity appear on the real plane.

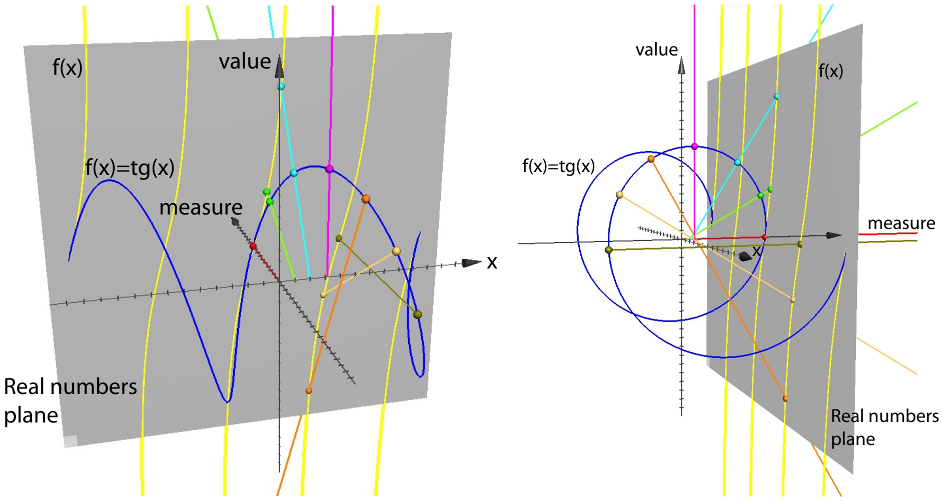


Fig. 73. Rational-number representation of $f(x) = \mathrm{tg}(x)$

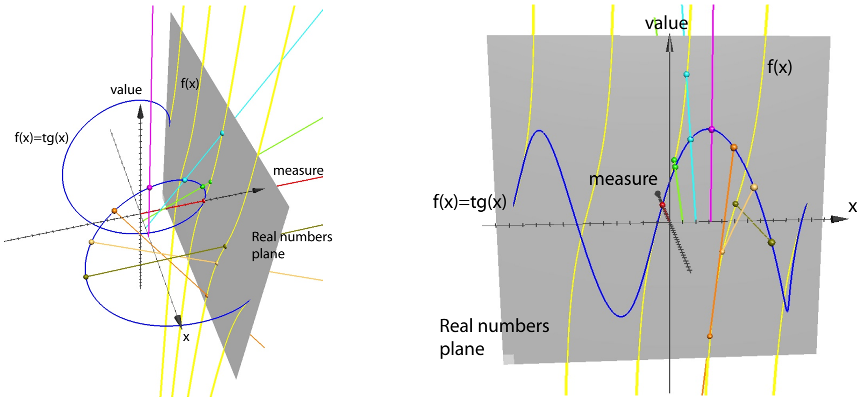


Fig. 74. The same graph from another perspective

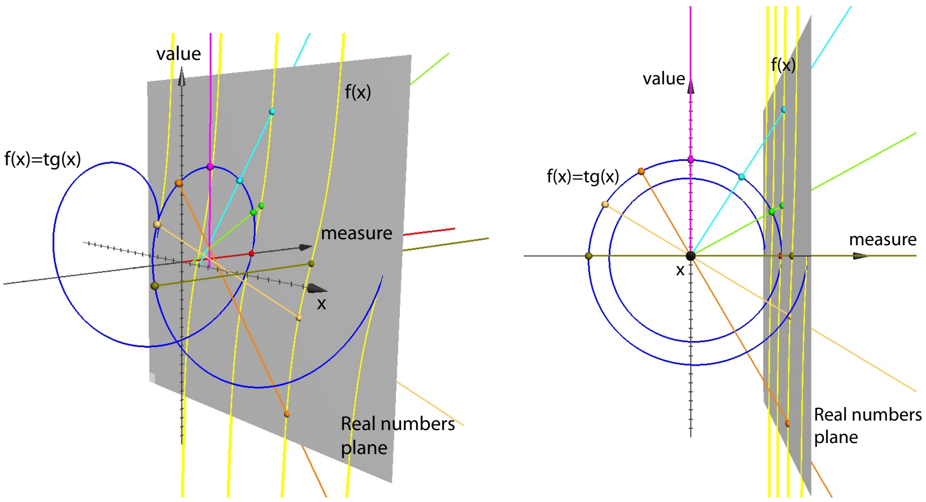


Fig. 75. The same graph from another perspective

22. Image of the world from the perspective of fractional numbers

If we look at all numbers through their true fractional, rational nature — that they are always in the form of the numerical ratio $\frac{\text{value}}{\text{measure}}$, or, in other words, the ratio between value and measure — we can represent this graphically and obtain certain new elements and relationships.

Note that a number represented as a fraction, the ratio of its value to a certain base measure, can also be understood as the ratio between the effect of a transformation and its beginning.

Value can also be interpreted as the end of the transformation, the final effect, what we want to obtain as the effect of the transformation, what we select, what we expect, what we get, what is at the end, and so on.

Whereas **measure** can also be interpreted as what we start from, the beginning of the transformation, the converted element, the transformed element, what we select from, what is given, what is at the beginning.

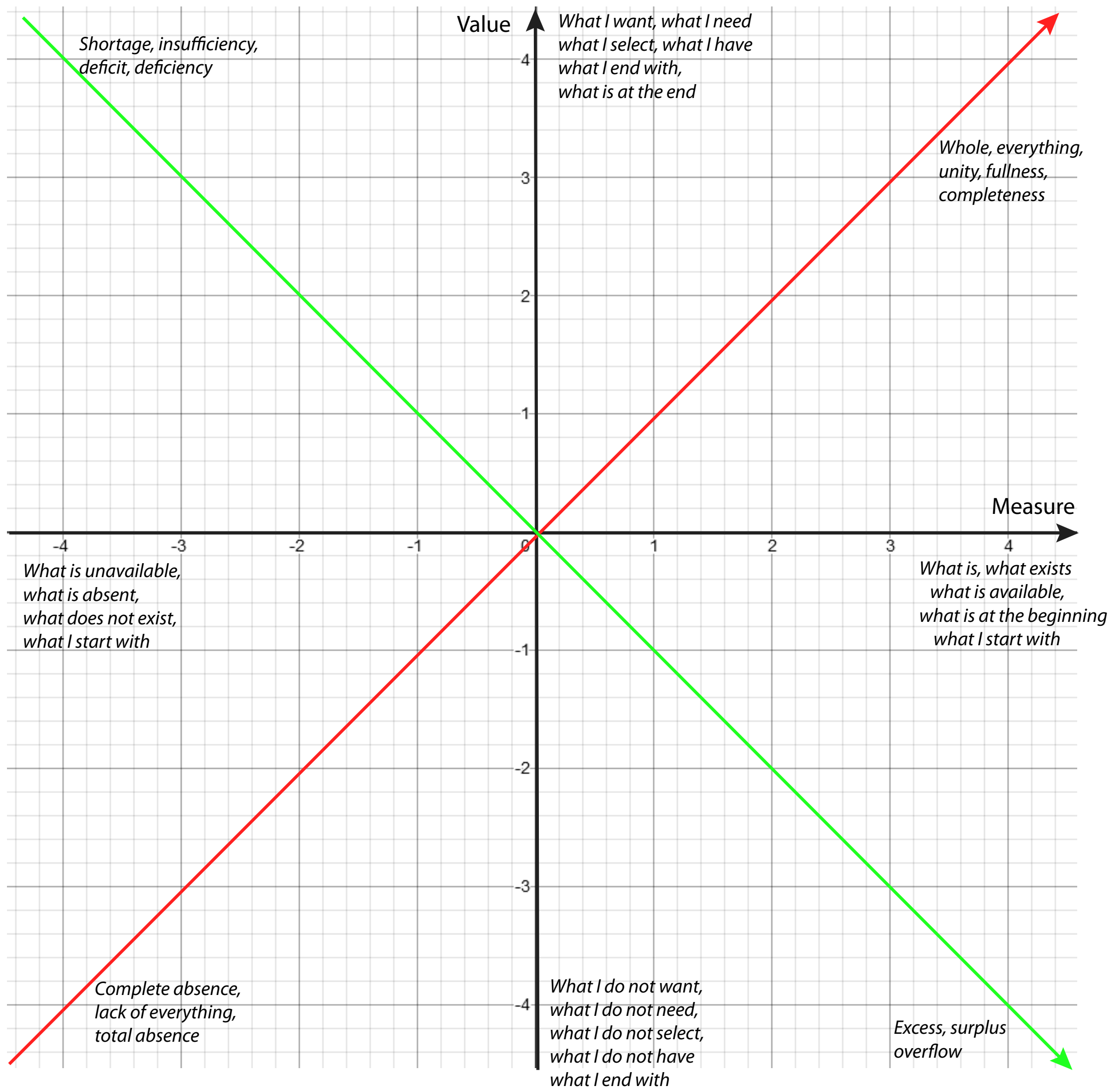


Fig. 76. The value–measure plane and interpretations of transformation states

23. Summary

Transformation is an operation that converts a pair of numbers into another pair of numbers. Division and multiplication are only certain special forms of transformation. Selection is another form of transformation.

A transformation can be total, as in the case of multiplication and division, or non-total, as in the case of selection. In every transformation we can identify a ratio between the result of that transformation and the element that undergoes the transformation. This ratio is also the element that performs the transformation. It converts the initial element of the transformation into the resulting element.

Every number in its most natural form is a numerical ratio between a certain value and a certain base measure to which that value refers. Contemporary mathematics makes an unjustified simplification by reducing all rational numbers (that is, numbers represented by the ratio of value to measure) to a fraction with denominator 1. In this way, the entire richness of rational numbers is flattened onto the real-number axis. This causes rational numbers representing completely different transformations to be equated with one another, losing their true meaning.

It also results in contemporary mathematics being unable to deal with division by zero, because the corresponding numbers of the form $\frac{x}{0}$ have no representation on the real-number axis and cannot be reduced to a fraction with denominator 1. Rational numbers of the form $\frac{x}{0}$ tell us about the numerical ratio between x and 0, about a transformation that begins at 0 and ends at x. We can say that they speak about "creation" or "expectation." Numbers of the form $\frac{0}{x}$, on the other hand, represent a transformation that ends at zero and begins at x. We can say that they speak about "annihilation" or a "lack of need."

24. Conclusion

The understanding of numbers, their relations, transformations, division, multiplication, and selection presented here brings with it a whole series of consequences and opens completely new possibilities for understanding mathematics, as well as having a significant impact on the understanding of physics and other related fields of science. This work makes it possible to understand the issue of division by zero and, through this, leads us to a completely new way of understanding numbers. I think it puts in order one of the foundations of mathematics, but because this foundation was previously defective at this point, it is difficult to predict all the consequences that this change may have for everything that has been built upon that foundation.

"This is not the end, it is not even the beginning of the end.
But it is, perhaps, the end of the beginning."

Winston Churchill