Lemma 1 If p is a prime number, then $p^2 \nmid p!$.
Proof.
Since $(p-1)! \equiv -1 \bmod p$ and $-1 \equiv p-1 \bmod p$, $(p-1)! \equiv p-1 \bmod p$. Moreover, $p! \equiv (p-1)p \bmod p^2$. Hence $p! = kp^2 + (p-1)p$ for some integer k. Since $p > 1, 0 < (p-1)p < p^2$. Thus $(p-1)p$ is the remainder when $p!$ is divided by p^2. Since the remainder is nonzero, $p^2 \nmid p!$.

Lemma 2 For all integers $n > 2$, $p^n \nmid p!$.
Proof.
Suppose $p^n \mid p!$. Since $p^2 \mid p^n$ and $p^n \mid p!$, $p^2 \mid p!$ which is a contradiction by Lemma 1.

Lemma 3 If G is a finite group and $H \neq G$ is a subgroup of G such that $|G| \nmid i(H)!$, then H must contain a nontrivial normal subgroup of G.
Proof.
This is Lemma 2.9.1 in [1].

Theorem 1 Any subgroup of order p^{n-1} in a group G of order p^n, p a prime number, is normal in G.
Proof.
The proof is by induction on n. Suppose the result is true for $n-1$. To show that it then must follow for n. Let G be a group of order p^n and H be its subgroup of order p^{n-1}. Since $|G| \nmid i(H)!$, that is $p^n \nmid p!$ by Lemma 2, H must contain a normal subgroup $N \neq (e)$ of G. Thus $|N| = p^k$ such that $1 \leqq k \leqq n-1$. Since p divides $|N|$, by Cauchy's theorem, N has an element $b \neq e$ of order p. Let B be the subgroup of G generated by b. So $|B| = p$. Since $b \in N$, B must be normal in G. Since G/B is a group of order p^{n-1} and H/B is its subgroup of order $p^{(n-1)-1}$, by the induction hypothesis H/B is normal in G/B. To conclude H is normal in G.

References

[1] I.N.Herstein, *Topics in Algebra*, John Wiley & Sons, New York, 1975.