Proof that there are no odd perfect numbers

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1. Abstract

For y to be a perfect number, if one of the prime factors is p, the exponent of p is an integer $n(n \geqq 1)$, the prime factors other than p are p_1,p_2,p_3,$\cdots p_r$ and the even exponent of p_k is q_k,

$$y/p^n = (1 + p + p^2 + \cdots + p^n)\prod_{k=1}^{r}(1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k})\,/(2p^n) = \prod_{k=1}^{r}{p_k}^{q_k}$$

must be satisfied. Let m and q be non-negative integers,

$$n = 4m + 1$$

$$p = 4q + 1$$

Letting b and c be odd integers, satisfying following expressions,

$$b = \prod_{k=1}^{r}{p_k}^{q_k}$$

$$c = \prod_{k=1}^{r}(1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k})/p^n$$

$$2b = c(p^n + \cdots + 1)$$

is established. This is a known content. By the consideration of this research paper, since it turns out that there is a solution at most one when a is a multiple of p^n and at this time the value of b becomes infinite, we have obtained the conclusion that there are no odd perfect numbers.

2. Introduction

The perfect number is one in which the sum of the divisors other than itself is the same value as itself, and the smallest perfect number is

$$1 + 2 + 3 = 6$$

It is 6. Whether an odd perfect number exists or not is currently an unsolved problem.

3. Proof

An odd perfect number is y, one of them is an odd prime number p, an exponent of p is an integer n (n ≧ 1). Let p_1,p_2,p_3,$\cdots p_r$ be the odd prime numbers of factors other than p, q_k the index of p_k, and variable a be the sum of product combinations other than prime p.

$$a = \prod_{k=1}^{r}(1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k}) \ \ldots ①$$

The number of terms N of variable a is

$$N = \prod_{k=1}^{r}(q_k + 1) \ \ldots ②$$

When y is a perfect number,

$$y = a(1 + p + p^2 + \cdots + p^n) - y \ (n > 0)$$

is established.

$$a\sum_{k=0}^{n} p^k /2 = y$$

$$a\sum_{k=0}^{n} p^k /(2p^n) = y/p^n \ \ldots ③$$

3.1. If q_k has at least one odd integer

Letting the number of terms where q_k is an odd integer be a positive integer u, because $y/p^n = \prod_{k=1}^{r} {p_k}^{q_k}$ is an odd integer, the denominator on the left side of expression ③ has a prime factor 2, from expression ② variable a has more than u prime factor 2 and variable a is an even integer. Therefore $\sum_{k=0}^{n} p^k$ must be an odd integer, n is an even integer and u is 1.

3.2. When all q_k are even integers

y/p^n is an odd integer, the denominator on the left side of expression ③ is an even integer, and since N is and odd integer when q_k are all even integers, variable a is and odd integer. Therefore $\sum_{k=0}^{n} p^k$ is necessary to include one prime factor 2, $\sum_{k=0}^{n} p^k \equiv 0 \ (mod\ 2)$ is established, and n must be an odd integer.

From 3.1, 3.2, in order to have an odd perfect number, only one exponent of the prime factor of y must be an odd integer and variable a must be an odd integer. We consider the case of 3.2 below.

In order for y to be a perfect number, the following expression must be established.

$$y/p^n = (1 + p + p^2 + \cdots + p^n)\prod_{k=1}^{r}(1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k})/(2p^n) = \prod_{k=1}^{r}{p_k}^{q_k}$$

However, $q_1, q_2, \ldots, q_r$ are all even integers.

Here, let b be an integer

$$b = \prod_{k=1}^{r}{p_k}^{q_k} \ \ldots④$$

A following expression is established.

$y/p^n = a(1 + p + p^2 + \cdots + p^n)/(2p^n) = b$

$a(p^{n+1} - 1)/(2(p - 1)p^n) = b$

$(a - 2b)p^{n+1} + 2bp^n - a = 0$ ...⑤

Because it is an $n + 1$ order equation of p, the solution of the odd prime p is $n + 1$ at most.

$(ap - 2bp + 2b)p^n = a$

Since $ap - 2bp + 2b$ is an odd integer, a/p^n is an odd integer, which is c.

$ap - 2bp + 2b = c\ (c > 0)$ ...⑥

$(2b - a)p = 2b - c$

Since variable a is an odd integer, $2b - a$ is an odd integer and $2b - a \neq 0$

$p = (2b - c)/(2b - a)$

Since $n \geqq 1$

$a - c = cp^n - c \geqq cp - c > 0$

$a > c$

is.

From equation ⑥

$2b(p-1) - (ap - c) = 0$

$2b - c(p^{n+1} - 1)/(p-1) = 0$

$(p^n + \cdots + 1)/2$ is an odd integer, $n = 4m + 1$ is required with m as an integer.

$2b(p-1) = c(p^{n+1} - 1)$

$2b = c(p^n + \cdots + 1)$

$2b = c(p+1)(p^{n-1} + p^{n-3} + \cdots + 1)$ ...⑦

b is an odd integer when $p + 1$ is not a multiple of 4. It is necessary that $p - 1$ be a multiple of 4. A positive integer is taken as q.

$p = 4q + 1$

is established.

When $p > 1$

$p^n - 1 < p^n$

$(p^n - 1)/(p-1) < p^n/(p-1)$

$p^{n-1} + \cdots + 1 < p^n/(p-1)$ ...⑧

Since p is an odd prime number satisfying $p = 4q + 1$ and $p \geqq 5$

$p^{n-1} + \cdots + 1 < p^n/4$

$2b - a = c(p^n + \cdots + 1) - cp^n = c(p^{n-1} + \cdots + 1)$

$2b - a < cp^n/4 = a/4$

$2b < 5a/4$

$a > 8b/5$ ...⑨

Let a_k and b_k be integers and if

$a_k = 1 + p_k + p_k^2 + \cdots + p_k^{q_k},\ b_k = p_k^{q_k}$,

$a_k - b_k < b_k/(p_k - 1)$

$a_k < b_k p_k/(p_k - 1)$

$$a = \prod_{k=1}^{r} a_k < \prod_{k=1}^{r} b_k p_k/(p_k - 1) = b \prod_{k=1}^{r} p_k/(p_k - 1)$$

$$a/b < \prod_{k=1}^{r} p_k/(p_k - 1)$$

When $r = 1$, since $a/b < 3/2$ is established, it becomes inappropriate contrary to inequality ⑨.

From expression ⑦,

$b = c(p + 1)/2 \times (p^{n-1} + p^{n-3} + \cdots + 1)$

holds. Since $(p + 1)/2$ is the product of only prime numbers of b, let d_k be the index,

$$(p + 1)/2 = \prod_{k=1}^{r} {p_k}^{d_k}$$

$$p = 2\prod_{k=1}^{r} {p_k}^{d_k} - 1$$

From $a = cp^n$ and expression ⑦,

$2bp^n = a(p^n + \cdots + 1)$

$a(p^n + \cdots + 1)/(2bp^n) = 1\ \ldots(A)$

When $r = 1$,

$a = ({p_1}^{q_1+1} - 1)/(p_1 - 1)$

$b = {p_1}^{q_1}$

Equation (A) does not hold since there is no odd perfect number when $r = 1$.

Let R be a rational number,

$R = a(p^n + \cdots + 1)/(2bp^n)$

Let b' be a rational number and let A and B to be an integer,

$b' = (p_k{}^{q_k+1} - 1)/(p_k{}^{q_k}(p_k - 1)) > 1$

$A = (p_k{}^{q_k+1} - 1)/(p_k - 1)$

$B = p_k{}^{q_k}$

Multiplying R by b', there are both cases that p_k increases p or does not change. When multiplied by b', the rate of change of R is $Ap^n(p'^n + \cdots + 1)/(Bp'^n(p^n + \cdots + 1))$, if p after variation is p'. If the rate of change of R is 1,

$Ap^n(p'^n + \cdots + 1)/(Bp'^n(p^n + \cdots + 1)) = 1$

$Ap^n\left(p'^n + \cdots + 1\right) = Bp'^n(p^n + \cdots + 1)$

This expression does not hold, since the right side is not a multiple of p when $p' > p$, and $A > B$ holds when $p' = p$. Due to this operation, R may be larger or smaller than the original value, since the rate of change of R does not become 1.

From $R \neq 1$ and $a = cp^n$ for some r, also multiplying fractions $b' = A_{r+1}/B_{r+1}$, $b'' = A_{r+2}/B_{r+2}$, $\cdots b''^{\cdots\prime} = A_x/B_x$, if $R = 1$ holds finally,

$$a(p^n + \cdots + 1)/(2bp^n) \times A_{r+1}p^n(p_{r+1}{}^n + \cdots + 1)/(B_{r+1}p_{r+1}{}^n(p^n + \cdots + 1))$$
$$\times A_{r+2}p_{r+1}{}^n(p_{r+2}{}^n + \cdots + 1)/(B_{r+2}p_{r+2}{}^n(p_{r+1}{}^n + \cdots + 1)) \dots A_x p_{x-1}{}^n(p_x{}^n$$
$$+ \cdots + 1)/(B_x p_x{}^n(p_{x-1}{}^n + \cdots + 1)) = 1$$

$a/(2b) \times A_{r+1}/B_{r+1} \times A_{r+2}/B_{r+2} \dots A_x(p_x{}^n + \cdots + 1)/(B_x p_x{}^n) = 1$

$a(p_x{}^n + \cdots + 1)A_{r+1}A_{r+2} \dots A_x = 2bp_x{}^n B_{r+1}B_{r+2} \dots B_x$

$cp^n(p_x{}^n + \cdots + 1)A_{r+1}A_{r+2} \dots A_x = 2bp_x{}^n B_{r+1}B_{r+2} \dots B_x$

When $p_x > p$, it becomes inconsistent since the right side of this expression does not include p as a factor.

When $p_x = p$,

$cp^n(p^n + \cdots + 1)A_{r+1}A_{r+2} \dots A_x = c(p^n + \cdots + 1)p^n$

$A_{r+1}A_{r+2} \dots A_x = 1$

It becomes contradiction, since this expression is not established. Therefore, $a = cp^n$ holds at one point where $R = 1$.

Assuming that $R = 1$ in some r by multiplying fractions $b' = A_{r+1}/B_{r+1}$, $b'' = A_{r+2}/B_{r+2}$, $\cdots b''^{\cdots\prime} = A_x/B_x$, if $R = 1$ holds,

$$1 \times A_{r+1}p^n({p_{r+1}}^n + \cdots + 1)/(B_{r+1}{p_{r+1}}^n(p^n + \cdots + 1)) \times A_{r+2}{p_{r+1}}^n({p_{r+2}}^n + \cdots + 1)/(B_{r+2}{p_{r+2}}^n({p_{r+1}}^n + \cdots + 1)) \ldots A_x{p_{x-1}}^n({p_x}^n + \cdots + 1)/(B_x{p_x}^n({p_{x-1}}^n + \cdots + 1)) = 1$$

$$A_{r+1}A_{r+2} \ldots A_x p^n({p_x}^n + \cdots + 1) = B_{r+1}B_{r+2} \ldots B_x{p_x}^n(p^n + \cdots + 1)$$

When $p_x > p$, it becomes inconsistent since the right side of this expression does not include p as a factor. When $p_x = p$,

$$A_{r+1}A_{r+2} \ldots A_x = B_{r+1}B_{r+2} \ldots B_x$$

is established. It becomes contradiction. Therefore, when $R = 1$ holds in some b, doing this operation by fixing n and changing b to a multiple of b larger than that value, y corresponding to b does not become an odd perfect number.

Assuming that $R = 1$ in some r by multiplying fractions $b' = A_{r+1}/B_{r+1}$, $b'' = A_{r+2}/B_{r+2}$, $\cdots b''^{\cdots\prime} = A_x/B_x$, if $R = 1$ holds. At this time, assuming that n also changes, the change rate when multiplying by A_{r+1}/B_{r+1} is

$$A_{r+1}p^n({p_{r+1}}^{n_{r+1}} + \cdots + 1)/(B_{r+1}{p_{r+1}}^{n_{r+1}}(p^n + \cdots + 1))$$

$$1 \times A_{r+1}p^n({p_{r+1}}^{n_{r+1}} + \cdots + 1)/(B_{r+1}{p_{r+1}}^{n_{r+1}}(p^n + \cdots + 1)) \times A_{r+2}{p_{r+1}}^{n_{r+1}}({p_{r+2}}^{n_{r+2}} + \cdots + 1)/(B_{r+2}{p_{r+2}}^{n_{r+2}}({p_{r+1}}^{n_{r+1}} + \cdots + 1)) \ldots A_x{p_{x-1}}^{n_{x-1}}({p_x}^{n_x} + \cdots + 1)/(B_x{p_x}^{n_x}({p_{x-1}}^{n_{x-1}} + \cdots + 1)) = 1$$

$$A_{r+1}A_{r+2} \ldots A_x p^n({p_x}^{n_x} + \cdots + 1) = B_{r+1}B_{r+2} \ldots B_x{p_x}^{n_x}(p^n + \cdots + 1) \ldots (B)$$

When $n = n_x$ it becomes contradiction like above proof.

When $n_x < n$, a contradiction arises in the case of $p_x \neq p$ since prime number p is not included in the right side. It becomes inconsistent in the case of $p_x = p$ since only the left side includes the prime number p. A contradiction arises similarly when $n_x > n$. In addition to this operation, exchanging the order of multiplications, suppose that division is performed until a can be divided by p^n. Dividing by A_s/B_s at $R = 1$,

$$1 \times B_s p^n({p_s}^{n_s} + \cdots + 1)/(A_s{p_s}^{n_s}(p^n + \cdots + 1))$$

When performing division, if the prime number of the denominator is included in the expression of p, it is excluded. At this time, assuming that the product of the numerators subjected to the division is $A_sA_{s+1} \ldots A_r$, since this term is the right side of the formula (B), it becomes contradiction similarly. Therefore, when $A_1A_2 \ldots A_{r-1}$ is a multiple of p^n, multiplying after this, the solution in which the value of R is 1 is at most one.

When $A_1A_2 \dots A_{r-1}$ can be divisible by p^n, the combinations of primes are infinite, and there is at most one solution for one of the combinations. Let a set having infinite number of elements which are odd prime multiples of the values of $B_1B_2 \dots B_{r-1}$ be a set A, and consider a set B with an infinite number of elements that are odd prime multiples of p^n. Set B includes all sets considered as set A. When b is included in the set A, the number of solutions is one and when b is included in the set B, the number of solutions is one. At this time, since the set A is a subset of the set B, there is at most one solution for all product sets of the set A. Therefore, even if an odd perfect number exists, since its value becomes infinite, there are no odd perfect numbers.

4. Complement

From equation ⑤,

$2bp^n(p-1) = a(p^{n+1}-1)$

$2 = a(p^{n+1}-1)/(bp^n(p-1))$

$2 = (p_1{}^{q_1+1}-1)(p_2{}^{q_2+1}-1)\ldots(p_r{}^{q_r+1}-1)(p^{n+1}-1)$

$\qquad /(p_1{}^{q_1}p_2{}^{q_2}\ldots p_r{}^{q_r}p^n(p_1-1)(p_2-1)\ldots(p_r-1)(p-1))$

$2(p_1{}^{q_1+1}-p_1{}^{q_1})(p_2{}^{q_2+1}-p_2{}^{q_2})\ldots(p_r{}^{q_r+1}-p_r{}^{q_r})(p^{n+1}-p^n)$

$\qquad = (p_1{}^{q_1+1}-1)(p_2{}^{q_2+1}-1)\ldots(p_r{}^{q_r+1}-1)(p^{n+1}-1)$

We consider when $r=2$.

$(p_1{}^{q_1+1}-1)(p_2{}^{q_2+1}-1)(p^{n+1}-1) = 2(p_1{}^{q_1+1}-p_1{}^{q_1})(p_2{}^{q_2+1}-p_2{}^{q_2})(p^{n+1}-p^n)$

Let s, t, u be integers,

$s = p_1{}^{q_1+1}-1$

$t = p_2{}^{q_2+1}-1$

$u = p^{n+1}-1$

are.

$stu = 2(p_1{}^{q_1+1}-1-(p_1{}^{q_1}-1))(p_2{}^{q_2+1}-1-(p_2{}^{q_2}-1))(p^{n+1}-1-(p^n-1))$

$stu = 2(s-(s+1)/p_1+1)(t-(t+1)/p_2+1)(u-(u+1)/p+1)$

$pp_1p_2stu = 2((s+1)p_1-(s+1))((t+1)p_2+(t+1))\big((u+1)p+(u+1)\big)$

$pp_1p_2stu = 2(s+1)(p_1-1)(t+1)(p_2-1)(u+1)(p-1)$

$stu/((s+1)(t+1)(u+1)) = 2(p_1-1)(p_2-1)(p-1)/(p_1p_2p)$

Since $stu/((s+1)(t+1)(u+1))$ is a monotonically increasing function for variables s, t and u, if

$s \geqq 3^{2+1}-1 = 26,\ p_1 = 3,\ q_1 = 2$

$t \geqq 7^{2+1}-1 = 342,\ p_2 = 7,\ q_2 = 2$

$u \geqq 5^2-1 = 24,\ p = 5,\ n = 1$

holds,

$stu/((s+1)(t+1)(u+1)) \geqq 26 \times 342 \times 24/(27 \times 343 \times 25) = 7904/8575$

$2(p_1-1)(p_2-1)(p-1)/(p_1p_2p) = 2 \times 2 \times 6 \times 4/(3 \times 7 \times 5) = 32/35$

Since $stu/((s+1)(t+1)(u+1))$ is limited to 1 when s, t and u are infinite,
$stu/((s+1)(t+1)(u+1)) < 1$

If $f(p_1, p_2, p) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)$ holds, it is sufficient to consider a combination where $f(p_1, p_2, p) < 1$.
$f(3,7,5) = 2 \times 2 \times 6 \times 4/(3 \times 7 \times 5) = 32/35$
$f(3,11,5) = 2 \times 2 \times 10 \times 4/(3 \times 11 \times 5) = 32/33$
$f(3,13,5) = 2 \times 2 \times 12 \times 4/(3 \times 13 \times 5) = 64/65$
$f(3,17,5) = 2 \times 2 \times 16 \times 4/(3 \times 17 \times 5) = 256/255$
$f(3,7,13) = 2 \times 2 \times 6 \times 12/(3 \times 7 \times 13) = 96/91$
$f(3,5,17) = 2 \times 2 \times 4 \times 16/(3 \times 5 \times 17) = 256/255$
From the above, when $r = 2$, a combination $(p_1, p_2, p) = (3,7,5), (3,11,5), (3,13,5)$ can be considered.

Let q_k be 2 and $n = 1$, if $g(p_1, p_2, p) = ({p_1}^3 - 1)({p_2}^3 - 1)(p^2 - 1)/({p_1}^3 {p_2}^3 p^2)$,
$g(3,7,5) = 26 \times 342 \times 24/(3^3 7^3 5^2) = 7904/8575 > 32/35$
$g(3,11,5) = 26 \times 1330 \times 24/(3^3 11^3 5^2) = 55328/59895$
$g(3,13,5) = 26 \times 2196 \times 24/(3^3 13^3 5^2) = 3904/4225$
Since the function g is the minimum in the case of $q_k = 2$ and $n = 1$, there is no solution q_k and n when $g > f$, so the case of $(p_1, p_2, p) = (3,7,5)$ becomes unsuitable.

$$stu/((s+1)(t+1)(u+1)) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)$$
$$({p_1}^{q_1+1} - 1)({p_2}^{q_2+1} - 1)(p^{n+1} - 1)/({p_1}^{q_1+1} {p_2}^{q_2+1} p^{n+1})$$
$$= 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)$$

If $F(p_1, p_2, p) = (p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)$,
$F({p_1}^{q_1+1}, {p_2}^{q_2+1}, p^{n+1}) = 2F(p_1, p_2, p)$

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6. References

Hiroyuki Kojima "The world is made of prime numbers" Kadokawa Shoten, 2017

Fumio Sairaiji·Kenichi Shimizu "A story that prime is playing" Kodansha, 2015