Remarks on Infinitesimal Amount of Riemann Zeta Zeros

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Abstract

This remarks proves, that Riemann zeta function has infinitesimal amount of zeros.

Riemann zeta function expressed as follow

$$\zeta(r,\theta)=\sum_{n=1}^{\infty}\frac{1}{n^{r\cos\theta+ir\sin\theta}} \tag{1}$$

According Abel's summation formula [1]

$$\zeta(r,\theta)=\sum_{n=1}^{\infty}\frac{1}{n^{r\cos\theta+ir\sin\theta}}=\lim_{x->\infty}\frac{\lfloor x\rfloor}{x^{r\cos\theta+ir\sin\theta}}-\int_{u=1}^{\infty}\frac{\lfloor u\rfloor du}{u^{r\cos\theta+ir\sin\theta+1}} \tag{2}$$

For $r>1,\cos\theta>0,\sin\theta>0$ the first term in (2) goes to zero and we can rewrite (2) without loosing of generalisation as follow

$$\zeta(r,\theta)=\gamma_{r\cos\theta+ir\sin\theta}-\int_{u=1}^{\infty}\frac{du}{u^{r\cos\theta+ir\sin\theta}} \tag{3}$$

where $\gamma_{r\cos\theta+ir\sin\theta}$ is constant for each r and θ. After integration we obtains

$$\zeta(r,\theta)=\gamma_{r\cos\theta+ir\sin\theta}-\left.\frac{u^{-r\cos\theta-ri\sin\theta+1}}{-r\cos\theta-ri\sin\theta+1}\right|_{u=1}^{u=\infty} \tag{4}$$

Let's restrict r and θ to $r\cos-1>0, r\sin\theta>0$. So,

$$\zeta(r,\theta)=\gamma_{r\cos\theta+ir\sin\theta}+\frac{1}{-r\cos\theta-ri\sin\theta+1} \tag{5}$$

or

$$\zeta(r,\theta)=\left((\Re\zeta(r,\theta))^2+(\Im\zeta(r,\theta))^2\right)\exp\left(i\arctan\frac{\Im\zeta(r,\theta)}{\Re\zeta(r,\theta)}+i2\pi n\right) \tag{6}$$

where

$$\Re\zeta(r,\theta)=\Re\gamma_{r\cos\theta+ir\sin\theta}+\frac{-r\cos\theta+1}{(1-r\cos\theta)^2+r^2\sin^2\theta} \tag{7}$$

$$\Im\zeta(r,\theta)=\Im\gamma_{r\cos\theta+ir\sin\theta}+\frac{r\sin\theta}{(1-r\cos\theta)^2+r^2\sin^2\theta} \tag{8}$$

Obviously, for $r >> 1$ we can find infinitesimal amount of Riemann zeta zeros by solving equation as follow

$$\frac{r^2 \Im \gamma_{r\cos\theta + ir\sin\theta} + r\sin\theta}{r^2 \Re \gamma_{r\cos\theta + ir\sin\theta} - r\cos\theta + 1} - 2\pi n = 0, \forall n \in (-\infty, \infty) \tag{9}$$

It implies that Riemann zeta function has infinitesimal amount of zeros.

References

[1] Apostol, Tom (1976), Introduction to Analytic Number Theory, Undergraduate Texts in Mathematics, Springer-Verlag.