Power law and dimension of the maximum value for belief distribution with the max Deng entropy

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Abstract

Deng entropy is a novel and efficient uncertainty measure to deal with imprecise phenomenon, which is an extension of Shannon entropy. In this paper, power law and dimension of the maximum value for belief distribution with the max Deng entropy are presented, which partially uncover the inherent physical meanings of Deng entropy from the perspective of statistics. This indicated some work related to power law or scale-free can be analyzed using Deng entropy. The results of some numerical simulations are used to support the new views.

Keywords: Deng entropy, Power law, Maximum Deng entropy, Dimension

1. Introduction

Deng entropy has been proposed by Prof. Deng to manage the uncertain information in the frame of Dempster-Shafer evidence theory (DST)[1], which has achieved plenty of attention recent years[2, 3, 4, 5, 6]. Deng entropy can be considered as an extension of Shannon entropy to deal with

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uncertain phenomenon in the probability field. In addition, Deng entropy can be applied to absorb the complex imprecise (or unknown) phenomenon in the belief filed (frame of DST) efficiently.

In this paper, the work focuses two investigations based on the maximum values of the belief distribution via the max Deng entropy with different scales of frame of discernment (FOD). One is the relation between the maximum value of belief distribution subjecting to the max Deng entropy and the scale of Deng information correspondingly. The other is dimension of the maximum value for belief distribution with the max Deng entropy. Some numerical simulations have been made to achieve the two discoverings, i.e., approximate power law and approximate constant dimension.

The paper is organized as follows. The preliminaries briefly introduce some concepts about Dempster-Shafer evidence theory, Deng entropy and max Deng entropy in Section 2. In Section 3, the new views about max Deng entropy are presented. One is the relation between the maximum value of belief distribution subjecting to the max Deng entropy and the scale of Deng information correspondingly. The other is dimension of the maximum value for belief distribution with the max Deng entropy. Finally, this paper is concluded in Section 4.

2. Preliminaries

In this section, some preliminaries are briefly introduced.

2.1. Frame of Dempster-Shafer evidence theory

Let X be a set of mutually exclusive and collectively exhaustive events, indicated by

$$X = \{\theta_1, \theta_2, \cdots, \theta_i, \cdots, \theta_{|X|}\} \tag{1}$$

where set X is called a frame of discernment (FOD). The power set of X is indicated by 2^X, namely

$$2^X = \{\emptyset, \{\theta_1\}, \cdots, \{\theta_{|X|}\}, \{\theta_1, \theta_2\}, \cdots, \{\theta_1, \theta_2, \cdots, \theta_i\}, \cdots, X\} \tag{2}$$

For a frame of discernment $X = \{\theta_1, \theta_2, \cdots, \theta_{|X|}\}$, a mass function is a mapping m from 2^X to $[0, 1]$, formally defined by:

$$m : \quad 2^X \rightarrow [0, 1] \tag{3}$$

which satisfies the following condition:

$$m(\emptyset) = 0 \quad and \quad \sum_{A \in 2^X} m(A) = 1 \tag{4}$$

where A is a focal element if $m(A)$ is not 0.

2.2. Deng entropy

With the range of uncertainty mentioned above, Deng entropy [1] can be presented as follows

$$E_d = -\sum_i m(F_i) \log \frac{m(F_i)}{2^{|F_i|} - 1} \tag{5}$$

where, F_i is a proposition in mass function m, and $|F_i|$ is the cardinality of F_i. As shown in the above definition, Deng entropy, formally, is similar with the classical Shannon entropy, but the belief for each proposition F_i is divided by a term $(2^{|F_i|} - 1)$ which represents the potential number of states in F_i (of course, the empty set is not included).

Specially, Deng entropy can definitely degenerate to the Shannon entropy if the belief is only assigned to single elements. Namely,

$$E_d = -\sum_i m(\theta_i) \log \frac{m(\theta_i)}{2^{|\theta_i|} - 1} = -\sum_i m(\theta_i) \log m(\theta_i)$$

Next, the condition of the maximum Deng entropy is discussed[7].

2.3. The maximum Deng entropy

Assume F_i is the focal element and $m(F_i)$ is the basic probability assignment for F_i, then the maximum Deng entropy for a belief function happens when the basic probability assignment satisfy the condition $m\left(F_i\right) = \frac{2^{|F_i|}-1}{\sum\limits_i 2^{|F_i|}-1}$, where $i = 1, 2, ..., 2^X - 1$, and X is the scale of the frame of discernment.

Theorem 1 (The maximum Deng entropy)**.** *The maximum Deng entropy:* $E_d = -\sum\limits_i m(F_i) \log \frac{m(F_i)}{2^{|F_i|}-1}$ *if and only if* $m\left(F_i\right) = \frac{2^{|F_i|}-1}{\sum\limits_i 2^{|F_i|}-1}$

More information refers to Appendix A. As shown in Fig. 1, belief distributions with the maximum Deng entropy are changing with the scale of FOD, $|X|$=1,..8. The point in this paper lies in the maximum value of each belief distribution.

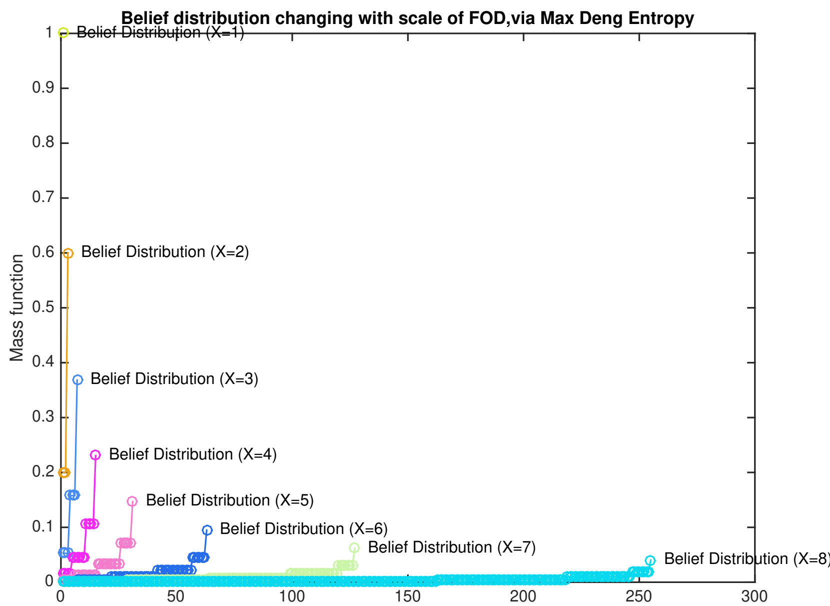

Figure 1: Belief distribution with the maximum Deng entropy changing with the scale of FOD, $|X|$=1,..8.

Next, Two focuses are presented. One is the relation between the maximum value of belief distribution subjecting to the max Deng entropy and the scale of Deng information correspondingly. The other is dimension of the maximum value for belief distribution with the max Deng entropy.

3. Power law and dimension of the maximum value for belief distribution with the max Deng entropy

In statistic, a power law is a relationship in which a relative change in one quantity gives rise to a proportional relative change in the other quantity, independent of the initial size of those quantities. Power law is a pervasive phenomenon in many fields, such as complex network (scale-free network) [8], Fractals[9].

Firstly, the power law function is established between the maximum value for belief distribution via max Deng entropy and the maximum Deng information scale correspondingly.

3.1. Power law of the maximum value for belief distribution with the max Deng entropy

Suppose the maximum value for belief distribution via max Deng entropy $\max\left[m\left(F_i\right)\right]$ relates to a function $P(r)$. In addition, assume the corresponding maximum Deng information scale $\sum\limits_i \left(2^{|F_i|}-1\right)$ relates to the variable r. A pow law function $P\left(\sum\limits_i \left(2^{|F_i|}-1\right)\right)$ with a scale invariance ($d \approx 0.59$) is established using Eq. (6).

$$P\left(r\right) = r^{-d} \tag{6}$$

where $r = \sum\limits_i \left(2^{|F_i|} - 1\right)$, $P\left(r\right) = \max\left[m\left(F_i\right)\right]$, $m\left(F_i\right) = \frac{2^{|F_i|} - 1}{\sum\limits_i \left(2^{|F_i|} - 1\right)}$,$d \approx 0.59$.

As shown in Fig. 2, when the scales of FOD ($|X|$) change from 1 to 10, all the few high belief (the maximum values of the belief distribution via max Deng entropy, $\max\left[m\left(F_i\right)\right]$) are contained in the front of the plane, most of the other low belief (the maximum values of the belief distribution via max Deng entropy, $\max\left[m\left(F_i\right)\right]$) are distributed in the following wide plane. This is an approximate power law distribution. A pow law function $P\left(\sum\limits_i \left(2^{|F_i|} - 1\right)\right)$ with a scale invariance (d) is easily observed by Fig. 2. What is more, the scale invariance $d \approx 0.59$, which will be discussed in the next section.

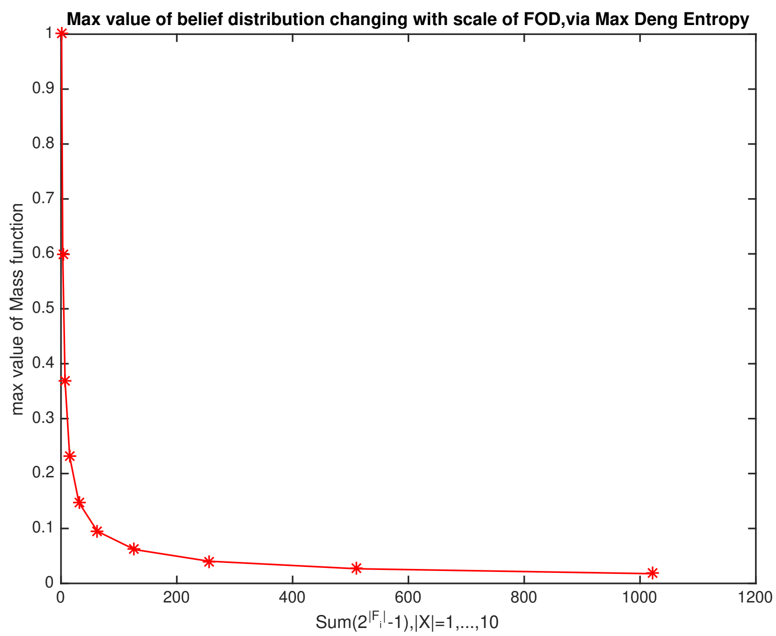

Figure 2: The maximum value of belief distribution with the maximum Deng entropy changing with the maximum scale of Deng information, the scale of FOD, $|X|$=1,..10. This is an approximate power law distribution.

Next, dimension of the maximum value for belief distribution with the max Deng entropy is presented.

3.2. Dimension of the maximum value for belief distribution with the max Deng entropy

The scale invariance d in Eq.(6) is equal to the dimension of the maximum value for belief distribution with the max Deng entropy. By the $polyfit$ function in the Matlab, the dimension $d \approx 0.59$ after investigating the data of belief distributions with max Deng entropy (scale of FOD, $|X| = 1, 2, ..., 25$). The result is shown in Fig. 3, which indicates the log values trending between the maximum value of belief distribution with the maximum Deng entropy and the maximum amount of Deng information, the scale of FOD, $|X|$=1,..25. As shown in Fig. 3, an approximate linear relation is obtained, which indicate the scale-free and power law.

$$d = -\lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(\varepsilon)} = \lim_{i \to \infty} \frac{\log_2 \max\left[m(F_i)\right]}{\log_2 \sum\limits_i (2^{|F_i|} - 1)} \approx 0.59 \tag{7}$$

$$s.t. \quad m(F_i) = \frac{2^{|F_i|} - 1}{\sum\limits_i (2^{|F_i|} - 1)} \tag{8}$$

4. Conclusion

In this paper, power law and dimension of the maximum value for belief distribution with the max Deng entropy are presented, which partially uncover the inherent physical meanings of Deng entropy from the perspective of statistics. The results of some numerical simulations are used to support

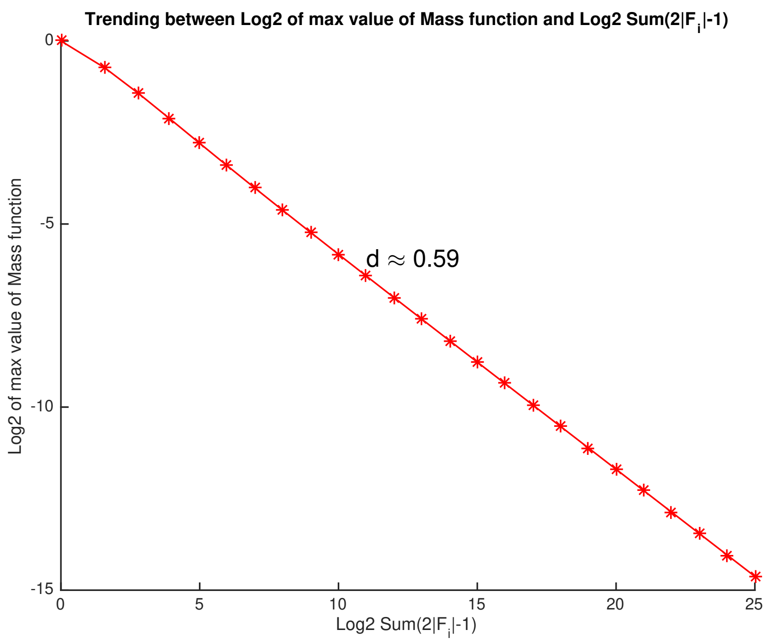

Figure 3: Dimension of the maximum value of belief distribution with the maximum Deng entropy and the maximum amount of Deng information, the scale of FOD, $|X|$=1,..25.

the new views. This indicated some work related to power law or scale-free can be analyzed using Deng entropy.

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Appendix A. The maximum Deng entropy

Assume F_i is the focal element and $m(F_i)$ is the basic probability assignment for F_i, then the maximum Deng entropy for a belief function happens when the basic probability assignment satisfy the condition $m\left(F_i\right) = \frac{2^{|F_i|}-1}{\sum\limits_i 2^{|F_i|}-1}$, where $i = 1, 2, ..., 2^X - 1$, and X is the scale of the frame of discernment.

Theorem 2 (The maximum Deng entropy)**.** *The maximum Deng entropy:* $E_d = -\sum\limits_i m(F_i) \log \frac{m(F_i)}{2^{|F_i|}-1}$ *if and only if* $m\left(F_i\right) = \frac{2^{|F_i|}-1}{\sum\limits_i 2^{|F_i|}-1}$

Proof. Let

$$D = -\sum_i m\left(F_i\right) \log \frac{m\left(F_i\right)}{2^{|F_i|}-1} \tag{A.1}$$

$$\sum_i m\left(F_i\right) = 1 \tag{A.2}$$

Then the Lagrange function can be defined as

$$D_0 = -\sum_i m(F_i) \log \frac{m(F_i)}{2^{|F_i|} - 1} + \lambda \left(\sum_i m(F_i) - 1 \right) \tag{A.3}$$

Now we can calculate the gradient,

$$\frac{\partial D_0}{\partial m(F_i)} = -\log \frac{m(F_i)}{2^{|F_i|} - 1} - m(F_i) \frac{1}{\frac{m(F_i)}{2^{|F_i|}-1} \ln a} \cdot \frac{1}{2^{|F_i|} - 1} + \lambda = 0 \tag{A.4}$$

Then Eq. (A.4) can be simplified as

$$-\log \frac{m(F_i)}{2^{|F_i|} - 1} - \frac{1}{\ln a} + \lambda = 0 \tag{A.5}$$

From Eq. (A.5), we can get

$$\frac{m(F_1)}{2^{|F_1|} - 1} = \frac{m(F_2)}{2^{|F_2|} - 1} = \cdots = \frac{m(F_n)}{2^{|F_n|} - 1} \tag{A.6}$$

Let

$$\frac{m(F_1)}{2^{|F_1|} - 1} = \frac{m(F_2)}{2^{|F_2|} - 1} = \cdots = \frac{m(F_n)}{2^{|F_n|} - 1} = k \tag{A.7}$$

Then

$$m(F_i) = k \left(2^{|F_i|} - 1 \right) \tag{A.8}$$

According to Eq. (A.2), we can get

$$k = \frac{1}{\sum_i 2^{|F_i|} - 1} \tag{A.9}$$

According to Eq. (A.7), we can get

$$m\left(F_i\right)=\frac{2^{\left|F_i\right|}-1}{\sum\limits_i 2^{\left|F_i\right|}-1} \tag{A.10}$$

Hence, the maximum Deng entropy $E_d=-\sum\limits_i m(F_i)\log\frac{m(F_i)}{2^{|F_i|}-1}$ if and only if $m\left(F_i\right)=\frac{2^{\left|F_i\right|}-1}{\sum\limits_i 2^{\left|F_i\right|}-1}$

$\square$

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