Using Cantor's Diagonal Method to Show $\zeta(2)$ is Irrational

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Abstract

We look at some of the details of Cantor's Diagonal Method and argue that the swap function given does not have to exclude 9 and 0, base 10. We then give a application of Cantor's Diagonal Method that shows $\zeta(2)$ is irrational.

Introduction

Cantor's diagonal method is typically used to show the real numbers are uncountable [1, 2]. Here is the reasoning.

If the reals are countable they can be listed. List their base ten decimal representations and starting with the upper left hand corner digit, construct, going down the upper left to lower right diagonal, a decimal not in the list. Use the following guide: if the decimal is 7 make your decimal 5 and if it is anything other than 7 make it 5. The number you construct is not in the list and therefore the real numbers are uncountable.

There are some points (fine print) to this argument. You can't use 0 and 9 in the argument. We show here that this is not really true. This is not to say that there is anything wrong with Cantor's Diagonal Method. If one does use 0 and 9 the argument is lengthened. You might call it less – or more – elegant.

It seems curious that mathematical proofs typically take fast detours around sticking points. Why bother with convoluted reasoning, if you don't have to. Reductio ad absurdum proofs seem to be like this. But in this particular case of Cantor's Diagonal Method, going into the weeds does produce a pertinent generalization. We can show $\zeta(n)$, n a natural number greater than 1, is irrational.

Why not 0 and 9

Note that if one used the guide if 0 change it to 9 and if 9 change it to 0, one could construct 0. For example,

$$
\begin{array}{l}
.3 \\
.04 \\
.005 \\
.0006 \\
.0000x \vdots
\end{array}
$$

then, as long as x is not 0, we get $.\overline{0} = 0$. If we constructed, using another list, $.000\overline{9}$, this would also be .001, a number in the list – it's a real number.

By making the swap with numbers like 5 and 4 or 3 and 7 or any two that are not 9 and 0, we don't run into this problem. But, for the sake of argument are we really assured that these patterns can be maintained? No that can't be. A little observation yields that any list will only be able to maintain some property of decimal position for a finite number. Any repeated pattern with 9 at position 1, for example can only work 1/10th of time in the list. Given an nth position eventually it will have to vary. The infinite number possible can't be only at the head of the list.

What about convergence

Cantor's diagonal method does not address the convergence of the decimal representation of a real number generated. Could it be all 5's ($.\overline{5}$) and hence converging to a rational number – a number in the list. A combination of 4's and 5's that represent a infinitely repeating decimal? These observations are of no concern because the argument is that the number's representation is not in the list. Statements beyond this seem irrelevant.

Of course if we suppose that ambiguity of representation is not allowed: only finite decimal representations are given of numbers like .5 and $.4\overline{9}$, then the infinite decimal we construct might be an excluded infinite decimal version of a number included in the list. This is when the use of not 9 and not 0 fix the situation fast. One could do a reductio ad absurdum argument. Suppose the constructed number converges to a number in the list, but the number in the list differs by at least one decimal point. So how close can $.5554445454\ldots$ get to say $.555444454\ldots$ – they differ at the 7th place. The numbers must differ by at least .0000001. Another argument: decimal representations are unique, excluding representations like $.5\overline{9} = .6$, but such a situation is impossible when 9 is not used in the swap function.

Proving $\zeta(2)$ is irrational

In Table 1 is a modified Cantor's Diagonal Table. The symbols D_{n^2} give single decimal points in base n^2. So, for example $D_4 = \{.1, .2, .3\}$ in base 4. How to read the table: All previous columns (left to right) pertain to the new, right most partial. For example $1/4 + 1/9 + 1/16$ is not in D_4, D_9, or D_{16}. So, like Cantor's diagonal method as applied to a list of base ten decimals, we build, not with a swap function, but with an addition, a number not in any decimal base given by a single decimal base n^2.

$+1/4$							
$+1/9$	$+1/4$	$+1/4$	$+1/4$	$+1/4$	$\ldots$	$+1/4$	
$\notin D_4$	$+1/9$	$+1/9$	$+1/9$	$+1/9$	$\ldots$	$+1/9$	
	$\notin D_9$	$+1/16$	$+1/16$	$+1/16$		$\vdots$	
		$\notin D_{16}$	$+1/25$	$+1/25$		$\vdots$	
			$\notin D_{25}$	$+1/36$		$\vdots$	
				$\notin D_{36}$			
						$+1/(k-1)^2$	
						$+1/k^2$	
						$\notin D_{k^2}$	
							$\ddots$

Table 1: A list of all rational numbers between 0 and 1 modified to exclude them all via partial sums of $\zeta(2) - 1$.

If this is true, can we conclude that $z_2 = \zeta(2) - 1$ must be irrational?

Note: for any rational $0 < p/q < 1$, $(pq)/q^2 \in D_{q^2}$. Thus all partials with a last term of $1/q^2$ can't be given with any single decimal base m^2 with $m < q$. Are we building a infinite series, like the infinite decimal of Cantor's original argument, that must be excluded from the list of all rationals (here-to-for all reals) and thus be irrational? Does the elimination element of Cantor's Diagonal Method force an irrational sum? Like CDM can we ignore the convergence point of the built infinite series?

Well, to play it safe, can we prove the convergence point is not in our list? Consider the following use of the triangle inequality: let C_x be a single decimal rational in some D_{m^2}, the best, meaning closest to z_n in D_{m^2}, then for all n large enough, either

$$\left| C_x - \sum_{k=2}^{n} \frac{1}{k^2} \right| = 0$$

or

$$0 < \left| C_x - \sum_{k=2}^{n} \frac{1}{k^2} \right| < \epsilon/2.$$

and

$$0 < \left| \sum_{k=2}^{n} \frac{1}{k^2} - z_2 \right| < \epsilon/2.$$

So, in all cases,

$$0 < |C_x - z_2| < \epsilon.$$

But this says z_2 is not rational.

Same technique base 4

In Table 2, the same technique as given for z_2 is used to show $\overline{1}$ base 4 is not a finite decimal base 4. We know this, apart from Table 2, because the geometric series associated with this infinite, repeating decimal is

$$\sum_{k=1}^{\infty} \frac{1}{4^k} = \frac{1}{3}.$$

$+1/4$	$+1/4$	$+1/4$	$+1/4$	$+1/4$	$\ldots$	$+1/4$	
$+1/4^2$	$+1/4^2$	$+1/4^2$	$+1/4^2$	$+1/4^2$	$\ldots$	$+1/4^2$	
$\notin D_4$	$+1/4^3$	$+1/4^3$	$+1/4^3$	$+1/4^3$	$\ldots$	$+1/4^3$	
	$\notin D_{4^2}$	$+1/4^4$	$+1/4^4$	$+1/4^4$		$\vdots$	
		$\notin D_{4^3}$	$+1/4^5$	$+1/4^5$		$\vdots$	
			$\notin D_{4^4}$	$+1/4^6$		$\vdots$	
				$\notin D_{4^5}$			
						$+1/4^{(k-1)^2}$	
						$+1/4^{k^2}$	
						$\notin D_{4^{(k-1)^2}}$	
							$\ddots$

Table 2: A list of all finite decimals base 4. The decimal number $.\overline{1}$, base 4 is generated by the sums.

Looking again at Table 1, one can see why the technique shows that z_2 is irrational. In order for the partials to be converging to a rational, for every ϵ, there would have to be a rational that all partials with a given upper limit greater than n gets close to. But the partials always exceed, with their denominator, all $\{2, 4, \ldots, n^2\}$ denominators. There are denominators further out from any given, fixed denominator that get consistently closer to such partials. Limits are unique, so the limit point can't be both approaching a previous finite decimal excluded earlier and some other later decimal.

Evens and Bertrand's postulate

It remains to show that partials escape the denominators of their terms: the columns of Table 1. This is the juncture of the argument where the fact that all numbers (greater than 1, to a power of 2) are included in the sum of z_2. Every other number is even so the power of 2 in the denominator is always greater than 2 and so the reduced denominator always has at least 2 in it. For example,

$$\frac{1}{4}+\frac{1}{9}+\frac{1}{16}=\frac{61}{144}$$

and

$$\frac{61}{144}+\frac{1}{25}=\frac{25\times 61+1\times 144}{144\times 25}$$

shows how the pattern continues.

Using Bertrand's postulate, we know there exists a prime p between $n^2/2$ and n^2.

Putting these two results together,

$$\sum_{k=2}^{n}\frac{1}{k^2}=\frac{a}{b},$$

where a/b is a reduced fraction with $b>n^2$. This established the set exclusions in the columns of Table 1 are correct. For details, see [3].

Conclusion

Cantor's diagonal method applied to show the existence of an irrational number and the proof given here for the irrationality of $\zeta(2)$ can be viewed as the same. The negations of set inclusions in Table 1 show that somewhere the decimal associated with the partial is not the same as those in each D_{k^2} set. As the union of all such sets give all the rationals the irrationality of $\zeta(2)$ follows. All decimal bases n^2 are replicated, including base 10 – just like Cantor's original idea.

References

[1] T. M. Apostol, *Mathematical Analysis*, 2nd ed., Addison Wesley, Reading, Massachusetts, 1974.

[2] R. Courant, H. Robbins, *What is Mathematics*, Oxford University Press, London, 1948.

[3] T.W. Jones, A Simple Proof that $\zeta(n)$ is Irrational (2017), available at `available at http://vixra.org/abs/1801.0140`