Solution of Erdös-Moser equation

$$1 + 2^p + 3^p + ... + (k)^p = (k + 1)^p$$

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Abstract

I will provide the solution of Erdös-Moser equation based on the properties of Bernoulli polynomials and prove that there is only one solution satisfying the above-mentioned equation.

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1 Notation

$1 + 2^p + 3^p + ... + (k)^p = (k + 1)^p$ represents Erdös-Moser equation, where $k, p \in \mathbb{N}$. Let b_n denotes Bernoulli numbers and $B_n(x) = \sum_{k=0}^{n} \binom{n}{k} b_{n-k} x^k$ denotes Bernoulli polynomials for $n \geq 0$.

2 Introduction

The Erdös-Moser equation (EM equation) named after Paul Erdös and Leo Moser has been studied by many number theorists through history since combines addition, powers and summation together. The open and very interesting conjecture of Erdös-Moser states that there is no other solution of EM equation than the trivial $1 + 2 = 3$. Investigation of the properties and identities of the EM equation and ultimately providing the proof of this conjecture is the main purpose of this article.

3 Solution

Lemma 3.1 *The EM equation is equivalent of*

$$\sum_{k=0}^{x} k^p \equiv \frac{B_{p+1}(x+1)}{p+1} = (x+1)^p \tag{1}$$

$x, p \in \mathbb{N} \wedge x > 2 \wedge p > 1$ *since we are seeking other solution than trivial.*

Proof Sum of pth powers is defined as

$$\sum_{k=0}^{x} k^p = \frac{B_{p+1}(x+1) - B_{p+1}(0)}{p+1}$$

Leo Moser proved that for another solution of EM equation two must divide p, see [1], what yields that $p+1$ must be odd and $B_{p+1}(0)$ with odd subscripts is equal to zero.

Lemma 3.2

$$B_{p+1}(x+1) - B_{p+1}(x) = (p+1)x^p \tag{2}$$

$$B_{p+1}(x+2) - B_{p+1}(x+1) = (p+1)(x+1)^p \tag{3}$$

Proof Relation of Bernoulli polynomials given by Whittaker and Watson, see [2], in general form is defined as $B_n(x+1) - B_n(x) = nx^{n-1}$.

Lemma 3.3 *Eq. (1) in combination with rearranged Eq. (2) gives a relation*

$$\frac{B_{p+1}(x+1)}{B_{p+1}(x)} = \frac{(x+1)^p}{(x+1)^p - x^p} \tag{4}$$

Proof Let us express $p+1$ from Eq. (2) as

$$\frac{B_{p+1}(x+1)}{x^p} - \frac{B_{p+1}(x)}{x^p} = p+1 \tag{5}$$

then by putting LHS of Eq. (5) in Eq. (1) we get

$$B_{p+1}(x+1) = (x+1)^p \left(\frac{B_{p+1}(x+1)}{x^p} - \frac{B_{p+1}(x)}{x^p} \right)$$

and after elementary rearrangements we can rearrange Eq. (1) to the form defined in Lemma (3.3).

Theorem 3.4 *The EM equation has other solution than trivial if and only if holds the following equation.*

$$\frac{B_{p+1}(x+2)}{B_{p+1}(x+1)} = 2 \tag{6}$$

$x, p \in \mathbb{N} \wedge x > 2 \wedge p > 1$.

Proof Let us rearrange Eq. (1) as

$$B_{p+1}(x+1) = (p+1)(x+1)^p \tag{7}$$

the RHS of Eq. (3) and Eq. (7) are equal so we can define

$$B_{p+1}(x+2) - B_{p+1}(x+1) = B_{p+1}(x+1)$$
$$B_{p+1}(x+2) = 2B_{p+1}(x+1)$$
$$\frac{B_{p+1}(x+2)}{B_{p+1}(x+1)} = 2$$

Lemma 3.5 *Let us define the set*

$$Z = \left\{ \frac{B_{p+1}(x_z+1)}{B_{p+1}(x_z)} = \frac{(x_z+1)^p}{(x_z+1)^p - x_z^p} \mid x_z, p \in \mathbb{N} \land p > 1 \right\}$$

containing Eq. (4) stated in Lemma (3.3) and the set

$$F = \left\{ \frac{B_{p+1}(x_f+2)}{B_{p+1}(x_f+1)} = 2 \mid x_f, p \in \mathbb{N} \land x_f > 2 \land p > 1 \right\}$$

containing all Eq.(6) with all possible non-trivial solutions x_f satisfying this equation then

$$F \subseteq Z$$

Proof The rules in the sets Z and F are sufficient to prove Lemma (3.5) since we are seeking other solution than trivial and for $x_f > 2 \land p > 1$. It is trivial to see that $F \subseteq Z$ since considering the fact that x_z, x_f are the variables of related elements and

$$\forall x_f : x_f = x_z - 1 \tag{8}$$

then the elements containing the variables x_z, x_f in relation (8) in both sets are equal and that proves Lemma (3.5).

Example 3.6 *Assuming that $x_f = 4$ would be the non-trivial solution. This example demonstrates the fact that $F \subseteq Z$ which follows from Lemma (3.5) since the elements in both sets containing the variables x_z, x_f in relation (8) are equal. In this case when $x_f = 4$ according to relation (8) $x_z = 5$ and related elements are equal (see below).*

x_z	Elements of set Z	x_f	Elements of set F
	$\frac{B_{p+1}(x_z+1)}{B_{p+1}(x_z)} = \frac{(x_z+1)^p}{(x_z+1)^p - x_z^p}$		$\frac{B_{p+1}(x_f+2)}{B_{p+1}(x_f+1)} = 2$
3	$\frac{B_{p+1}(4)}{B_{p+1}(3)} = \frac{(4)^p}{(4)^p - 3^p}$		
4	$\frac{B_{p+1}(5)}{B_{p+1}(4)} = \frac{(5)^p}{(5)^p - 4^p}$	4	$\frac{B_{p+1}(6)}{B_{p+1}(5)} = 2$
5	$\frac{B_{p+1}(6)}{B_{p+1}(5)} = \frac{(6)^p}{(6)^p - 5^p}$		
$\vdots$	$\vdots$		

Theorem 3.7 *There is no element in the set Z which is equal to two for $x_z > 2 \wedge p > 1$ and since $F \subseteq Z$ the EM equation does not have any other solution than trivial.*

Proof From Lemma (3.5) follows $F \subseteq Z$. It is clear that the elements of each set are an equations and if the elements containing variables x_z, x_f in relation (8) are equal these equations must be equal as well. Let us recall that every element in the set Z is defined as $\frac{B_{p+1}(x_z+1)}{B_{p+1}(x_z)} = \frac{(x_z)^p}{(x_z)^p - x_z^p}$ and every element in the set F is defined as $\frac{B_{p+1}(x_f+2)}{B_{p+1}(x_f+1)} = 2$ (see definitions of the sets in Lemma (3.5)). Since $F \subseteq Z$ and every element in the set F is equal to two in order to prove Theorem (3.7) it is enough to prove that no element in the set Z has an integral solution equal to two for $p > 1$ since it will be in contradiction. It is trivial to see that the expression $\frac{(x_z)^p}{(x_z)^p - x_z^p}$ has an integral solutions for $x_z > 1$ if and only if $0 < p < 2$ since by using the binomial expansion of the elements in the set Z we get

$$\frac{B_{p+1}(x_z+1)}{B_{p+1}(x_z)} = \frac{(x_z+1)^p}{(x_z+1)^p - x_z^p} = \frac{x_z^p + px_z^{p-1} + ... + 1}{px_z^{p-1} + ... + 1} = \frac{x_z^p}{px_z^{p-1} + ... + 1} + 1$$

where is clear that $(px_z^{p-1} + ... + 1) \nmid x_z^p$ for $p > 1$. In other words there is no element in the set Z which is equal to two for $p > 1$ and that is in contradiction with the fact that $F \subseteq Z$. On the basis of this facts we can state that there is only trivial solution of the EM equation when $p = 1$ as it follows from the basic formula of summation $\sum_{k=0}^{x} k^1 \equiv \frac{x*(x+1)}{2} = x + 1 \Rightarrow \frac{x}{2} = 1$ where x must be equal to two. All of the above-mentioned facts unconditionally prove Theorem (3.7) and at the same time the Erdös-Moser conjecture.

Example 3.8 *Let us assume that $x_f = 5$ is the non-trivial solution then Eq.(6) after substitution $\frac{B_{p+1}(7)}{B_{p+1}(6)} = 2$ holds for this x_f and this Eq.(6) is an element of the set F. Since $F \subseteq Z$ and thanks to the relation (8) we are able to define $x_z = 6$ and the element of the set Z as $\frac{B_{p+1}(7)}{B_{p+1}(6)} = \frac{(7)^p}{(7)^p - 6^p}$ (LHS of elements in both sets are equal so RHS must be equal as well) but this element is not equal to two for $p > 1$ which is in contradiction and therefore $x_f = 5$ can not be the non-trivial solution of EM equation.*

References

[1] L.Moser, On the Diophantine Equation $1^k + 2^k + ...(m-1)^k = m^k$, *Scripta Math. 19,* (1953), 84-88.

[2] E. T. Whittaker, G. N. Watson, A course of MODERN ANALYSIS, *Cambridge University Press 3rd edition,* (1920), 127.