We report a mathematical construction that is autological, universal and tautological. The autological property allows the derivation of the laws of physics from the construction, while the other two properties explain why the construction is universally and necessarily applicable to the world. The construction can explain the World at its most fundamental level up to and including the derivation of the familiar laws of physics. It is formulated as a statistical ensemble of feasible mathematics (an algorithmic information theory analog to statistical physics) and its domain is the set of all analytical statements of logic —here they take the role of the permissible facts of the world. The familiar laws of physics are emergent from the ensemble as entropic laws; this includes special relativity, general relativity, dark energy, the Schrödinger equation, the Dirac equation, quantum field theory, the speed of light as a maximal speed, and the space-time background. Furthermore, the construction provides tentative solutions in regard to the quantum measurement problem, the unification of physics and the arrow of time. In this context, the laws of physics are interpreted as the limits applicable to the feasible verification of the elements of the set of all analytical facts (they limit the actual facts of the world). The scope is universal (applies to the whole world), it is constructed tautologically (is indubitable) and it is autological (allows the derivation of the laws of physics) —thus, it is a tentative final theory.

1 Definitions

The construction that we introduce has three properties that will allow us to understand the World at its most fundamental level. The three properties are a) tautological, b) universal and c) autological. What is the role of each property?

(a) Tautological: First, it is well known that all tautologies are necessarily valid. Second, as it does not leave the first cause unexplained, a construction which is tautological does not suffer from the infinite regress problem common to axiomatic constructions. For these two reasons, the role of the tautological property is to make the construction philosophically indubitable.

(b) Universal\(^1\): With this property, the World can never be "out-of-scope" with respect to the construction. Thus, the construction applies to the whole World. The construction, because it is both universal and tautological, is thus an indubitable description of the World.

\(^1\) in the Turing sense of universal
(c) **Autological**: With this property, the World described by the construction is capable of self-explanation. It is able to recover the familiar laws of physics as its emergent properties. Thus, with all three properties, the construction is an indubitable *self-explaining* description of the World.

As the usage of the word autological is new in physics, let us now define it:

**Definition 1.0.1** (Autological). From linguistics, we have the following preexisting definitions:

- **(Linguistics)** Of a word, especially an adjective: having or representing the property it denotes, as opposed to "heterological".\(^2\)

- **(Linguistics)** An autological word (also called a homological word or autonym) is a word that expresses a property that it also possesses (e.g., the word "short" is short, "noun" is a noun, "English" is English, "pentasyllabic" has five syllables, "word" is a word). The opposite is a heterological word, which does not apply to itself (e.g., "long" is not long; "monosyllabic" has five syllables).\(^3\)

These definitions of "autological" are used in logic and philosophy and they apply to linguistics. In the context of a physical system, we can say that its description is autological if:

- **(Physics)** The description exhibits the same properties (e.g., limits, symmetries, etc.) as the system it describes.

For short, we may refer to a description of the world that is both tautological and universal as *Tauto-universal*, and to a formal theory that has all three properties as *auto-tauto-universal*.

We will show that in the case of a tauto-universal description of the world, the consequence is the autological emergence of the laws of physics from the description. With these three properties, the full circle is completed and the question, "Why is the world what it is?" has a short answer: its laws are an autological consequence of its tautological and universal description. We will review these terms in greater detail as we formalize the construction. But first, let us review the prior art and the differences from the auto-tauto-universal (ATU) approach.

### 2 Prior art

The construction of a modern physical theory usually proceeds as follows. First, a background to reality is assumed. This is usually taken to be space-time, with one dimension of time and three or
more dimensions of space. Symmetries are then associated with this space-time. According to Noether’s theorem, each symmetry gives rise to a conserved quantity that becomes a law of physics. Along with potentially complicated initial conditions, these symmetries are responsible for describing the state of the universe over time and deciding the facts.

\[
\begin{align*}
\text{Background (Space-time)} & \Downarrow \\
\text{Symmetries (Lorentz, Poincaré, Gauge, ...)} & \Downarrow \\
\text{Laws (Special relativity, general relativity, ...)} & \Downarrow \\
\text{Initial conditions} & \Downarrow \\
\text{Facts (The chicken crossed the road)} & \\
\end{align*}
\]

Figure 1: The logical flowchart of a modern theory of physics.

Here, the background is taken as axiomatic—assumed to be true but not proven from first principles. The background leads to symmetries—some of which are taken as axiomatic—and those symmetries lead to laws. The laws and initial conditions decide the facts of the universe over time (see Figure 1).

As appealing as this picture might be, the method has logical weaknesses. First, as it is not tautological, the method cannot explain that which it assumes. In the example given, this would be the background and some of its symmetries. Second, it is quite difficult to probe the universe at all interesting scales to identify all of the correct symmetries.

A theory of everything (ToE) that is constructed with even just one axiom cannot explain everything. At best, it can explain everything but the one axiom it assumes. As a result, it cannot successfully defend itself against all philosophical attacks targeting the validity of its axioms. An adherent of an axiomatic ToE will always wonder why it and not something else is actual. Ultimately, it cannot present reality as indubitable. This ambiguity is unacceptable.

We have found that these weaknesses can be resolved by rethinking the process via the ATU program. Without axioms, a logical system becomes tautological. In traditional philosophy, this is believed to be too limiting, but here we have found a way to construct a universal tautological system. It is from this discovery that autology
becomes possible.

Under the program, any philosophical attacks against the theory are negated by its tautological property and thus cannot succeed. Thus, the World and its structure can be presented as indubitable. This makes the theory final in the sense of Steven Weinberg4.

Let us introduce the starting point of the program, and then we rearrange the logical flowchart appropriately.

3 Introduction

Consider an idealist observer pondering about the world. For brevity, let us call this observer Alice. We ask, “Given no experimental knowledge of any kind, can Alice correctly describe the world in which she is embedded?” To illustrate, suppose that Alice believes that her physical senses (sight, sound, touch, etc.) are unreliable. As such, she would not trust the process of falsification. Thus, acting rationally, she should only believe in logical conclusions that are entirely indubitable. To answer the question rigorously, we consider the problem in only the context of formal systems.

Definition 3.0.1 (Formal system). A formal system is any well-defined system of abstract thought based on the model of mathematics.5 In formal logic, a formal system consists of a formal language and a set of inference rules that are used to derive an expression from one or more other premises (to make conclusions) that are antecedently supposed (axioms) or derived (theorems). The axioms and rules may be called a deductive apparatus. A formal system may be formulated and studied for its intrinsic properties, or it may be intended as a description (i.e., a model) of external phenomena.6

Within these restrictions, Alice can still deduce the correct structure of the world by producing a purely mathematical construction that has the ATU properties. Indeed, the ATU program is quite possibly her only way to do so. To introduce the program, let us compare it to its scientific analogue:

<table>
<thead>
<tr>
<th>Scientific</th>
<th>Auto-tauto-universal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Experimental evidence</td>
<td>Tauto-universal statements</td>
</tr>
<tr>
<td>↓</td>
<td>↓</td>
</tr>
<tr>
<td>Falsifiable theory explaining all known evidence</td>
<td>Autological theory resulting from a group of tauto-universal statements</td>
</tr>
<tr>
<td></td>
<td>Optionally, falsifiability is recovered by corroborating the theory using experimental evidence</td>
</tr>
</tbody>
</table>
In the scientific case, we first gather experimental evidence, then produce (or attempt to produce) a theory capable of explaining all known experimental evidence.

Under the ATU program, we carefully construct a group of statements such that the group is universal and tautological, then we recover the laws of physics via autology.

The correctness of the program most directly depends on the completeness of the group of tautological statements. The theory is tautological only if the group of tautologies is universal. Essentially, we have found a way to construct a minimalist formal system that is both tautological and universal. The formal system is obtained by stripping first-order logic of its deductive apparatus. We call this the formal system Tauto-universal logic, and we give its construction in section 5.3. The following tautological statements are theorems:

\[
\begin{align*}
tautology_1 &:= [(\text{deductive-apparatus})_1 \vdash \text{theorem}_1] \quad (3.0.5) \\
tautology_2 &:= [(\text{deductive-apparatus})_2 \vdash \text{theorem}_2] \quad (3.0.6) \\
& \vdots \\
tautology_n &:= [(\text{deductive-apparatus})_n \vdash \text{theorem}_n] \quad (3.0.7) \\
& \vdots
\end{align*}
\]

Each tautology is simply a claim that a certain deductive apparatus is able to prove a certain theorem. The list contains all theorems for all deductive apparatuses, and thus, the formal system of Tauto-universal logic is actually a universal function in disguise. This concept is key: Tauto-universal logic only contains tautologies, yet its tautologies are sufficiently general to describe a universal function. Therefore, it can describe any system (including those of arbitrary logical complexity) using only tautologies.

Special care must be taken when constructing Tauto-universal logic and when listing its tautologies. Notably, we must sacrifice the philosophical notion of a synthetic fact, and thus, we must take the position that all facts are provable within and up to some deductive apparatus (i.e., they are analytical). We discuss this in section 5.8. From this, we can adopt a working definition of a possible world and then of the actual world.

**Definition 3.0.8 (Possible world).** A possible world is a world in which each of its facts is an element of the set of theorems of Tauto-universal logic. Thus, the theorems of Tauto-universal logic restrict what the facts of any possible world can be; they are the permissible facts.

**Proof.** The proof is trivial. As Tauto-universal logic contains all facts for all deductive apparatuses (it is universal), then necessarily, the
world cannot have a fact that is not a theorem of Tauto-universal logic.

Which world is the actual world? Using principles of feasible mathematics (an algorithmic information theory (AIT) analog to statistical physics), we found that the notion of an actual world can be introduced and corresponds to a statistically emergent behavior over the possible arrangements of feasibly-verifiable facts. In section 6, we explicitly derive the actual world by constructing such a statistical ensemble. Then, in part II, we show how the laws of physics are emergent from the ensemble. Since the result is statistical, we expect fluctuations around the average. These fluctuations connect to the laws of quantum physics. Indeed, this is how we recover the Schrödinger equation, the Dirac equation, and the Feynman path integral formulation of quantum field theory. We adopt the following definition of the actual word:

**Definition 3.0.9** (Actual world). The actual world is the result of a statistically emergent behavior over the ensemble of feasibly-verifiable facts. Using the tools of feasible mathematics, it is described as a partition function over the theorems of Tauto-universal logic — from which the laws of physics are emergent. The actual world is given by equation 4.0.1.

The general interpretation of the derivation is explained in the context of statistical physics. Consider an ideal gas, which is a typical example in statistical physics. In this scenario, we construct a partition function to describe the energy of the gas molecules. When the number of particles is very large (e.i., in the thermodynamic limit), we recover the ideal gas law as a macroscopic law that is emergent from the microscopic behavior of the molecules.

The interpretation presented here is similar. We consider all possible theorems of Tauto-universal logic as the possible facts of the world. These analytical facts play the role of the "microscopic" description of the world. Using feasible mathematics, we then recover the "macroscopic" laws that are emergent from the properties of these analytical facts. As the list of facts applies to the entire world, the laws that emerge thus apply to the whole universe.

The autological property connects to the other two (universal and tautological) via feasible mathematics. As we apply feasible mathematics to a universal and tautological theory (e.g. we make Tauto-universal logic feasible), the theory becomes autological:
The derivation of the laws of physics is not a mere coincidence but a direct result of the autological property of the construction (section 5, 6 and 7). Rigorously deriving these relationships is the main goal of this manuscript.

4 Mathematical formulation

At the end of the proof (part I), we will find that the function \( Z \) is the ATU construction:

\[
Z = \sum_{q \in Q} e^{-Fx(q)} - Wt(q)
\]  

The function describes a statistical ensemble of analytical facts \( q \) from within the set \( Q \) of all possible analytical facts. Each possible analytical fact is a micro-state of the statistical ensemble. Furthermore, each is statistically weighted by the length of its proof \( t(q) \) and by the length of its description \( x(q) \). The following function is the probability that the analytical fact \( q \in Q \) is actual in the universe.

\[
p(q) = \frac{1}{Z} e^{-Fx(q)} - Wt(q)
\]
The world is an autology derived from all tautologies

\[ Z = \sum_{q \in Q} e^{-F(x(q)) - W_t(q)} \]

\[ G = \sum_{x \in X} e^{-\beta E(x) - \gamma V(x)} \] (4.0.3)

This similarity should be reassuring as it shows that the program has not caused the apple to fall too far away from the tree of traditional physics. Here, the fact takes the role of that simplest and most fundamental constituent of reality. The universe is, quite literally, the sum of its facts.

The laws derived from \( Z \) limit the scope of the facts that can be verified in the world (section 6) — and they correspond to the laws of physics (part II)!

Remark: The function \( Z \) models a single fact. To model a world with more than one fact, as \( Z \) is a micro-canonical ensemble, it suffices to consider \( N \) independent systems each described by \( Z \)—one for each fact of the world. Alternatively, \( Z \) can be extended to a grand canonical ensemble by adding the usual \( \mu N \) thermodynamic conjugate pair. In this case, we obtain a world with a variable amount of analytical facts for a cost per fact of \( \mu \). However, we found that the micro-canonical representation was sufficient to recover the laws of physics. Thus, the implication is that the laws of physics are valid independently from the number of facts described by them. In what follows, we will examine only \( Z \) as micro-canonical while keeping in mind that it can be extended to \( N \) independent systems without changing the laws derived from it. For \( N = 1 \), we interpret each emergent law as a limit applicable to a fact \( q \in Q \). When and if we need to describe a world with many facts, we will take the thermodynamic limit by posing \( N \gg 1 \).

Part I

Proof

The main steps of the proof are:

Section 5: Listing all tautologies

(a) We will rigorously define Tauto-universal logic.

(b) We will further develop the intuition behind the proof.

(c) We will list all analytical facts as tautologies of Tauto-universal logic.
Section 6: **Feasible mathematics**

(a) We will derive feasible mathematics as the theory which allows the statistical derivation of laws from a set of logical statements. Feasible mathematics is how autology is formalized.

Section 7: **Model of the World**

(a) We will present a model of the world in which facts compete with other facts for a share of the explanatory pie.

(b) We will show how feasible mathematics connects the logical verification of analytical facts to the main notions of statistical physics.

(c) We will explain how to connect the quantities of the partition function to known physical quantities.

5 **Listing all tautologies**

Consider brute facts:

**Definition 5.0.1 (Brute Fact).** A statement of a language that is true without formal justification. In physics, the axioms of the eventual theory of everything along with any possible initial conditions would generally be considered to be the brute facts of reality. Brute facts are not justified by any other more fundamental principles.

A tauto-universal description cannot be contingent on brute facts, or it would cease to be tautological. The key to constructing one is to avoid importing any brute facts while maintaining the universality of the theory. This can be done by importing all theorems for all formal theories as tautologies of a minimalistic formal theory. We will perform a preliminary philosophical exercise to build the intuition, and then we will show how to do it explicitly.

5.1 **The universal doubt method of Descartes**

We will recall the philosophy of René Descartes (1596-1650), the famous 17th-century French philosopher most well-known for his derivation of *cogito ergo sum* - I think, therefore I am. As we will soon see, we can guarantee the elimination of all brute facts when we modernize his universal doubt method into a formal logic system such as first-order logic. But first, let us recall what the universal doubt method is.
Descartes’ main idea was to come up with a test that every statement must pass before it will be accepted as true. The test will be the strictest test imaginable. Any reason to doubt a statement will be a sufficient reason to reject it. Then, any statement which survives the test will be considered indubitable.

Using this test, and for a few years, Descartes rejected every statement he considered. The laws and customs of society, as they have no logical justifications, are obviously the first to be rejected. Then, he rejects any information that he collects with his senses; vision, taste, hearing, etc, on the grounds that a "demon" (think hallucinogens) could trick his senses without him knowing. He also rejects the theorems of mathematics on the grounds that axioms are required to derive them, and such axioms could be wrong. For a while, his efforts were fruitless and he doubted if he would ever find an indubitable statement.

But, Eureka! He finally found one which he published in 1641. He doubts about things! The logic goes that if he doubts everything, then it must be true that he doubts. Furthermore, to doubt he must think and to think, he must exist (at least as a thinking being). Hence, cogito ergo sum, or I think, therefore I am.

5.2 Repeating Descartes’ result in formal logic

To repeat Descartes’ results in formal logic, it is not enough to simply import the syntax of the argument; one must instead replicate the methodology. Let’s see why that is by first investigating how importing the syntax fails. Indeed, most criticisms against the cogito are constructed as an attack against the syntax of the argument once imported into some system of formal logic. Consider two well-known criticisms:

- Hidden Premises: Bertrand Russell argues that the Cogito contains two separate premises: 1) “There is a thought going on” and 2) these thoughts are attached to something called “me”. The first can be proved, but the second isn’t logically necessary.
- Circularity. Descartes is trying to prove the “I” exists, but his first premise is “I doubt”. It is an invalid argument to assume the truth of the conclusion in one or more of the premises. As Leibniz wrote: “I am thinking is already to say I am”.

—Summary by: ([Thinkyt, 2012])

In each of these cases the author imports the syntax of the cogito (but not its meaning) into a preferred logical system, then finds the
imported argument to be lacking in some way. These criticisms are usually countered by claiming that the cogito can be expanded above and beyond all possible systems of logic; for instance, one would counter with: "one cannot talk about circularity or find a hidden premise unless one first exist to talk about it! Thus, the (idea of the) cogito holds even if its implementation within your chosen logical system fails."

But, if the cogito holds, why is it so difficult to import it into a formal system of logic such that it retains its value? Can we formalize the intuition that the cogito is more fundamental than the logical systems it is imported in, and if so, how? As we will see, the solution is to focus on the methodology (universal doubt method) instead of the syntax (cogito ergo sum).

5.3 Tauto-universal logic

We now refocus our interest to Descartes’ universal doubt method itself and not so much in the cogito. To identify the theorems of the theory which describes the universe, we will repeat Descartes’ universal doubt method within the context of a formal logic system. The method will produce a minimal set of rules whose theorems are tautologies. Since the resulting theory will also be universal (section 5.7), we named it: Tauto-universal logic.

Tauto-universal logic is, in many ways, similar to the constructivism project in mathematics but taken to the extreme. We select first-order logic as our starting point. Then, as we do not know which axioms are the true axioms of the universe, we remove all formal axioms from first-order logic on the ground that they carry doubt. Then, we further remove all rules of inference with the exception of conditional introduction. This method parallels Descartes’ universal doubt method within first-order logic. The main argument is that if we remove from first-order logic all formal axioms and all rules of inference which could potentially be controversial, then whatever theorem is left will surely be indubitable. The result is a system of logic which, essentially, does not deceive its user.

Like Descartes with the cogito, we will also obtain statements that cannot be doubted of, but since we have formalized Descartes’ method within first-order logic, the indubitable statements obtained will be logic statements and are therefore mathematically usable. Specifically, the indubitable statements obtained are the theorems of Tauto-universal logic.

To write sentences in a clear and unambiguous manner, Tauto-universal logic preserves the syntax of first-order logic but does not retain its rules of inference (with the exception of conditional intro-
As only conditional introduction remains, let us recall its definition.

**Definition 5.3.1 (Conditional introduction).** *Conditional introduction formalizes the idea that proving a theorem using a set of assumptions is valid within these assumptions. It shows that if by assuming \( A \) one can show that \( A \vdash B \), then \( A \rightarrow B \) is a theorem of the logical system. It is often considered one of the most obvious rules of inference of logic, as without it we cannot extend a logical system with new axioms/assumptions. Using conditional introduction, we can start from seemingly nothing and rebuild any of the familiar logic systems such as Peano’s axioms (PA) or set theory (ZFC) by assuming their axioms.*

Why keep conditional introduction? For the simple reason that using it does not introduce doubt but removing it would. It is the only standard rule that has this property. If \( A \) is an axiom, then \( A \) is true—but what if we suddenly decide to doubt \( A \). Then we can still use \( A \) but to do so we must assume it true or assume it false, then use it. Doubting of an axioms consists of conditionally introducing it instead of positing it. What if we tried to completely eliminate conditional introduction as a rule? The cost of eliminating conditional introduction is an infinite doubt; to do so we must set all possible axioms to either true or false apriori so as to leave no places for statements to be conditionally introduced. Thus, a logical system with no doubt is one whose only rule is that of conditional introduction.

For example, let’s consider another rule, say the law of excluded middle. Adopting this rule in the foundation of the theory would increase doubt as it is impossible to determine apriori if this is a valid rule of the universe or not. However, conditionally introducing it would be fine. Indeed, in the latter case, we would say "we can prove by contradiction that \( \sqrt{2} \) is irrational conditionally on the law of excluded middle". It only affects the branch of the tree under which it is assumed and not the whole system.

In Tauto-universal logic, no theorem stands on its own. Any theorem must include, within its description, the list of assumptions that are required to prove it. The user of Tauto-universal logic is always reminded that the theorems that he proves are of the form ‘Assume \( A \), then \( A \) proves \( C \)’. Hence, via conditional introduction, \( A \implies C \) is a theorem, but \( C \) by itself never is. Tauto-universal logic can be interpreted as the starting point of all logical work—it is the state of mind a logician is in before having morning coffee and selecting a specific system of axioms to work with. As a result and compared to other logic systems, it more accurately represents reality as it reflects the full freedom available to the logician to select any set of axioms prior to working.
5.4 Similarities between the cogito and Tauto-universal logic

An interesting pattern emerges as a result of the similarities between the 'I think' in Descartes' cogito and 'conditional introduction' in Tauto-universal logic. Indeed, in Tauto-universal logic, conditional introduction allows one to formulate logical proofs for arbitrary logical systems —similarly, in Descartes' cogito, the ability to think allows one to formulate arbitrary thoughts:

<table>
<thead>
<tr>
<th>Tauto-universal logic:</th>
<th>conditional introduction</th>
<th>[\Rightarrow] I can formulate proofs for arbitrary formal systems</th>
</tr>
</thead>
<tbody>
<tr>
<td>Descartes' cogito:</td>
<td>I think</td>
<td>[\Rightarrow] I can formulate arbitrary-thoughts</td>
</tr>
</tbody>
</table>

Furthermore, the rule of conditional introduction is the only rule of Tauto-universal logic (we have eliminated all other axioms or rules from it) —similarly, in Descartes' cogito, one's ability to think is the only certain fact (we can doubt of everything else):

<table>
<thead>
<tr>
<th>Tauto-universal logic:</th>
<th>I attempt to doubt of all formal axioms</th>
<th>[\Rightarrow] Conditional introduction cannot be eliminated because doubting of an axiom converts it to a conditionally introduced statement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Descartes' cogito:</td>
<td>I attempt to doubt of everything</td>
<td>[\Rightarrow] I cannot doubt that I doubt</td>
</tr>
</tbody>
</table>

But here is where it gets really interesting. The main result of Descartes' cogito is that "I think" is irrefutably proven; thus 'I exist' follows. Considering the similarities, what can we expect the main result of the ATU program to be? Should we also expect to find a proof of 'I exist' (or some formal equivalent) within it —and if so, what would the equivalent of "I exist" be in a formal system?

To answer the question we must first understand what Tauto-universal logic gives us. It gives us the list of all statements which cannot be doubted of (are indubitable) in the World. Surely the properties and/or symmetries of this group of statements must interesting in some way —e.i. it would be quite the prank if they turn out to be completely irrelevant. Indeed, it may be surprising to hear but the properties of this group of tautologies are identical to the laws of physics.

We ask rhetorically; why would the laws of physics have an origin in the group of all indubitable statements of the World?
Main result of the cogito:  
I think $\implies$ I exist

Main result of the ATU program:  
Tauto-universal logic $\implies$ A mathematical construction describing the origins of the laws of physics from that which is indubitable in the World

From this program, the World can be understood as an autology derived from the tautologies of Tauto-universal logic (function $Z$, equation 4.0.1). As the construction is mathematical, the main result of the program is more much specific than simply: "I exist". What justifies this increased specificity? First, Tauto-universal logic is a reproduction of Descartes’ cogito using the modern tools of formal logic (e.i. it is mathematically precise). Second, Tauto-universal logic is a constructive mathematical theory (e.i. it does not adopt the law of excluded middle in its deductive apparatus thus every proof that it admits must construct the subject of the proof). Therefore, if Descartes’ cogito is sound, then the reproduction of the cogito within formal logic should preserve the scope of Descartes argument—but, with the added benefit of increased mathematical precision.

Thus the statement ‘I exist’ of the cogito, will, in Tauto-universal logic, be replaced by a mathematical construction describing the origins of the laws of physics from the properties of its tautologies.

5.5 Discussion on metaphysics

To build the intuition, let us have a small conversation on metaphysics. The goal of this section is to construct a bridge from metaphysics to physics. The completion of the exercise will identify all theorems applicable to the universe. To iron out the subtleties we will present, in the long-standing tradition of philosophy, a hypothetical dialogue about the thesis. The dialogue is based on a number of real conversations which has been edited and combined to remove repetitions, to accelerate the flow and to help illustrate the points being made.

Bob: - I believe in empiricism. To derive the laws of physics, one must make observations. Without these observations, there is no way to know which of many possible worlds is the actual world. For example, is the geometry of the universe Euclidean or hyperbolic? Is the speed of light maximal? Does the microscopic world obey the Schrödinger equation? Etc. Pure reason alone cannot prove these to be actual. Only continual observations followed by refinements or falsifications can improve our degree of confidence in a scientific theory.

I understand your point of view, but I believe I have found a bridge between metaphysics and physics that allows one to obtain

---

7 Specifically, when Bob’s dialogue is taken verbatim from a conversation with Toid Boigler, it will be side-noted with the initials TB.
indubitable knowledge about the universe. I will try to explain the bridge from the following angle. First, let’s assume that the cogito is true: I think, therefore I exist. Do you agree with the cogito?

Bob: - Yes.

Then, for I to exist, the universe must be restricted in some way. At the very least, it must be such that it does not contradict the existence of thought. We have now essentially reformulated the anthropic principle as an extension to the cogito. I think, therefore I exist - and to exist, I must actually exist in a universe capable of supporting such existence. Would you agree that this argument rules out some universes?

Bob: - Fair enough, yes - it rules out the [...] universes incompatible with the existence of thought [...].

OK. From that, we already have a slight connection between metaphysics and physics. An argument from pure reason, the cogito extended with the anthropic principle, can be used to place restrictions on what the universe can be. As it contains very little information, the restriction is very loose, but it is nonetheless a restriction.

Bob: - I agree that the anthropic principle rules out universes that are not capable of producing an observer. But, a scientific theory should make precise and falsifiable predictions and the anthropic principle is not sufficiently specific for that.

Now we enter the core of the argument. We will use Tauto-universal logic to improve the specificity of the anthropic principle. Each theorem of Tauto-universal logic that we can provide will serve to further restrict what the universe is. For example, using my mind I can prove the sentence "PA implies that two plus two equals four", and since my mind is in the universe, then the sentence must be a theorem of both my mind and the universe. This is how we find the theorems applicable to the universe. We have now restricted the universe by two statements instead of one. So the previously poorly defined bound is now slightly better defined. Agree, or disagree?

Bob: - Well, you want the phrase "theorems [applicable to the] universe" to be telling us something about the physical world; to put it in your own words, "...this is how we bridge metaphysics to physics." But how does this work? If "true in the universe" just means provable in PA or ZFC or whatever (as you seem to have just said), how does this provide any link with physical reality at all?  

Hold on, it appears that you have missed a subtlety. "Provable in the universe" means provable in the Tauto-universal logic system I defined earlier. If you use another logic system than Tauto-universal logic (such as PA) then the argument does not work. If you use PA
or ZFC, then the theorems rely on the axioms PA or ZFC. As the universe might not have the axioms of PA or ZFC, we cannot prove that PA’s or ZFC’s theorems are indeed the theorems of the theory which explains the universe. However, Tauto-universal logic teaches us that the theorems of the universe are not "two plus two equals four", they are "Assume PA, then two plus two equals four". The "Assume PA" prefix is what the subtlety is all about. "Assume PA, then 2+2=4" is a theorem of the universe because it is true that in the universe if you assume PA you can prove (within PA) that 2+2=4. You can easily do the exercise in your head to prove that it can be done in the universe.

- Bob: OK, so you want to think of all mathematical proofs as conditional - if certain axioms hold, then certain consequences follow. Fine. How does that provide any connection with physics or the physical world?10 TB

Yes, mathematical proofs that are explicitly conditional on assumptions derived exclusively from conditional introduction are theorems of the universe. Whereas those that do not meet this condition are theorems of their respective logic system. For example, "2+2=4" is a theorem of Peano’s axioms. But, "Assume PA, then 2+2=4" is a theorem of the universe. So all worlds where "Assume PA, then 2+2=4" is not true are ruled out.

- Bob: This is one point where I am a little confused. Pure logic (call it Tauto-universal logic if you want) guarantees that PA implies 2+2=4. So it’s hard to see what worlds it rules out - unless you mean worlds in which there is a mind, but that [this] mind is too [primitive] to realize that PA implies 2+2=4. Is that the kind of world that you take to have been ruled out? If so, I am OK with what you have said.11 TB

Yes, it is part of what I am ruling out. Generally speaking, I am ruling out any world which does not embed universal reason. I also rule out worlds for which logic would be incomplete and worlds which would contradict logic by say, letting you prove any theorems regardless of the axioms that you assume.

Since our mind is able, in principle, to explore all branches of Tauto-universal logic and since the universe must embed our mind, we can precisely identify all the theorems of the universe: The ultimate theory which describes the universe must have, as its theorems, all theorems of Tauto-universal logic.

Bob: Here I really don’t know what you mean, unless you are just saying that there are no ‘violations’ of [Tauto-universal] logic in the world. If that’s what you mean, I’m happy with that claim.12 TB

I am indeed claiming that there are no violations of Tauto-universal logic in the universe, but I am also claiming something additional.
What I am claiming is that we can use Tauto-universal logic and the anthropic principle to completely restrict reality to a single solution. Think of it as a totalitarian version of the anthropic principle.

Consider the following; each theorem of Tauto-universal logic that we supply can be used to restrict the universe further. In principle, we can supply arbitrarily many theorems. PA has $2+2=4$ as a theorem, but it also has $2+3=5$, etc. Then, ZFC also has infinitely many theorems as well. If we keep supplying theorems, we will eventually supply all theorems for all branches of Tauto-universal logic\(^{13}\). Furthermore, as Tauto-universal logic is universal, all possible theorems for all possible sets of assumptions will eventually be supplied. No branches of theorems will be left out by the process.

As a result, we will have maximally restricted what the World can be. Indeed, the World cannot be simpler than Tauto-universal logic because that would mean leaving a theorem out (but we already said the work will eventually supply all possible theorems so none can be left out). What about complexity - can the World be more complex then Tauto-universal logic? The World cannot be more complex than Tauto-universal logic either because that would mean the World has theorems that Tauto-universal logic hasn’t (but this cannot be the case because Tauto-universal logic is universal —it embeds all possible theorems within its branches).

Therefore, as the World is restricted both from the perspective of increasing its complexity as well as reducing it, the bound cannot be improved furthermore. The method herein described fully restricts the World to a single solution.

Bob: *I am not sure [I see where you are going with this]. I’m happy to say that the universe must allow for the possibility of a mind that, in principle, can verify all the theorems of [Tauto-universal] logic. But what follows from that?\(^{14}\)*

Usually, a theory is first defined by a set of axioms, then the theorems follow from them. In our present situation, we have the reverse; we have a list of theorems but we do not have the theory which neatly explains such theorems. Nonetheless, the specification of the theory is complete. To compress it to a neat package of axioms, we will use a meta-theory such as first-order arithmetic or set theory to study Tauto-universal logic. The process is somewhat reminiscent of how we have invented and are now using mathematics to describe the physical universe we live in.

Bob: *What I mean is that I don’t understand [the connection to physics] at all. What we have now are all the tautologies of [Tauto-universal] logic. What connection is there between that and a physical theory?\(^{15}\)*

The connection is that, for the reasons stated, the theorems of
Tauto-universal logic are the theorems of the universe. Hence Tauto-universal logic, as its theorems are identical to those of the universe, must be autological with respect to the universe (e.i. it must be governed by the same laws).

Bob: You say "The theorems of [Tauto-universal] logic are theorems of the universe." If by this you just mean that the universe obeys the laws of [Tauto-universal] logic, then yes, I agree. Then you say "Hence Tauto-universal logic, as its theorems are identical to those of the universe, [must be governed by the same laws]." This seems clearly wrong. It is true in the universe that there [is the law of gravity]. That there [is the law of gravity] is, however, not a theorem of [Tauto-universal] logic. Thus, the theorems of [Tauto-universal] logic do not fully describe the universe.\textsuperscript{16}

There is a misunderstanding. I am not claiming that the laws of the universe can be found within Tauto-universal logic under a certain set of assumptions. What I am claiming is that Tauto-universal logic is autological; e.g. it possesses the properties of reality because it is both universal and tautological — thus, is itself a necessarily complete and valid description of reality.

Studying Tauto-universal logic with a meta-theory is equivalent to trying to make sense of the universe using mathematics. Except, as the theorems of Tauto-universal logic are precisely defined and listed, correctly deriving the proper laws and symmetries from them will be easier then it is in the physical case.

Bob: Can you spell out the [connection] you have in mind [between the theorems of Tauto-universal logic and the laws of physics]?\textsuperscript{17}

Yes, let us explicitly enumerate the facts of reality, then we will be ready for step 2.

5.6 Enumeration of all possible facts of the World

We can make this rigorous using the standard tools of formal logic. Let us do a recap of formal logic.

Definition 5.6.1 (Symbol). For our purposes, a symbol is a reproducible mark or shape that can be distinguished from other marks or shapes. For example, 0 and 1 are the symbols of the binary language.

Symbols can be grouped as

Definition 5.6.2 (Sentences). A sentence is a group of symbols written one after the other. For completeness, we consider groups of one symbol to also be sentences. The absence of symbols will be the empty sentence \( \in \). A sentence is of finite length. For example, 000110 and 111 are sentences of the binary language.
A notion of truth can be imported into a formal theory. To do so, a list of rules is first defined:

**Definition 5.6.3 (Rules of inference).** Typically, a rule of inference will be truth-preserving. A rule of inference takes sentences as input, then produces a conclusion. For example, \((p \land q) \implies p\) is a rule of inference in propositional logic.

**Definition 5.6.4 (Axioms).** A sentence that is considered true by definition.

**Definition 5.6.5 (Theorem).** A sentence that is proven to be true as a result of a valid proof. A valid proof is obtained by applying in succession either an axiom or a rule of inference to the previous line until the sentence is recovered.

**Definition 5.6.6 (Deductive apparatus).** The axioms plus the rules of inference of a formal system constitutes the deductive apparatus of a formal system.

We can convert all theorems for all deductive apparatus to a list of tautologies as follows: First, we filter the sentences of a language based on their provability within a certain deductive apparatus, then we construct them as tautologies. More precisely, we perform these steps:

\[
\begin{align*}
\text{IF} & \quad \text{deductive-apparatus}_a \vdash \text{sentence}_b \\
\text{THEN} & \quad \text{sentence}_b :\iff \text{theorem}_b \\
\text{AND} & \quad \text{tautology}_c ::= \left[\left(\text{deductive-apparatus}_a \vdash \text{theorem}_b\right)\right]
\end{align*}
\]

What we mean is that IF a certain deductive apparatus is able to prove a sentence (5.6.7) THEN the sentence is a theorem of the deductive apparatus (5.6.8). Thus, it is tautological to claim that the deductive apparatus proves the theorem (5.6.9).

We can abstractly create the list by applying the process to all sentences of a language. The list is as follows:
We note that each tautology so defined embeds the set of axioms and the rules of inference (the deductive apparatus) which permits the verification of the theorem it pertains to. The list allows repetitions of the theorems and of the deductive apparatuses. For example, a certain theorem might be provable from multiple deductive apparatuses, and a certain deductive apparatus might prove many theorems.

To show that this list can indeed be produced, we consider that we can enumerate all sentences of a language. Then, as the tautologies of the list are sentences of a language, they too are part of this enumeration. For the proof, we select the binary language with the symbol 0 and 1. As every language can be encoded in binary, the choice of language has no impact on the generality of the derivation. We can enumerate all sentences of the binary language in short-lex (sorted by length and then alphabetically) as:

\[
\begin{align*}
0 \quad & (5.6.14) \\
1 \quad & (5.6.15) \\
00 \quad & (5.6.16) \\
01 \quad & (5.6.17) \\
10 \quad & (5.6.18) \\
11 \quad & (5.6.19) \\
000 \quad & (5.6.20) \\
& \vdots
\end{align*}
\]

Most of these sentences will be nonsensical, but once in a while, a sentence does make sense within the chosen formal theory. We can see this more clearly if we consider listing all sentences of the English language including special characters. We list them as \(a, b, c, \ldots, aa, ab, \ldots, ba, bb, \ldots\). Eventually, all sentences, all books, and all manuscripts will be enumerated. Then, as the English language can be encoded in binary, the binary language is equivalent in scope to the English language.
The next step will be to extract from the enumeration only the valid tautologies and to eliminate non-facts and nonsensical sentences. To do so, we will have to think of facts as abstract computer programs. This will allow us to define a computational cost required to build the list.

### 5.7 Facts as abstract computer programs

To distinguish the theorems from the sentences, we define a function \( F : S \rightarrow \mathbb{Z}_2 \). The function returns 1 if the sentence \( S \) is a theorem and 0 otherwise. This function is a universal function. Gregory Chaitin\(^{18}\) has shown that for a universal formal system, such a function exists but is non-computable.

<table>
<thead>
<tr>
<th>( S )</th>
<th>( F(S) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>00</td>
<td>1</td>
</tr>
<tr>
<td>01</td>
<td>0</td>
</tr>
<tr>
<td>:</td>
<td>:</td>
</tr>
</tbody>
</table>

The theorems of Tauto-universal logic can be interpreted formally as computer programs running on a universal Turing machine. Indeed, it suffices to consider that the program script is the deductive apparatus and the program input is the theorem. Then the universal Turing machine takes as input the concatenation of the program script (deductive apparatus) with the program input (theorem) and outputs the proof. It halts once the proof is complete (the theorem is verified from the axioms) or runs forever (the theorem is not provable from the axioms).

As Tauto-universal logic describes a universal system, the function \( F(s) \) associated with it is non-computable. Non-computable systems do have a computable sub-domain. To properly study this sub-domain, we will introduce feasible mathematics and use it in step 2 of the proof. But before we show that, let us have a small discussion on the nature of facts.

### 5.8 Discussion on the nature of facts

What then is a fact in this context? This construction is likely to be at least somewhat controversial amongst philosophers because it eliminates a commonly used class of facts from our definition of phys-
ical reality. Specifically, philosophy considers the existence of synthetic facts. These are facts related to the current world. For example, "Julius Caesar was the emperor of Rome" relates to the present world as it has unfolded historically, but is unlikely to be true for all possible world. Thus, it is a synthetic fact but not an analytical fact.

In the present construction of Tauto-universal logic, all such synthetic statements are converted to analytic statements. Thus, there is no place left for synthetic statements as carriers of truth. Indeed, our understanding of any facts is now always contingent on a set of assumptions. The validity of the fact can be verified based on the assumptions, but cannot be proven to stand on its own as a brute fact. The claim of the existence of Julius Caesar must be corrected to include the assumption which permits the deduction; somewhat informally we can say "If we find evidence of an emperor named Julius Caesar, such as signed parchments and historical records, and that we assume that these records are valid and representative, then Julius Caesar was the emperor of Rome". The fact "Julius Caesar was the emperor of Rome" is no more certain that the assumptions the claim depends upon.

Indeed, I would argue that this representation is more legitimate than the synthetic representation which ignores (or obscures) the underlying assumptions. We could one day find that the historical record has been tampered with (or that perhaps Caesar was just the pawn of a shadow government); thus this construction can account for the possible future discovery of tampering (by changing the assumptions). The possibility of tampering with the historical record (or even the senses) is suggestive that all synthetic facts are no more than analytical facts in disguise and contingent upon some set of assumptions.

Do not fear the loss of synthetic facts; as by removing them, we are rewarded with a tauto-universal description.

Bob: How can an actual world arises purely from analytic facts?

We will first need a theory by which "macroscopic" laws emerge from a set of theorems. This will be introduced in the next section as feasible mathematics. We can introduce the idea of a world which is actual as follows: The world that we perceive as actual will be the statistical average associated with the greatest amount of equivalent arrangements of analytical facts that produce the same "macroscopic" description in the sense given by feasible mathematics. The absence of a precisely defined actual world (inherently statistical) is the reason why our universe is fundamentally quantum mechanical. Indeed, fluctuations around the average explicitly connect to quantum fluctuations. In the last chapter, we recover the Feynman path integral
formulation of quantum mechanics from fluctuations over analytical facts arrangements, as well as the Schrödinger and the Dirac equations. What we think of as the actual world are average properties emergent from the set of all analytical facts.

6 Feasible mathematics

Some research has been done in the area of feasible numbers. Perhaps the most promising is from Vladimir Yu. Sazonov’s paper on feasible numbers\(^{19}\). He suggests that feasible numbers are intuitively the set of numbers \(F\) which satisfies \(0 \in F\), \(F + 1 \subseteq F\) and \(2^{1000} \notin F\). Then, he goes on to investigate various constructions which would allow the consistent treatment of such a set.

Here, we take a different approach. We recognize that \(2^{1000}\) is a large number but nonetheless, it can be compressed to a short representation. Thus, we accept that theorems featuring this number can be proven even in the context of limited resources. Hence, a more general approach to feasibility is required. We propose a method to treat feasibility as a limit applicable to the complexity of the proofs themselves.

Mathematical proofs come in various sizes and have various indicators of complexity. By bounding proofs based on such indicators, the proof landscape available to a mathematician with limited resources is reduced (made feasible). We believe that a representation of mathematical feasibility based on limited proof complexity more accurately describe the intuitive notion of feasibility. After all, a theorem whose shortest proof requires \(2^{1000}\) bits will surely never be proven in our lifetime, but the number \(2^{1000}\) is easily representable even in simple proofs.

What is feasible mathematics? Feasible mathematics is an alternative to (and in some permissive sense, a generalization of) computational complexity theory (CT). Let us first see what CT is. CT is the study of the inherent difficulty of computational problems; as such, CT classifies problems by the increase in difficulty associated with an increase in the input size. For example, a binary search algorithm will have a difficulty of \(O(\log n)\), and thus it has a logarithmic complexity. In this example, the number of steps required to find an item amongst \(n\) sorted items grows proportionally to the logarithm of \(n\).

Why bother with an alternative? The problem is that CT does not correctly distinguish between all indicators of complexity. As an example, the difficulty between, say, an exponential problem with a small multiplication constant \(0.0001 \times O(2^n)\) and a polynomial problem with a large multiplication constant \(10^{999999} \times O(n^2)\) is incorrectly classified. As far as CT is concerned, the latter problem is much

simpler than the first as it only grows in \( n^2 \) versus \( 2^n \). However, in practice, the latter might never be solved because there might not be enough resources in the observable universe to do so (even for \( n=1 \)). Thus, to truly represent reality, something is missing from CT.

This is where feasible mathematics comes in. Feasible mathematics treats computational complexity according to the absolute difficulty of problems. Thus, it is not “fooled” by, say, a mere multiplication constant. Its domain is defined as a restriction on the domain of the halting probability \( \Omega \) of computer science. Using a similar construction, we define a probability \( Z \) that represents the probability that a random program will halt within some available computing resources. These resources can be time, memory, clock speed, etc. \( Z \) does for feasible mathematics what \( \Omega \) does for "universal mathematics"; it can decide all the programs that halt within the available computing resources. When the limits are made to vanish, \( Z \) converges to \( \Omega \).

As the construction of \( Z \) is meta-logically applicable to an arbitrary set of formal axioms, we introduce a distinction between feasible mathematics and universal mathematics. Universal mathematics is made feasible when, intuitively, the proof landscape of the mathematician is bounded by computational limits. In this sense, all practical work in mathematics is feasible.

To formalize feasible mathematics, we will consider mathematical proofs as computer programs that are executed on a self-delimiting universal Turing machine. We will then construct a statistical ensemble able to feasibly decide mathematics.

### 6.1 Main problem

Suppose a research group with access to a supercomputer. Bob has been granted a fixed amount of computing resources to use on the supercomputer. He has further been instructed to run a program \( q_A \). With no prior knowledge of \( q_A \), what is the probability that the program will halt within the allocated resources?

Answering this question will require notions of algorithmic thermodynamics and statistical physics.

### 6.2 Statistical physics

We will provide a brief recap of statistical physics. In statistical physics, we are interested in the distribution that maximizes entropy,

\[
S = -k_B \sum_{x \in X} p(x) \ln p(x)
\]  

(6.2.1)
subject to the fixed macroscopic quantities. The solution for this is the Gibbs ensemble. Typical thermodynamic quantities are:

<table>
<thead>
<tr>
<th>quantity</th>
<th>name</th>
<th>units</th>
<th>type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T = 1/(k_B\beta)$</td>
<td>temperature</td>
<td>$K$</td>
<td>intensive (6.2.2)</td>
</tr>
<tr>
<td>$E$</td>
<td>energy</td>
<td>$J$</td>
<td>extensive (6.2.3)</td>
</tr>
<tr>
<td>$p = \gamma/\beta$</td>
<td>pressure</td>
<td>$J/m^3$</td>
<td>intensive (6.2.4)</td>
</tr>
<tr>
<td>$V$</td>
<td>volume</td>
<td>$m^3$</td>
<td>extensive (6.2.5)</td>
</tr>
<tr>
<td>$\mu = \delta/\beta$</td>
<td>chemical potential</td>
<td>$J/kg$</td>
<td>intensive (6.2.6)</td>
</tr>
<tr>
<td>$N$</td>
<td>number of particles</td>
<td>$kg$</td>
<td>extensive (6.2.7)</td>
</tr>
</tbody>
</table>

Taking these quantities as examples, the partition function becomes:

\[
Gibbs\ ensemble
Z = \sum_{x \in X} e^{-\beta E(x) - \gamma V(x) - \delta N(x)} \quad (6.2.8)
\]

The probability of occupation of a micro-state is:

\[
Gibbs\ measure
p(x) = \frac{1}{Z} e^{-\beta E(x) - \gamma V(x) - \delta N(x)} \quad (6.2.9)
\]

The average values and their variance for the quantities are:

\[
\bar{E} = \sum_{x \in X} p(x) E(x) \quad \bar{E} = -\frac{\partial \ln Z}{\partial \beta} \quad (\Delta E)^2 = \frac{\partial^2 \ln Z}{\partial \beta^2} \quad (6.2.10)
\]

\[
\bar{V} = \sum_{x \in X} p(x) V(x) \quad \bar{V} = -\frac{\partial \ln Z}{\partial \gamma} \quad (\Delta V)^2 = \frac{\partial^2 \ln Z}{\partial \gamma^2} \quad (6.2.11)
\]

\[
\bar{N} = \sum_{x \in X} p(x) N(x) \quad \bar{N} = -\frac{\partial \ln Z}{\partial \delta} \quad (\Delta N)^2 = \frac{\partial^2 \ln Z}{\partial \delta^2} \quad (6.2.12)
\]

The laws of thermodynamics can be recovered by taking the following derivatives

\[
\left. \frac{\partial S}{\partial E} \right|_{V,N} = \frac{1}{T} \quad \left. \frac{\partial S}{\partial V} \right|_{E,N} = \frac{p}{T} \quad \left. \frac{\partial S}{\partial N} \right|_{E,V} = -\frac{\mu}{T} \quad (6.2.13)
\]

, which can be summarized as

\[
dE = TdS - pdV + \mu dN \quad (6.2.14)
\]
This is known as the equation of state of the thermodynamic system. The entropy can be recovered from the partition function and is given by:

\[ S = k_B \left( \ln Z + \beta E + \gamma V + \delta N \right) \]  

(6.2.15)

6.3 Algorithmic thermodynamics


\[ S = -k_B \sum p_i \ln p_i \]

and the entropy in information theory

\[ S = -\sum p_i \log_2 p_i \].

Furthermore, the similarity between the halting probability \( \Omega \) and the Gibbs ensemble of statistical physics has also been studied. Tadaki suggests to augment \( \Omega \) with \( D \) which acts as a compression term on \( \Omega \).

<table>
<thead>
<tr>
<th>Chaitin construction</th>
<th>Tadaki ensemble</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Omega = \sum_{q\in\text{halts}} 2^{-</td>
<td>q</td>
</tr>
</tbody>
</table>

(6.3.1)

With this change, the Gibbs ensemble compares to the Tadaki ensemble as follows;

<table>
<thead>
<tr>
<th>Gibbs ensemble</th>
<th>Tadaki ensemble</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z = \sum_{x\in X} e^{-\beta E(x)} )</td>
<td>( \Omega_D = \sum_{q\in\text{halts}} 2^{-D</td>
</tr>
</tbody>
</table>

(6.3.3)

Interpreted as a Gibbs ensemble, the Tadaki construction forms a statistical ensemble where each program corresponds to one of its micro-state. The Tadaki ensemble admits a single quantity; the prefix code length \( |q| \) conjugated with \( D \). As a result, it describes the partition function of a system which maximizes the entropy subject to the constraint that the average length of the codes is some constant \( |q| \):

\[ |q| = \sum_{q\in\text{halts}} |q| 2^{-|q|} \]  

from 6.2.10  

(6.3.4)

The entropy of the Tadaki ensemble corresponds to the average length of prefix-free codes available to encode programs.

\[ S = k_B \left( \ln \Omega + D|q| \ln 2 \right) \]  

from 6.2.15  

(6.3.5)
The constant $\ln 2$ comes from the base 2 of the halting probability function instead of base $e$ of the Gibbs ensemble.

John C. Baez and Mike Stay\textsuperscript{21} take the analogy further by suggesting an interpretation of algorithmic information theory based on thermodynamics, where the characteristics of programs are considered to be thermodynamic quantities. Starting from Gregory Chaitin’s $\Omega$ number, the Chaitin construction

$$\Omega = \sum_{q \in \text{halts}} 2^{-|q|}$$

(6.3.6)

is extended with algorithmic quantities to obtain

\begin{align*}
\text{Gibbs ensemble} & \quad Z = \sum_{x \in \mathcal{X}} e^{-\beta E(x) - \gamma V(x) - \mu N(x)} \quad \text{Baez-Stay ensemble} & \quad \Omega' = \sum_{q \in \text{halts}} 2^{-\beta E(q) - \gamma V(q) - \delta N(q)}
\end{align*}

(6.3.7)

(6.3.8)

Noting the similarity between the Gibbs ensemble of statistical physics (6.2.8) and (6.3.8), these authors suggest an interpretation where $E$ is the expected value of the logarithm of the program’s runtime, $V$ is the expected value of the length of the program and $N$ is the expected value of the program’s output. Furthermore, they interpret the conjugate variables as (quoted verbatim from their paper);

"1. $T = 1/\beta$ is the algorithmic temperature (analogous to temperature). Roughly speaking, this counts how many times you must double the runtime in order to double the number of programs in the ensemble while holding their mean length and output fixed.

2. $p = \gamma/\beta$ is the algorithmic pressure (analogous to pressure). This measures the trade-off between runtime and length. Roughly speaking, it counts how much you need to decrease the mean length to increase the mean log runtime by a specified amount while holding the number of programs in the ensemble and their mean output fixed.

3. $\mu = -\delta/\beta$ is the algorithmic potential (analogous to chemical potential). Roughly speaking, this counts how much the mean log runtime increases when you increase the mean output while holding the number of programs in the ensemble and their mean length fixed.


"–John C. Baez and Mike Stay

From equation (6.3.8), they derive analogs of Maxwell’s relations and consider thermodynamic cycles, such as the Carnot cycle or
Stoddard cycle. For this, they introduce the concepts of algorithmic heat and algorithmic work.

Other authors have suggested other alternative mappings in other but related context\(^{22}\).

6.4 Derivation of feasible mathematics

1. We start from the standard Chaitin construction applicable to a self-delimiting universal Turing machine\(^ {23}\).

\[
\Omega = \sum_{q \in \text{halts}} 2^{-|q|} \tag{6.4.1}
\]

where:

- \(\Omega \in (\mathbb{R} \cap [0, 1])\) numerical value of the sum \(\tag{6.4.2}\)
- \(q \in \Sigma_b\) binary program (encoded as a prefix-free code) \(\tag{6.4.3}\)
- \(|q| : \Sigma_b \rightarrow \mathbb{N}\) length of the program’s code \(\tag{6.4.4}\)

2. We augment \(\Omega\) with a multiplication constant \(D\); we obtain the Tadaki ensemble\(^ {24}\).

\[
\Omega_D = \sum_{q \in \text{halts}} 2^{-D|q|} \tag{6.4.5}
\]

where:

- \(\Omega_D \in (\mathbb{R} \cap [0, 1])\) numerical value of the sum \(\tag{6.4.6}\)
- \(D \in \mathbb{R}\) Conjugate to program length \(\tag{6.4.7}\)

3. With this addition, \(\Omega_D\) has the same mathematical structure as a Gibbs ensemble of statistical physics\(^ {25}\).

Gibbs ensemble
\[
G = \sum_{x \in X} e^{-S(x)} \tag{6.4.8}
\]

Tadaki ensemble
\[
\Omega_D = \sum_{q \in \text{halts}} 2^{-D|q|} \tag{6.4.8}
\]

where:

- \(\left(2^{-D|q|}\right)\) micro-state representing a program \(\tag{6.4.9}\)
- \(D \in \mathbb{R}\) algorithmic decompression \(\tag{6.4.10}\)

4. We interpret the Tadaki ensemble within the context of algorithmic thermodynamics\(^ {26}\). We can introduce a probability distribution for \(\Omega\) and \(\Omega_D\) that maximizes the entropy of the system. This


probability distribution is a standard result of statistical physics (see 6.2.9).

\[
\text{Halting measure} \quad \text{Halting measure with fixable } |q| \\
p(q) = \frac{1}{\Omega} 2^{-|q|} \quad p(q, D) = \frac{1}{\Omega_D} 2^{-D|q|} \quad (6.4.11)
\]

In the case of \( \Omega_D \), \( D \) is a Lagrange multiplier and \( p(q, D) \) is the probability measure that maximizes the entropy subject to the constraint that the average program length is \( |q| \).

\[
|q| = \sum_{q \in \text{halts}} p(x)|q| \quad (6.4.12)
\]

, where

\[
|q| \in \mathbb{R}_{\geq 0} \quad \text{average program length} \quad (6.4.13)
\]

5. Finally, to obtain feasible mathematics, we introduce into \( \Omega_D \) a new quantity \( t(q) \) the runtime of program \( q \) and pair it with its conjugate \( -W \) (we will explain the minus sign in a moment). We obtain the construction \( Z \).

\[
Z = \sum_{q \in \text{halts}} 2^{Wt(q) - D|q|} \quad (6.4.14)
\]

, where

\[
Z \in \mathbb{R}_{\geq 0} \quad \text{numerical value of the sum} \quad (6.4.15)
\]

\[
t(q) : q \rightarrow \mathbb{N} \quad \text{number of iterations required for } q \text{ to halt} \quad (6.4.16)
\]

\[
W \in \mathbb{R} \quad \text{conjugate to } t(q) \text{ in units of } (\text{iterations})^{-1} \quad (6.4.17)
\]

\[
|q| : q \rightarrow \mathbb{N} \quad \text{number of bits of program } q \quad (6.4.18)
\]

\[
D \in \mathbb{R} \quad \text{conjugate to } |q| \text{ in units of } (\text{bits})^{-1} \quad (6.4.19)
\]

The corresponding probability measure is:

\[
p(q, W, D) = \frac{1}{Z} 2^{Wt(q) - D|q|} \quad (6.4.20)
\]

It maximizes the entropy subject to the following constraints:

\[
|q| = \sum_{q \in \text{halts}} p(q, W, D)|q| \quad \text{average program length } \bar{|q|} \quad (6.4.21)
\]

\[
\bar{t} = \sum_{p \in \text{halts}} p(q, W, D)t(q) \quad \text{average program runtime } \bar{t} \quad (6.4.22)
\]

\( W \) is defined to be negative as a convention and is justified on the grounds that most programs stop quickly or never halt\(^{27}\). Thus, compatible micro-states begin to rarefy as the average program-runtime is increased. From the ensemble of feasible mathematics we

can produce the following equation of state for the system: \( dS = -Wd\tilde{t} + Dd|q| \). Posing \( W \) to be negative illustrate clearly that we expect the entropy to decrease as \( \tilde{t} \) is increased.

Let us now study this equation in more detail in the following section.

6.5 Results

We interpret the supercomputing research group as taking a similar role to the role taken by the various baths in thermodynamics (heat bath, particle bath). For example, in thermodynamics, we would say that a system which can exchange energy with its environment is in a heat bath. Its temperature will be constant but its total energy would fluctuate as it is exchanged with the bath. By analogy, in feasible mathematics, we would imagine that a computation occurs in a supercomputer which schedule priority, assigns memory, etc. so has to maintain various computing resources fixed during the calculation. This is analogous the role of the thermodynamic baths.

To make this more precise, let us define what we mean by fixed resources.

6.6 Fixed resources

Each Lagrange multiplier of the partition function \( Z \) is a computing resource fixed by the supercomputer. In the provided definition of \( Z \), there are two such constants: \( W \) and \( D \). They can be interpreted as follows:

\[
\begin{align*}
\text{Halting-power} & \quad \mathcal{P} = -\frac{1}{W} \\
\text{Halting-force} & \quad \mathcal{F} = -\frac{D}{W}
\end{align*}
\]

- The halting-power counts how much the runtime must be doubled in order to double the entropy of the ensemble while holding the mean length fixed.

- The halting-force counts how much the average length must be decreased to increase the average runtime by a specified amount while holding the entropy in the ensemble fixed.

By adjusting the halting-power and the halting-force, the supercomputer is able to control the value of the constraints of the system \( \tilde{t} \) and \(|p|\). Thus, the halting probability of Bob’s program \( q_A \) depends on the halting-power allocated by the supercomputer. In the supercomputer analogy, the halting-power can be understood as related to
the clock speed of the processor(s), and the halting-force as a compression algorithm applied to input memory.

6.7 Alternative formulations

There exist alternative constructions of $Z$ such that other resources are fixed by the supercomputer.

**Action-frequency formulation:**

$$Z' = \sum_{q \in \text{halts}} 2^{-Af(q) - D|q|} \quad (6.7.1)$$

The supercomputer must fix

$$S = \frac{1}{A} \quad (6.7.2)$$

- The halting-action counts how much the action must be doubled in order to double the entropy of the ensemble while holding the mean length fixed.

**Time-power formulation:**

$$Z'' = \sum_{q \in \text{halts}} 2^{-IP(q) - D|q|} \quad (6.7.3)$$

The supercomputer must fix

$$t \quad (6.7.4)$$

- The halting-time counts how much the time must be doubled in order to double the entropy in the ensemble while holding the mean length fixed.

This formulation does not describe a time cutoff (see next formulation). Rather, it describes a system where all programs halt at the same time. To guarantee that the work on each program terminates simultaneously (e.g. there are no partial executions), the supercomputer must adjust the computation power on a per program basis.

**Time-cutoff formulation:**

$$Z''' = \sum_{q \in \text{halts}; t(q) \leq k} 2^{-D|q|} \quad (6.7.5)$$

The sum $Z'''$ is performed only on programs that halt within a time cutoff $k$. Thus, $Z'''$ contains no halting information and is computable. $\Omega$ is recovered in the limit when $k \to \infty$. 
Size-cutoff formulation:

\[ Z''' = \sum_{q \in \text{halts}, |q| \leq k} 2^{-D|q|} \] (6.7.6)

The sum \( Z''' \) is performed only on programs with sizes less or equal to \( k \). \( \Omega \) is recovered in the limit when \( k \to \infty \). \( Z''' \) represents the first \( n \) bits \( \Omega \) up to the cutoff \( k \).

6.8 Relation to \( \Omega \)

**Theorem 6.8.1.** \( Z \to \Omega_D \) as the amount of available resources is increased arbitrarily.

\[
\lim_{P \to \infty} Z \to \Omega_D
\] (6.8.2)

**Proof.** First, we rewrite \( \Omega_D \) as:

\[
\Omega_D = \sum_{i=1}^{\infty} 2^{-H(q_i) - D|q_i|} \quad \text{where } H(q_i) := \begin{cases} 0 & q_i \text{ halts} \\ \infty & \text{otherwise} \end{cases}
\] (6.8.3)

Second, we note that the runtime \( t(q_i) \) of a program \( q_i \) will be finite if it halts and infinite otherwise.

\[
t(q_i) = \begin{cases} t_i \in \mathbb{R}_{\geq 0} & q_i \text{ halts} \\ \infty & \text{otherwise} \end{cases}
\] (6.8.4)

Then taking the limit of \( Z \),

\[
\lim_{P \to \infty} \frac{1}{P} t(q_i) = \begin{cases} 0 & q_i \text{ halts} \\ \infty & \text{otherwise} \end{cases}
\] (6.8.5)

This is the definition of \( H(q_i) \). Therefore,

\[
\lim_{P \to \infty} \frac{1}{P} t(q_i) \to H(q_i)
\] (6.8.6)

Thus,

\[
\lim_{P \to \infty} Z \to \Omega_D
\] (6.8.7)
Theorem 6.8.8. \( Z \) monotonically converges towards \( \Omega_D \) as the available resources are increased.

Proof. Without loss of generality, let us now expand \( Z \) explicitly with an example. Assume a system comprised of three micro-states with prefix code-length \( |q_1| = 1, |q_2| = 2 \) and \( |q_3| = 3 \) and with the run-times \( t_1 = 5, t_2 = \infty \) and \( t_3 = 5 \). In this example, \( q_1 \) and \( q_2 \) halt and \( q_3 \) does not. For the purposes of simplicity, we can assume that all other programs do not halt. In this case, the system is not universal but let us nonetheless use it as a simplified numerical example. The sum \( Z \) becomes:

\[
Z(W) = 2^{-1+5W} + 2^{-2+\infty W} + 2^{-3+5W}
\]  

(6.8.9)

We will now produce a series of numerical calculations with progressively smaller values of \( W \) and we will look at the evolution of the error rate \( \xi(W) = \Omega - Z(W) \). For this system, \( \Omega = 0.10101_2 \).

\[
\begin{array}{cccc}
(-W) & Z(W) & \xi(W) & \text{error} \\
\infty & 0 & \Omega & \text{max} \\
1 & 0.000000101\ldots_2 & 0.10011011_2 & \approx 2^{-1} \\
0.1 & 0.011100010\ldots_2 & 0.00101110\ldots_2 & \approx 2^{-3} \\
0.01 & 0.100110101\ldots_2 & 0.000000010\ldots_2 & \approx 2^{-6} \\
0.001 & 0.011100010\ldots_2 & 0.000000000\ldots_2 & \approx 2^{-9} \\
\vdots & \vdots & \vdots & \vdots \\
0 & \Omega & 0 & \text{none} \\
\end{array}
\]

(6.8.10)

As we can see, increasing the halting-power \( (P = -1/W) \) causes the value \( Z \) to monotonically converges towards \( \Omega \). The error rate decreases as more valid bits of \( \Omega \) are obtained.

Theorem 6.8.16. An observer knowing \( n \) bits of \( Z \) will be able to decide at most \( 2^N \) programs.

Proof. We consider a numerical value for \( Z \) whose first \( k \) bits correspond to the bits of \( \Omega \). We look at two cases: 1) For the first \( k \) bits, \( Z \) (as with \( \Omega \)) can decide \( 2^N \) programs per bit. 2) For the bits after \( k \), the situation is a bit more complex:

To recover the feasible programs beyond \( k \), an observer can execute programs on a universal Turing machine in dovetail. As they halt, the observer adds their contribution to \( Z \). Once the value of \( Z \) is recovered, then all programs taking longer to halt are beyond the feasible bound, regardless of whether they ultimately halt or not.
7 Model of the World

To reap the benefits of the ATU program, we must commit to describing the World exclusively using Tauto-universal logic. This means that whenever there is a clash between what we intuit the World to be and what Tauto-universal logic allows us to describe, we must give the win to Tauto-universal logic and adjust our preconceived notions accordingly. The temptation to extend Tauto-universal logic by introducing new axioms must be resisted.

To lay out the model, we will look at the properties of facts objectively: e.i. we only look at those properties which are measurable such as the program-length or the program-runtime and avoid "subjective" properties such as how meaningful, interesting a fact is. This is the analytical-fact interpretation (section 5.8).

Each fact has a cost. A world formally describable by facts is not immune to this cost. Thus, it must bear the cost of each fact that comprises it.

7.1 The role of conditional introduction

What is the role played by the rule of conditional introduction in Tauto-universal logic and what is its connection to feasible mathematics? The rule of conditional introduction is a rule of inference and like all other such rules, it informs us of the truth value of a statement. When we ask if theorem \( \text{b} \) (of deductive-apparatus \( a \)) is true in Tauto-universal logic, we are asking if theorem \( \text{b} \) is a theorem of Tauto-universal logic. The answer is almost always no because Tauto-universal logic’s deductive apparatus is minimal. Nonetheless, truth claims can still be made via conditional introduction. Indeed, consider the following steps which are used to introduce theorems into Tauto-universal logic conditionally upon their deductive apparatuses:

\[
\text{IF} \quad \text{deductive-apparatus}_a \vdash \text{sentence}_b \quad \text{(7.1.1)}
\]

\[
\text{THEN} \quad \begin{align*}
\text{sentence}_b & : \Leftrightarrow \text{theorem}_b \\
\text{AND} & \\
\text{deductive-apparatus}_a & \implies \text{theorem}_b
\end{align*} \quad \text{(7.1.2, 7.1.3)}
\]

What we mean here is that IF a certain deductive apparatus is able to prove a sentence \( \text{(7.1.1)} \) THEN the sentence is true on the condition

---

\[\text{A small exception exists because Tauto-universal logic can refer to itself with the rule of conditional introduction; by "conditionally introducing conditional introduction" into itself.}\]
that the deductive-apparatus is also true (7.1.3). The truth of the sentence is conditional upon the truth of the deductive apparatus it is provable in.

Feasible mathematics can be applied to any field of mathematics, in the sense that any proof system can be made feasible by it, but due to some unique features it connects best to Tauto-universal logic. Indeed, Tauto-universal logic precisely meets the requirements of a partition function of feasible mathematics.

7.2 Native two-state system

The first feature that Tauto-universal logic has that other systems of logic might not have is that it natively defines two tautological states for each of its theorems. These two states are analogous to the occupied vs non-occupied state of the micro-state of a partition function of feasible mathematics. Let’s investigate in more detail.

As each theorem of Tauto-universal logic is constructed as an implication, let us review its truth table (Table 1). From the table, we conclude that an implication cannot be a tautology in the general case because of the possibility of scenario no. 2, which can be false.

However, to construct the theorems of Tauto-universal logic, the sentences are first filtered by step 7.1.1. This filter eliminates scenario no. 2 and no. 4 from the truth table. Specifically, B is false if B is not-provable, but the filter only keeps the sentence if it is provable in its associated deductive apparatus, thus B cannot be false.

Within Tauto-universal logic, the possible scenarios of the implication are thus restricted to scenario 1 and 3 (Table 2). With only scenario 1 and 3, the implication in Tauto-universal logic is a tautology (A \implies B is always true). We recall that as with all tautologies, the implication in Tauto-universal logic is necessarily valid and implied. How are we to make sense of this?

We first define state 1 and 2 for each tautology:

State 1: The deductive apparatus is true (the micro-state of Z corresponding to this theorem is "occupied").

State 2: The deductive apparatus is false (the micro-state of Z corresponding to this theorem is "non-occupied").

7.3 Facts are existential

Let’s take an example. Say an observer notices that "the chicken crossed the road". As Tauto-universal logic list all analytical facts, this fact can be associated to one of its theorems. Then, the question "Why did the chicken cross the road?" can be explained within and up to a certain deductive apparatus.
Hypothetically, an observer could spend the better part of his life trying to understand the motivation of the chicken, but if he does there will be an opportunity cost. While he focuses on the chicken, he is not focusing on other problems. For example, he might have trouble "proving" to an employer that he can work productively.

The observer might determine that the connectome’s explanation of the chicken’s motivation is not elegant enough for his taste. He might want to derive the structure of the chicken’s connectome as directly implied by the big bang:

\[(\text{deductive-apparatus}) \quad (\text{theorem})\]
\[
\text{the Big-Bang} \quad \Rightarrow \quad \text{The structure of the chicken’s connectome}
\]

\[(7.3.2)\]

Rigorously, the opportunity cost paid by the observer for working on one problem is based on the properties (program-runtime and program-length) of the questions the observer is trying to explain and is limited by its available resources (in the computing sense). To provide a deeper explanation of a question (one that is mathematically elegant), the observer must spend considerably more computational resources than he would need to if he were to provide a short one. For example, explaining the chicken’s motivation by appealing to the presence of "peanuts on the other side" is simpler than the full-blown; big-bang implies connectome implies motivation.

\[7.4 \quad \text{Entropy}\]

The consequence of limited computing resources means that multiple configurations of explanations are possible. Should the observer spend all of his resources trying to explain the chicken’s motivation, or should he distribute his resources to many independent facts so as to maximize his global understanding of the World (for some sensible definition of global understanding)? If the later, what distribution strategy should the observer follow to achieve the goal?

I am glad that you asked because we have the function for you! \(Z\) describes the optimal strategy to maximize our understanding of the World. The peak of understanding coincides with numerical knowledge of the constant \(Z\) and occurs when the entropy of the system is maximal. Facts with exceedingly long explanations or descriptions
are exponentially discouraged in just the right amount to maximize the entropy (i.e. the informational content) of the explanatory system.

Given some computational resources, an observer can produce multiple sets of explanations for a given set of facts. As a result, a system of facts admits multiple valid configurations for its explanations. As each explanatory configurations are compatible with the computational resources, this translates to an entropy.

We can now introduce the AUT program as a model of the universe, given the following definition:

**Definition 7.4.1 (Universe).** The universe (at least in the epistemological sense) is as a tautological mathematical structure which maximizes its self-explanation given the general constraints of formal logic.

### 7.5 Main result of feasible mathematics

The main result of feasible mathematics is that the ensemble of abstract programs, organized by program-runtime and program-length will produce when all other things are equal, two equilibrium quantities that are constant throughout the system; namely the halting-density $\mathcal{F}$ and the halting-power $\mathcal{P}$. These properties characterize the system. If the universe is indeed describable by analytical facts, evidence for these two quantities should empirically be plentiful.

For notation consistency, we will pose $x(q) := |q|$. The function $Z$ of feasible mathematics

$$Z = \sum_{q \in Q} e^{Wt(q) - Dx(q)} \quad (7.5.1)$$

describes a statistical ensemble of analytical fact $q$ from within the set $Q$ of all such facts. Each micro-state is statistically weighted by its runtime $t(q)$ (the size of its proof) and by its length $|q|$ (the size of its description). Furthermore, the function

$$p(q) = \frac{1}{Z} e^{Wt(q) - Dx(q)} \quad (7.5.2)$$

is the probability that a fact $q \in Q$ is actual in the universe.

**Remark.** Assuming the shortest proof, the program-runtime of a theorem is irreducible. Indeed, reducing its size can only be done by taking steps out (and then it no longer a valid proof). The program-length is also irreducible for the reason that changing it will transform it to another theorem. Thus, the limits associated with these quantities will be what we would call hard limits. Physically, we can think of hard limits as the speed of light or as a horizon applicable to the observable universe—all of which are inviolable.
Other properties of facts and proofs may, I hypothesize, give rise to softer limits. For example, a solution requiring lots of memory and time would require the contribution of a large amount of physical substrate (many molecules working together) for a long time. Thus, the computation of such fact is likely to be destroyed by fluctuations of the environment. This makes it unlikely that such high memory long running statement be verified early in the history of the universe. These types of limits are statistical and yield soft laws and can be investigated by usual complexity class theory and probability theory. As a concrete example, we can think of the evolution of life and its many near extinction events.

However interesting soft limits may be, in this manuscript, we will only be concerned with deriving the hard limits.

7.6 Dovetailing

Figure 2 shows the diagram of a universe entirely defined by the facts that comprise it. As no brute facts are permitted, all facts of the world are restricted to those that can be verified within some deductive apparatus of logic (group b). The group c corresponds to the universe as we know it. Indeed, in part II, we show that it is isomorphic to the laws of physics.

We can intuitively understand why group c corresponds to the universe in many ways, but the most insightful explanation is perhaps in the context of dovetailing and universal Turing machines. Consider an observer attempting to verify the theorems of Tauto-universal logic. The goal of the observer could be to build the list of theorems from scratch. To do it, the observer could pick a binary sentence at random, run it on a universal Turing machine and wait until it halts. If it halts, the sentence is considered verified and is added to the list. Then work begins on the second sentence, and so on. So far all is fine and good, but the problem occurs when a sentence is non-provable. In this case, the universal Turing machine will hang attempting to prove it and work on new sentences will never begin.

So what is an observer to do to avoid hanging? Let’s review a few alternative options. The observer could try to run each binary sentences for just one iteration each. Then, return to the beginning sentence and run a second set of iterations, and so on. However, as you might have guessed, as there are infinitely many sentences he will never return to the beginning. For a countably infinite set of sentences which includes non-provable sentences, the correct solution is to dovetail the work.

Definition 7.6.1 (Dovetailing). Dovetailing is a proof-finding strategy for a system of logic to guarantee that progress will be made on arbitrarily many
A specific implementation is the simple dovetailing algorithm:

**Definition 7.6.2** (Simple Dovetailing). Consider the case of simple dovetailing. First, an observer runs one iteration for the first sentence. Then, the observer runs one iteration for both the first and the second sentence. Then, the observer runs one iteration for the first, the second and the third sentence. And so on. The observer stops running iterations for sentences whose execution has halted. In the case of a non-provable sentence, the observer would continue to run iterations for it forever but this would not cause the work to hang. Using this method, progress will eventually be made for every sentence and no sentence will cause the progress to hang indefinitely.

This dovetailing algorithm is simple to understand and simple to implement, but it is not the most efficient. Thus, we can ask: is there a most efficient dovetailing algorithm?

Let’s investigate Z — does it describes a dovetailing algorithm? Indeed it does; as the available computing resources are increased, the numerical value of the normalization constant Z converges towards Ω (an efficient way to encode the theorems of Tauto-universal logic). From Z, we can now define a new form of dovetailing that we will call native dovetailing.

**Definition 7.6.3** (Native dovetailing). A type of Dovetailing performed so as to maximize the entropy in the output of the calculation. Z defines the structure of the algorithm. As the entropy in the output is maximized, natural dovetailing packs the most bang for the buck (it is the most efficient use of computing resources for calculating Ω).

**Proof.** Recall theorem 6.8.16. n bits of Z is able to decide at most \(2^n\) programs from its domain. Since the entropy of the system described by Z is maximal, the bits of Z can decide the highest number of configuration of facts compatible with its estimation of Ω.

Native dovetailing justifies why the correct formulation of Z based on program-runtime and program-length is enough and why additional computational quantities are not required. The two quantities are sufficient to make Z converge towards Ω. Thus, the dovetailing algorithm is "eventually universal".

### 7.7 Proving randomness

One of the central assumptions of the Gibbs ensemble is that the compatible configurations are equiprobably and random. Randomness, when introduced into a formal physical theory, is almost always introduced axiomatically. For example, one often reads "states that
are equivalent are assumed to be random and equiprobable”. The randomness is never proven.

But here, we can do much better. Due to the connection between \(Z\) and \(\Omega\) we can formulate a formal proof of randomness applicable to the system.

**Proof.** To see how this is possible, consider that \(\Omega\) (its infinite expansion) is provably algorithmically random. To construct the infinite expansion of \(Z\), knowledge of \(\Omega\) is required, but nonetheless \(Z\) is less compressed than \(\Omega\). \(Z\) is algorithmically D-random\(^{29}\).

7.8 *Mapping to the space-time background*

We will show in the next part that the laws implied by \(Z\) correspond to the familiar laws of physics. Specifically, we will derive the following laws: special relativity, general relativity, dark energy, the arrow of time, the second law of thermodynamics, the Schrödinger equation, the Dirac equation and the Feynman path integral formulation of quantum mechanics from \(Z\).

How do these laws come out of \(Z\)? The first step will be to connect the quantities of \(Z\) to physical quantities. Our strategy to do so will be borrowed from introductory statistical physics. We will adopt the same line of reasoning which allows the Lagrange multiplier \(\beta\) of statistical physics to be connected to the notion of a physical temperature. As you may recall, in introductory statistical physics:

1. The Gibbs ensemble is first derived from statistical arguments as the ensemble which maximizes the entropy subject to fixed quantities. The process introduces a multiplication constant known as the Lagrange multiplier and is designated by \(\beta\).

2. From the Gibbs ensemble, a relation between \(\beta\), energy and entropy is obtained: \(\beta dE = dS\).

3. Then, it is shown that this relation recovers a well known and empirically-uncontested law; such that the two are exact replicas if and only if \(\beta\) is defined using the temperature \(T\). In this case, \(S = \ln\Omega\) is used to connect \(\beta\) to \(T\) via \(\beta = 1/(k_B T)\).

4. Thus, we conclude that the Gibbs ensemble is a description of a physical system involving energy, entropy, and temperature.

We adopt the same line of reasoning for the derivation of the space-time background from \(Z\). Our goal is to derive as many laws of physics as we can from \(Z\) so as to show the extent of the physical connection. Specifically, we will show that some quantity of \(Z\) corresponds to the time in the equations for Special relativity, the Dirac...
equation, etc. and that some other quantity of Z corresponds to space in those same equations.

The validity of the mapping between the quantities of Z and the physical notion of space-time is ultimately the conclusion of this manuscript and rests on deriving overwhelmingly many known laws of physics from Z and to a degree such that it exceeds that which would be expected from a mere coincidence.

For pedagogical reasons, we now explicitly state the mapping obtained at the conclusion of this manuscript, then derive the laws of physics within the context of the mapping. Each law of physics that we derive from Z adds weight to the thesis of the mapping.

<table>
<thead>
<tr>
<th>Property</th>
<th>Variable</th>
<th>Mapping</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>program-runtime</td>
<td>( t(q) )</td>
<td>( t(q) \rightarrow \text{time} )</td>
<td>seconds</td>
</tr>
<tr>
<td>program-length</td>
<td>( x(q) )</td>
<td>( x(q) \rightarrow \text{space} )</td>
<td>meters</td>
</tr>
</tbody>
</table>

**Part II**

The laws of physics

Using the language of physics, we consider an interpretation of Z isomorphic to its interpretation in Tauto-universal logic. Consistent with Z, we inject as thermodynamic conjugate-pairs the two quantities that are mapped to time and space, respectively program-runtime and program-length. Interpreted as time and space and to recover the units of energy (appropriate for statistical physics), time must be multiplied by a power and space (e.g. a length) must be multiplied by a force; thus, the partition function describes arbitrary micro-states in terms of both space and time. Our goal will be to show that this mapping is valid by deriving a plurality of well-known laws of physics from Z. Due to the simplicity and generality of the construction, it is perhaps reassuring that the relations of space-time; special relativity, general relativity, and dark energy are provable solutions of its equation of state. Furthermore, thermal fluctuations along the time and space quantities produce the Schrödinger and Dirac equations as thermo-statistical extensions to classical analogs. The notion of temperature is recovered simply as the proportion between entropy and energy as per the standard definition. The construction suggests that both general relativity and the quantum world emerge from a more fundamental statistical physics construction.
8 The Gibbs ensemble of Time and Space itself

Consistent with the function $Z$ applicable to analytical truth and to the physical mapping, we propose the following partition function, constructed as a Gibbs ensemble:

$$Z(\beta, P, F) = \sum_{q \in Q} e^{-\beta[Fx(q) - Pt(q)]} \quad (8.0.1)$$

where

<table>
<thead>
<tr>
<th>quantity</th>
<th>name</th>
<th>units</th>
<th>type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>entropic power</td>
<td>$J/s$</td>
<td>intensive</td>
</tr>
<tr>
<td>$F$</td>
<td>entropic force</td>
<td>$J/m$</td>
<td>intensive</td>
</tr>
<tr>
<td>$t(q)$</td>
<td>thermal time</td>
<td>$s$</td>
<td>extensive</td>
</tr>
<tr>
<td>$x(q)$</td>
<td>thermal space</td>
<td>$m$</td>
<td>extensive</td>
</tr>
</tbody>
</table>

The introduction of the term $\beta$ is done to cancel out whatever units we ascribed to $Fx(q)$ and $Pt(q)$ by the mapping —in this case, energy in Joules.

The partition function includes the somewhat familiar entropic force and the completely unfamiliar entropic power. Its equation of state is:

$$TdS = -PdT + Fdx$$

We can convert it to an equivalent representation by converting the time to a frequency and the power to an action. Let us do that now.

$$TdS = -Pdf + Fdx$$

$$TdS = -Pd(\int^{-1}f) + Fdx \quad [t := 1/f] \quad (8.0.8)$$

$$TdS = Pf^{-2}df + Fdx \quad [d(f^{-1}) = -f^{-2}df] \quad (8.0.9)$$

$$TdS = Sdf + Fdx \quad [S := Pf^{-2}] \quad (8.0.10)$$

This representation introduces two new quantities, defined as:

<table>
<thead>
<tr>
<th>quantity</th>
<th>name</th>
<th>units</th>
<th>type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>entropic action</td>
<td>$Js$</td>
<td>intensive</td>
</tr>
<tr>
<td>$f(q)$</td>
<td>thermal frequency</td>
<td>$s$</td>
<td>extensive</td>
</tr>
</tbody>
</table>

Thus, the equation of state admits these two formulations:
\[ TdS = S\underaccent\check{d}f + F\underaccent\check{d}x \quad \text{action-frequency formulation} \quad (8.0.13) \]
\[ TdS = -Pd\tilde{t} + Fd\tilde{x} \quad \text{power-time formulation} \quad (8.0.14) \]

which we will refer to throughout this part.

Note that this "physical" \( Z \) is of the same mathematical structure as the one obtained from feasible mathematics. Hence, they describe the same mathematical object, precisely as we would expect from the autological property.

\[
\begin{align*}
\text{Physical} & \quad Z(\beta, F, P) = \sum_{q \in \mathcal{Q}} e^{-\beta[Fx(q) - Pt(q)]} \\
\text{Analytic-facts} & \quad Z(D, W) = \sum_{q \in \mathcal{Q}} e^{-[Dx(q) - Wt(q)]} \\
\end{align*}
\]

(8.0.15)

The only difference is the introduction of a multiplication constant \( \beta \) to cancel out the units (meters, seconds, Watts, Newtons, etc) introduced by the physical mapping (negligible).

8.1 Regimes and cycles

We will derive the familiar laws of physics by studying the equation of state in terms of its regimes. To do so, we will fix some derivatives (e.g. \( dS = 0 \)) and analyze what happens when we let the others vary.

8.2 Special relativity

Here, we use the power-time formulation and pose \( dS = 0 \). We obtain the fundamental relation of special relativity linking space to time.

\[
\begin{align*}
0 &= -Pd\tilde{t} + Fd\tilde{x} \quad (8.2.1) \\
F\tilde{d}x &= Pd\tilde{t} \quad (8.2.2) \\
d\tilde{x} &= \frac{P}{F}d\tilde{t} \quad (8.2.3) \\
\end{align*}
\]

As the power \( P \) and the force \( F \) are Lagrange multipliers of the partition function, they are constant throughout the system. Therefore, their quotient is also a constant.

\[
c := \frac{P}{F} \quad (8.2.4)
\]

Therefore,

\[
d\tilde{x} = cd\tilde{t} \quad (8.2.5)
\]

As the units of \( P/F \) are meters per second, \( c \) will be our working definition of the speed of light.
Remark: When $P$ is the Planck power and $F$ is the Planck force, we do indeed recover the speed of light:

$$P\left(\frac{1}{F}\right) = \frac{c^5}{G\left(\frac{G}{c^4}\right)} = c \quad (8.2.6)$$

8.3 Light cones as thermodynamic cycles

In this section, we look at the thermodynamic cycle of the system transiting through time and space starting at $O$ to $A$ to $B$ and back to $O$, as illustrated on Figure 3. During the transitions and to keep the energy constant, trade-offs must be made between time, distance and entropy. This cycle is reminiscent of other thermodynamic cycles, such as those involving pressure and volume. Interestingly, the cycles can also be interpreted as light cones.

$O$ to $A$: As $O$ is translated forward in time to $A$ while keeping the distance constant ($d\tau = 0$), the entropy decreases over time.

$$\left. (TdS = Fd\tau - PdT) \right|_{d\tau = 0} \quad (8.3.1)$$

$$\implies \frac{dS}{dt} = \frac{-P}{T} \quad (8.3.2)$$

$A$ to $B$: As $A$ is translated forward in space to $B$ while keeping the time constant ($dt = 0$), the entropy increases over the distance.

$$\left. (TdS = Fdx - Pdt) \right|_{dt = 0} \quad (8.3.3)$$

$$\implies \frac{dS}{dx} = \frac{F}{T} \quad (8.3.4)$$

$O$ to $B$: As $O$ is translated forward both in time and in space to $B$ while keeping the entropy constant ($dS = 0$), the system has a speed of $c$.

$$\left. (TdS = Fd\tau - PdT) \right|_{dS = 0} \quad (8.3.5)$$

$$\implies \frac{dx}{dt} = \frac{P}{F} = c \quad (8.3.6)$$

We conclude that an object traveling at speed $c$ is neither encouraged nor discouraged by entropy. The speed of light represents an inflection point in the rate of entropy production over time. We will return to that notion in the section on the arrow of time.
8.4 Lorentz’s transformation

To recover the Lorentz’ factor $\gamma$, let us consider figure 4. Two observers start at the origin $S$ and travel in space-time respectively to $O$ and $O’$. We regard $O’$ as traveling at speed $|v|$ in the reference frame of $O$. From standard trigonometry, we derive the following values for the length of the segment:

<table>
<thead>
<tr>
<th>Segment</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>SO</td>
</tr>
<tr>
<td>$</td>
<td>SO’</td>
</tr>
<tr>
<td>$</td>
<td>O’O</td>
</tr>
</tbody>
</table>

From the Pythagorean theorem and solving for $\cos \theta$, we obtain:

\[
\sqrt{1 - (\sin \theta)^2} = \cos \theta
\]

We consider that the distance between two observers moving at constant speed is simply $vt$. Hence, $|O’O| = vt$. Solving for $\sin \theta$, we obtain:

\[
|O’O| = vt = L \sin \theta
\]

\[
\Rightarrow \sin \theta = \frac{vt}{L}
\]

From equation (8.4.7) and (8.4.9), we get the reciprocal of the Lorentz factor:

\[
\gamma = \frac{1}{\sqrt{1 - \frac{v^2 t^2}{L^2}}}
\]

Finally, we consider that $L$ is the distance traveled by $O$ in the reference frame of $O’$ such that the entropy of $O$ is constant over time. According to the relation $dx = c dt$, for this to be the case, the speed of $O$ must be $c$. Thus, the distance traveled by $O$ during time $t$ is $L = ct$. We obtain:
\[ \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \]  

(8.4.12)

which is the well-known Lorentz factor and is the multiplication constant connecting \(|\mathcal{S}O|\) to \(|\mathcal{S}O'|\).

9 Macroscopic limit

One of the requirements of any complete theory is that General Relativity is recovered in the macroscopic limit. To investigate, we will now look at the World far away from the microscopic features of the bits — the macroscopic World. We will show that in the macroscopic limit, we can recover a smooth inertia \((F = ma)\), General relativity and dark energy.

To approximate, we will introduce a "scaling constant" \(\Delta B\) into \(x\) which scale the gap between the bits, then we will pose \(\Delta B \to 0^+\). Thus, the separation effects between the bits will become negligible. As we do so, we must adjust our notation as it pertains to the facts.

We create an uncountable analog to a fact \(q\) and we name it \(L\). \(L\) is an element of an uncountable set \(\mathcal{L}\); the new domain of the partition function \(Z\). The units of \(|L|\) are the nats instead of the bit. A fact \(q\) can be made to correspond to a segment of \(L\). The approximation holds macroscopically but would fail on a short section of \(L\) as it is continuous, and thus can erroneously describe sub-facts as an artifact of the approximation. This approximation allows us to expand \(x(L)\) into a Taylor series.

9.1 Taylor series

We haven’t discussed this in great detail so far, but the length of the sentences of a language do not grow linearly with the number of bits. Indeed, we recall that \(n\) bits can encode \(2^n\) sentences at its maximum. Further requirements on the function required for a convergent definition of \(\Omega\) such as the prefix-free property might also be required and will reduce the number of sentences available from \(n\) bits to a total less than \(2^n\). Absent of an exact specification for \(x(L)\), we consider it to be amenable to an expansion as a Taylor series. The Taylor series of \(F x(L)\) is:

\[
F x(L) = F x(0) + F x'(0)L + \frac{F x''(0)}{2}L^2 + \frac{F x'''(0)}{6}L^3 + O(L^4) \quad (9.1.1)
\]
In the macroscopic approximation, the thermodynamic quantity $x(L)$ can be expanded as a Taylor series (as $L$ is uncountable), and thus its derivative with respect to $L$ is:

$$F dx(L) = Fx'(0)dL + Fx''(0)LdL + \frac{Fx''''(0)}{2}L^2dL + 4O(L^3)dL \quad (9.1.2)$$

Then, injecting it into the power-time formulation, we obtain:

$$TdS = -Pd\bar{t} + Fd\bar{x} \quad (9.1.3)$$

$$TdS = -Pd\bar{t} + Fx'(0)d\bar{L} + Fx''(0)d\bar{L} + \frac{Fx''''(0)}{2}L^2d\bar{L} + 4O(L^3)d\bar{L} \quad (9.1.4)$$

Something interesting happens with the units of the Taylor expansion. Let us investigate:

<table>
<thead>
<tr>
<th>Taylor term</th>
<th>quantity</th>
<th>units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Fx'(0)d\bar{L}$</td>
<td>$F$</td>
<td>$N$</td>
</tr>
<tr>
<td>$Fx''(0)d\bar{L}$</td>
<td>$x'(0)$</td>
<td>$\text{?}$</td>
</tr>
<tr>
<td>$Fx'''(0)Ld\bar{L}$</td>
<td>$d\bar{L}$</td>
<td>$m$</td>
</tr>
<tr>
<td>$Fx'''''(0)L^2d\bar{L}$</td>
<td>$x''(0)$</td>
<td>$1/m$</td>
</tr>
<tr>
<td>$Fx''''''(0)L^3d\bar{L}$</td>
<td>$Ld\bar{L}$</td>
<td>$m^2$</td>
</tr>
</tbody>
</table>

Since $Ld\bar{L}$ has units $m^2$ and $L^2d\bar{L}$ has units $m^3$, we pose $\gamma d\bar{A} := Ld\bar{L}$ and $ad\bar{V} := L^2d\bar{L}$. Furthermore, as $x'(0)$ has no units, we define it as the baseline $x'(0) := 1$ and we define $x''(0) := l_A/L_0$ and $x''''(0) := l_V/A_0$ as they respectively have units $m^{-1}$ and $m^{-2}$. For empirical reasons (e.g., the observable universe is a sphere), we consider that $\gamma d\bar{A}$ describes the surface of a sphere and that $ad\bar{V}$ describes the volume of a sphere. Therefore, to properly link $\gamma d\bar{A}$ to $Ld\bar{L}$, the factor $\gamma$ must be $1/(4\pi)$ and the factor $a$ must be $3/(4\pi)$. Finally, we absorb 4 into $O(L^3)d\bar{L}$. Performing these replacements, the equation of state becomes:

$$TdS = -Pd\bar{t} + Fd\bar{x} + l_A\frac{F}{4\pi L_0}d\bar{A} + l_V\frac{3F}{8\pi A_0}d\bar{V} + O(L^3)d\bar{L} \quad (9.1.14)$$
where \( l_A \) and \( l_V \) are leftovers of the Taylor coefficients. We can recover three relations by varying the intensity of the Taylor approximation.

\[
T dS = -P d\bar{t} + F d\bar{\mathcal{L}} + O(\bar{\mathcal{L}}) d\bar{\mathcal{L}} \tag{9.1.15}
\]

\[
T dS = -P d\bar{t} + F d\bar{\mathcal{L}} + l_A \frac{F}{4\pi L_\text{0}} d\bar{A} + O(\bar{\mathcal{L}}^2) d\bar{\mathcal{L}} \tag{9.1.16}
\]

\[
T dS = -P d\bar{t} + F d\bar{\mathcal{L}} + l_A \frac{F}{4\pi L_\text{0}} d\bar{A} + l_V \frac{3F}{8\pi A_\text{0}} d\bar{V} + O(\bar{\mathcal{L}}^3) d\bar{\mathcal{L}} \tag{9.1.17}
\]

\[
9.2 \quad \text{Inertial mass}
\]

In this section, we will need to use the Unruh temperature\(^{30}\). As can be reviewed in the citations provided, the Unruh temperature is an exact result obtained from special relativity. The Unruh effect is the prediction that an accelerating observer will observe blackbody radiation (at the Unruh temperature), whereas an inertial observer would observe none. The Unruh temperature is:

\[
T = \frac{\hbar a}{2\pi c k_B} \tag{9.2.1}
\]

The Unruh temperature connects acceleration to the temperature. We will use it here to convert an entropic force expressed in terms of a temperature to an entropic force expressed in terms of acceleration.

Furthermore, we start from the power-time formulation and pose \( d\bar{t} = 0 \). As originally done by Erik Verlinde\(^{31}\), from these starting points, we can derive \( F = ma \) as follows:

\[
T dS = F d\bar{\mathcal{L}} + O(\bar{\mathcal{L}}) d\bar{\mathcal{L}} \tag{9.2.2}
\]

Posing the approximation of short distances \( O(\bar{\mathcal{L}}) d\bar{\mathcal{L}} \to 0 \), we obtain:

\[
T dS = F d\bar{\mathcal{L}} \tag{9.2.3}
\]

\[
F = T \frac{dS}{d\bar{\mathcal{L}}} \tag{9.2.4}
\]

\[
F = \left( \frac{\hbar a}{2\pi c k_B} \right) \frac{dS}{d\bar{\mathcal{L}}} \tag{9.2.5}
\]

\[
F = \left( \frac{\hbar}{2\pi c k_B} \frac{dS}{d\bar{\mathcal{L}}} \right) a \tag{9.2.6}
\]

This equation corresponds to \( F = ma \) provided that \( \left( \frac{\hbar}{2\pi c k_B} \frac{dS}{d\bar{\mathcal{L}}} \right) = m \). How reasonable is that? Well, for it to be the mass, it suffices that


\]
\( \frac{dS}{dL} \) is the inverse of the reduced Compton wavelength multiplied by a constant. Recall that the reduced Compton wavelength is \( \frac{\hbar}{(mc)} \). Let us investigate:

\[
\frac{\hbar}{2\pi ck_B} \frac{dS}{dL} = m \implies \frac{dS}{dL} = 2\pi k_B \left( \frac{mc}{\hbar} \right)
\]  

(9.2.7)

We obtain a relation between entropy and \( L \). What could this mean? It means two things.

1. The further away an object is from the origin, the higher it’s positional entropy is.

2. The more massive an object is, the higher its positional entropy.

Why then the factor \( 2\pi \)? The presence of \( \pi \) suggests a connection between a line and a circle. Therefore, a possible interpretation is that the entropy associated with positional entropy is scaled proportionally to the curvature of a circle (we can think of it as a one-dimensional case of the holographic principle). Then, as an object with a small Compton wavelength that can be more finely located, it requires more positional entropy to describe its position than an object with a large Compton wavelength. Why then the factor \( k_B \)? The factor \( k_B \) converts the reduced Compton wavelength to the units of entropy/length (joules per kelvin per meter).

*How does this connect to the analytical-fact interpretation?* Quite nicely in fact. A fact \( q \) is describable by a binary string. Longer descriptions require more bits, thus also have a higher entropy. This is why an object further away from an origin must have a higher entropy —we need more bits to accurately describe its position. A more massive object also requires more bits (its Compton wavelength is smaller): This role is assumed by the Lagrange multiplier \( F \) in the analytical-fact representation which compressed the bits together. The two interpretations are isomorphic.

### 9.3 Gravitational constant

To find a suitable definition for \( G \), we must derive Newton’s law of gravitation from the equation of states. A derivation of Newton’s law of gravitation from the entropic perspective has been done before by Erik Verlinde. His derivation can be imported into our equation of states. To obtain \( G \), we start from the power-time formulation expanded with two Taylor terms:

\[
TdS = -Pd\bar{t} + Fd\bar{L} + l_A \frac{F}{4\pi L_0} d\bar{A} + O(\bar{L}^2) d\bar{L}
\]  

(9.3.1)
Then, we pose \( d\bar{t} = 0 \) and \( O(\bar{L}^2)\, d\bar{L} \to 0 \). We obtain:

\[
TdS = Fd\bar{L} + l_A \frac{F}{4\pi L_0} d\bar{A} \quad (9.3.2)
\]

We notice that the term \( d\bar{L} \) grows linearly as the term \( d\bar{A} \) grows quadratically. Thus, as \( \bar{L} \) is increased, there will be a point where \( d\bar{A} \gg d\bar{L} \) (recall that \( d\bar{A} = Ld\bar{L} \)). The approximation yields:

\[
TdS = l_A \frac{F}{4\pi L_0} d\bar{A} \quad (9.3.3)
\]

This regime contains the holographic principle and, as a result, the entropy of the system grows proportional to \( \bar{L}^2 \), an area law. To recover Newton’s law of gravity, and consistent with the holographic principle, we further pose the assumption that an entropy is associated to this area law and is given by bits occupying a small area \( L^2 \) on the surface of a sphere. In this case, the total number of bits on the surface is given by:

\[
N = \frac{4\pi x^2}{L^2} \quad \text{holographic assumption} \quad (9.3.4)
\]

The term \( \bar{L}d\bar{L} \) of the equation of state is associated to \( L^2 / 2 \) in the partition function. As a result of the equipartition theorem, which applies to quadratic energy terms, the average energy will be \( \bar{E} = k_B T / 2 \). Multiplying \( \bar{E} \) by \( N \), we get the total energy associated with \( \bar{L}d\bar{L} \):

\[
E = \frac{1}{2} \left( \frac{4\pi x^2}{L^2} \right) k_B T \quad (9.3.5)
\]

\[
\Rightarrow T = \frac{L^2 \, E}{2\pi k_B \, x^2} \quad (9.3.6)
\]

Consistent with thermodynamic equilibrium, we obtain a temperature \( T \). As our goal is to recover the gravitational force, we inject this temperature in the entropic force relation.

\[
Fd\bar{L} = TdS \quad \text{entropic force} \quad (9.3.7)
\]

\[
Fd\bar{L} = \left( \frac{L^2 \, E}{2\pi k_B \, x^2} \right) dS \quad \text{derived temperature} \quad (9.3.8)
\]

\[
F = \left( \frac{L^2 \, E}{2\pi k_B \, x^2} \right) \frac{dS}{d\bar{L}} \quad (9.3.9)
\]
What then is \(dS/d\mathcal{L}\)? Recall equation 9.2.7; the connection between the reduced Compton wavelength and the distance entropy.

\[
F = \left( \frac{L^2}{2\pi k_B} \frac{E}{x^2} \right) \left( 2\pi k_B \frac{mc}{h} \right) \quad \text{Compton wavelength} \quad (9.3.10)
\]

\[
F = \left( \frac{L^2c^3}{h} \right) \frac{Em}{x^2} \quad \text{clean up} \quad (9.3.11)
\]

We then convert \(E\) to its rest mass via \(E = mc^2\).

\[
F = \left( \frac{L^2c^3}{h} \right) \frac{Mm}{x^2} \quad (9.3.12)
\]

We obtain the Newton’s law of gravitation along with a definition for \(G\).

\[
F = G \frac{Mm}{x^2} \quad (9.3.13)
\]

\[
\Rightarrow G := \frac{L^2c^3}{h} \quad (9.3.14)
\]

which further implies that

\[
L = \sqrt{\frac{hG}{c^3}} \quad \text{Planck’s length} \quad (9.3.15)
\]

### 9.4 Energy-to-frequency equation

We have previously shown that \(0 = Fd\mathcal{x} - Pd\mathcal{t}\) implies that all observers agree on a certain speed \(c\). We now ask; what is the energy relation of such a system with respect to frequency? To derive the result, we assume that the system has access to a pool of energy. The question, then becomes; how much energy must be taken from the pool increase the frequency to \(f\)? With access to a pool of energy and posing \(d\mathcal{x} = 0\), the power-time formulation becomes:

\[
TdS = d\mathcal{E} - Pd\mathcal{t} \quad (9.4.1)
\]

\[
\frac{TdS}{dt} = \frac{d\mathcal{E}}{dt} - P \quad (9.4.2)
\]

We consider the case of speed \(c\). Thus, as we have seen in the section on special relativity, this implies that \(dS/d\mathcal{t} = 0\).

\[
d\mathcal{E} = Pd\mathcal{t} \quad (9.4.3)
\]

We change to the action-frequency formulation by posing \(f = 1/t\) and \(-P = S\mathcal{t}^2\)

\[
-d\mathcal{E} = Sd\mathcal{f} \quad (9.4.4)
\]
Then, integrating:

$$- \int d\mathcal{E} = S \int d\tilde{f}$$  \hspace{1cm} (9.4.5)

$$-\bar{\mathcal{E}} + C_1 = S\tilde{f} + C_2$$  \hspace{1cm} (9.4.6)

Here, $\bar{\mathcal{E}}$ is the energy that must be taken from the pool for the system to occupy a micro-state with frequency $\tilde{f}$. Reversing the sign of $\bar{\mathcal{E}}$ and posing the integration constants to 0, we obtain the energy associated with it: $\mathcal{E} = Sf$. Furthermore, recall that we fixed $dS/d\tilde{f} = 0$ which is associated with speed $c$.

Indeed, posing $S = h$, the Planck action, we do recover the energy-to-frequency relation of a photon: $\mathcal{E} = hf$.

### 9.5 Planck units

We have now obtained a definition for three of the fundamental constants.

$$h := S \quad c := \frac{P}{F} \quad G := \frac{L^2 c^3}{\hbar}$$ \hspace{1cm} (9.5.1)

Thus, we can now show that the Lagrange multipliers of the equation of states $P$ and $F$ are indeed the Planck units.

<table>
<thead>
<tr>
<th>expression</th>
<th>quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G = \frac{L^2 c^3}{\hbar}$</td>
<td>$L = \sqrt{\frac{hG}{c^3}}$ Planck’s length (9.5.2)</td>
</tr>
<tr>
<td>$t = \frac{L}{c} = \sqrt{\frac{hG}{c^5}}$</td>
<td>Planck’s time (9.5.3)</td>
</tr>
<tr>
<td>$P = t^2 S = 2\pi c^5 \frac{G}{G}$</td>
<td>Planck’s power* (9.5.4)</td>
</tr>
<tr>
<td>$\frac{P}{F} = c \implies F = 2\pi c^4 G$</td>
<td>Planck’s force* (9.5.5)</td>
</tr>
</tbody>
</table>

*The reader will notice that we have obtained the definitions of $P$ and $F$ with an added multiplication constant $2\pi$; whereas in the literature these quantities are defined without it. The definitions we have here are actually the correct ones. Indeed, in the literature, the Planck time is connected to the Planck angular frequency via $\omega_P = 1/t_P$. In reality, however $\omega = 2\pi / t$. Thus, for our equations to balance out, we cannot ignore the factor $2\pi$ and must use the corrected value for the Planck units which are $P = 2\pi c^5 / G$ and $F = 2\pi c^4 / G$. 

9.6 General relativity

In this section, we will show how the term $dA$ suggests that general relativity is emergent and entropic. Our goal is to derive the Einstein field equation of general relativity, starting from the $dA$ regime. First, we start from the power-time formulation expanded with two Taylor terms:

$$TdS = -Pd\bar{t} + Fd\bar{L} + l_A\frac{F}{4\pi L_0}dA + O(\bar{L}^2)d\bar{L}$$  \hspace{1cm} (9.6.1)

Then, we pose $d\bar{t} = 0$ and $O(\bar{L}^2)d\bar{L} \to 0$. We obtain:

$$TdS = Fd\bar{L} + l_A\frac{F}{4\pi L_0}dA$$  \hspace{1cm} (9.6.2)

We notice that the term $d\bar{L}$ grows linearly and the term $dA$ grows quadratically. Thus, as $\bar{L}$ is increased, there will be a point where $dA \gg d\bar{L}$. The approximation yields:

$$TdS = l_A\frac{F}{4\pi L_0}dA$$  \hspace{1cm} (9.6.3)

Deriving general relativity from $TdS = l_A\frac{F}{4\pi L_0}dA$ has indeed been done before in the literature, notably by Ted Jacobson\textsuperscript{33}, then later (and differently) by Erik Verlinde\textsuperscript{34}. Furthermore, key insights were provided by Christoph Schiller\textsuperscript{35}. Here, we will provide a sketch of the proof by Ted Jacobson as summarized by Schiller.

First, the entropic force $F$ is constant throughout the system as a result of being a Lagrange multiplier. We have already shown that $F$ is the Planck force. This has allowed us to derive special relativity and the speed of light; therefore, we must continue to use $F$ as the Planck force here.

What then is $L_0$? Recall that earlier we used the Unruh temperature to link $T$ to an acceleration and derive $F = ma$. Here and likewise, we will use special relativity to derive a relation between length and acceleration and use it to replace $L$. As per Schiller’s paper, we select $L_0$ as the maximum length that an accelerated object can have under special relativity\textsuperscript{36}.

$$L_0 = \frac{c^2}{2a}$$  \hspace{1cm} (9.6.4)

$L_0$ is perhaps better understood as the acceleration of circular motion ($r = v^2/a$) at the speed of light ($v = c$). In the present context, $L$ is the length associated with the maximum force, the Planck force.
In the context of maximums, the force cannot accelerate the object beyond the speed of light, and therefore is best defined for a circular motion produced by a force perpendicular to the direction of motion. The maximum acceleration changes the direction of the motion, but does not increase the speed beyond the speed of light.

With $F = 2\pi c^4 / G$, we obtain:

$$TdS = l_A \frac{c^2}{G} ad\bar{A} \quad (9.6.5)$$

With this result, Jacobson’s proof directly follows. Starting from $dE = TdS$, he first connects $TdS$ to an arbitrary coordinate system and energy flow rates:

$$TdS = \int T_{ab} k^a d\Sigma^b \quad (9.6.6)$$

Here $T_{ab}$ is an energy-momentum tensor, $k$ is a killing vector field, and $d\Sigma$ the infinitesimal elements of the coordinate system. Jacobson then shows that the area part can be rewritten as follows:

$$ad\bar{A} = c^2 \int R_{ab} k^a d\Sigma^b \quad (9.6.7)$$

where $R_{ab}$ is the Ricci tensor describing the space-time curvature. This relation is obtained via the Raychaud-Huri equation, giving it a geometric justification. Combining the two with a local law of conservation of energy, he obtains

$$\int T_{ab} k^a d\Sigma^b = l_A \frac{c^2}{G} \int R_{ab} k^a d\Sigma^b \quad (9.6.8)$$

which can only be satisfied if

$$T_{ab} = l_A \frac{c^2}{G} \left[ R_{ab} - \left( \frac{R}{\Sigma} + \Lambda \right) g_{ab} \right] \quad (9.6.9)$$

Here, the full field equations of general relativity are recovered, including the cosmological constant (as an integration constant). Only the numerical value of $l_A$ remains. The exact formulation of the field equation is obtained by posing the numerical value to $l_A := 1 / (8\pi)$.

**Remark:** Had we not used the corrected Planck force ($F = 2\pi c^4 / G$), we would have a $2\pi$ term dividing $T_{ab}$, and $l_A$ would have been 1/4. Thus, the difference would have been absorbable. However, using the corrected Planck force has the consequence that all dimensionless numerical multipliers are attributed to the Taylor coefficient, making the derivation more aesthetically pleasing.
9.7 Dark energy

Connecting dark energy to a volumetric entropy has been suggested and discussed by other authors before\(^37\). First, we start from the power-time formulation expanded with three Taylor terms:

\[
TdS = -Pd\tau + Fd\bar{L} + l_A \frac{F}{4\pi L_0} d\bar{A} + l_V \frac{3F}{8\pi A_0} d\bar{V} + O(L^3) d\bar{L} \quad (9.7.1)
\]

Then we pose \(d\tau = 0\) and \(O(L^3) d\bar{L} \to 0\). We obtain:

\[
TdS = Fd\bar{L} + l_A \frac{F}{4\pi L_0} d\bar{A} + l_V \frac{3F}{8\pi A_0} d\bar{V} \quad (9.7.2)
\]

We notice that as \(d\tau\) grows linearly, \(d\bar{A}\) grows as the square and \(d\bar{V}\) as the cube. Thus, there will be a point where \(d\bar{V} \gg d\bar{A} \gg d\bar{L}\). The approximation yields:

\[
TdS = l_V \frac{3F}{8\pi A_0} d\bar{V} \quad (9.7.3)
\]

We notice that the factor \(F/A_0\) has the units of pressure. Hence, our goal will be to derive a value of the pressure \(p\) associated with volumetric entropy. As suggested by the factor \(F/A_0\) and in line with our earlier derivations, we will select \(F\) to be the corrected Planck force \((F = 2\pi^4 c^4/G)\) and will take \(A_0\) as the area of a sphere. In this case, the pressure relates to the force as

\[
F = -pA \quad (9.7.4)
\]

\[
\Rightarrow p = -\frac{F}{A} = -\frac{F}{4\pi x^2} \quad (9.7.5)
\]

\[
p = -\frac{c^4}{2Gx^2} \quad \text{entropic pressure} \quad (9.7.6)
\]

The sign of the force is negative because the force points in the direction of increased entropy, which is oriented outward of the enclosing area. Physically and as argued by Easson et al., it makes sense to connect the size of the sphere to the Hubble horizon. Therefore, we take the radius of the sphere to be the Hubble radius \(x := c/H\). Finalizing our derivation, we obtain:

\[
p = -\frac{c^2 H^2}{2G} \quad (9.7.7)
\]

This is close to the current measured value for the negative pressure associated with dark energy\(^38\). As we can see, the suggested entropic derivation of dark energy applies to the third term of the Taylor expansion.

---


10 Discussion —Arrow of time

Adding a time variable to a partition function adds a whole new dynamic to a thermo-statistical system. The system now becomes aware of future, past, and present configurations and can translate from time to space and from space to time for an entropic cost (provided that various limits are respected). By studying thermodynamic cycles involving space and time, we investigated what happens to the entropy when a system is translated forward or backward in time and draw conclusions that pertain to the arrow of time. In the model presented, space serves as an entropy sink for time; whose role is to power change in the universe by depleting the system of future alternatives (entropic power).

10.1 Negative power

In the power-time formulation, increasing $\tilde{t}$, while keeping the other variables constant, decreases the entropy. Indeed, starting with the power-time formulation and posing $\frac{dx}{dt} = 0$, we obtain:

$$TdS = -Pd\tilde{t} \quad (10.1.1)$$
$$\Rightarrow T\frac{dS}{d\tilde{t}} = -P \quad (10.1.2)$$

This result is expected for the following reason: to obtain the relation $dx = cd\tilde{t}$ with the correct signs, the power $P$ must have a different sign than the force $F$ in the equation of states. Thus, a positive force implies a negative power and vice versa. As we require a positive force to recover $F = ma$ (and not $F = -ma$), the sign of the force is already chosen for us. Therefore, the power must be negative.

We will now discuss this result in more detail.

What is a negative power? Let’s take an example. Consider the case of an electric car; whose engine is powered by a battery. To propel the car, the battery supplies power to the engine. If the driver hits the breaks, such that regenerative braking kicks in, the flow of power will reverse and the engine will supply power to the battery. Thus, the power is now considered to be negative and occurs when the engine depletes the energy of the system (e.g. the car slows down) to supply power to the battery.

Why does time have a negative power? Power is associated with time because it powers all changes that occur in the universe. To understand why it is negative, it helps to understand negative power in the
context of thermodynamics. To do so, let’s first recall its more familiar cousin: the negative temperature. If we understand temperature as the random movements of molecules, then a temperature is always equal to or above zero. However, statistical physics admits a generalized definition of temperature as the trade-off between energy and entropy. Most systems cannot admit a negative temperature because their entropy will always increase at higher energies; however, for some systems, e.g. the population inversion in a laser, the entropy saturates at higher energies. Thus, a negative temperature is possible.

In regards to time, the negative power has essentially the same interpretation; increasing time, while keeping the other variables constant, decreases the entropy. A decrease in entropy over time produces a negative entropic power.

Why are there fewer micro-states available in the future? Most programs stop quickly or never halt\footnote{Cristian S Calude and Michael A Stay. Most programs stop quickly or never halt. Advances in Applied Mathematics, 40(3):295–308, 2008}. This result can be intuitively understood as follows: We know that when \( t \to \infty \), the partition function has no available micro-states because non-halting programs are not part of \( Z \). Furthermore, for any finite set of programs as \( t \) is increased, there are fewer programs left in the set that have yet to halt. The implication is that the entropy of the system is reduced as \( t \) is increased.

10.2 The second law of thermodynamics as an opposition to negative power

**Question:** How does this result reconcile with the second law of thermodynamics, which states that entropy increases with time (or in some ideal cases stays constant)?

The power-time formulation admits other terms: \( dL, dA, \) and \( dV \). The term \( -Pdt \) encourages a reduction in the entropy over time, but the other variables, as their signs are positive, work in the other direction. Thus, the entropy of the system as a whole need not necessarily decrease over time. It is more accurate to say that increasing \( t \), while keeping the other variables constant, decreases the entropy. We will now study this in more detail.

To offset the decrease in entropy caused by the negative power, we suggest a proportional increase in the quantities \( L, A, \) and \( V \).

To simplify the power-time formulation, let us rename \( \kappa := \frac{F}{16\pi L} \) and \( p := \frac{3V\ell}{4\pi A} \) and pose \( O(L^3)dL \to 0 \). We obtain:

\[
TdS = -Pdt + FdL + \kappa dA + pdV
\] (10.2.1)

Dividing both sides by \( dt \), we obtain:

\[
\frac{T}{dt}dS = -P + \frac{F}{dt}dL + \kappa \frac{dA}{dt} + p \frac{dV}{dt}
\] (10.2.2)
This result puts in opposition the change of entropy caused by a change of $\bar{t}$ to the change in entropy caused by a change of $\bar{L}, \bar{A}$ and $\bar{V}$. To investigate this result, let us look at these three cases:

$$F \frac{d\bar{L}}{dt} + \kappa \frac{d\bar{A}}{dt} + p \frac{d\bar{V}}{dt} < P \implies \frac{dS}{dt} < 0 \quad \text{decreasing entropy}$$

(10.2.3)

$$F \frac{d\bar{L}}{dt} + \kappa \frac{d\bar{A}}{dt} + p \frac{d\bar{V}}{dt} = P \implies \frac{dS}{dt} = 0 \quad \text{constant entropy}$$

(10.2.4)

$$F \frac{d\bar{L}}{dt} + \kappa \frac{d\bar{A}}{dt} + p \frac{d\bar{V}}{dt} > P \implies \frac{dS}{dt} > 0 \quad \text{increasing entropy}$$

(10.2.5)

At (10.2.4), we have an inflection point and a shift occurs in the direction of the production of entropy over time. It is the point at which the production of entropy caused by the space quantities overtake and exceed the reduction in entropy caused by the time quantity. The second law of thermodynamics states that $\frac{dS}{dt} \geq 0$ and will hold for (10.2.4) and (10.2.5), but will be violated for (10.2.3).

10.3 Arrow of time

In this section, we will explain why these results provide us with an understanding of the arrow of time. Indeed, it links the arrow of time to three concepts: 1) a reduction in entropy over time caused by the negative power, 2) an increase in entropy over time caused by the space quantities, and 3) a closed system’s inability to reduce its own entropy. We will see how it corresponds to an observer’s perception of time.

Definition 10.3.1 (Future alternative). A future alternative refer to a micro-state $q$ that may be occupied at a future time and that has a $t(q) > \bar{t}$.

Definition 10.3.2 (Historical record). An historical record refer to a micro-state $q$ that may be occupied at a past time and that has a $t(q) < \bar{t}$.

1. At the beginning of time all micro-states are future alternatives.

   Thus, the pool of future entropy accessible to $\bar{t}$ is maximal. In contrast, as no programs have yet halted, the entropy associated with the space quantities is zero. This is compatible with our current empirical data regarding the Big Bang for which the entropy of space was very low and the entropy of time, as the future was as of yet undetermined, was very high.

2. During the evolution the future becomes past and the possible future alternatives are consumed and converted to a historical
record. This reduction in entropy caused by a growth in $I$ produces a negative entropic power that fuels the growth of entropy in the space quantities.

3. At the "end of time" there is no future alternatives. The full history of the system is now recorded. The system can no longer produce an entropic power to fuel changes and the entropy associated with the space quantities is at its maximum.

**Question:** The conventional wisdom is that the arrow of time is connected to an increase in entropy with time. Are you suggesting something else?

A partition function constructed without the use of a time quantity will follow the second law of thermodynamics. This statistical effect is partially explained by the H-theorem of Boltzmann; however, this changes when time is inserted as a thermodynamic quantity. Such a partition function then becomes aware of past, present, and future micro-states. The rarefaction of futures alternatives as time is increased is associable to a time which moves forward by closing future alternatives as it creates a historical record (a past). Thus, an increase in the time quantity, while keeping other quantities constant, must be followed by a decrease in entropy.

To help fixate the idea, let us look at an example:

### 10.4 The physics of future alternatives

Here, we give a simple system which follows the requirements of the equation of states.

Suppose a system with $n$ open binary future alternatives. At $\bar{t} = 0$, there are $2^n$ possible futures each equally compatible with the present macroscopic state. Thus, the entropy of the system (which includes a description of its possible futures) is equal to $S = k_B n \ln 2$. As time is increased, events occur: a future alternative is closed and converted to a historical record. Say, at $\bar{t} = 1$, one event occurs: Thus, one future alternative becomes a historical record and the entropy of the system’s future is reduced to $S = k_B (n-1) \ln 2$.

For instance, we might have:

<table>
<thead>
<tr>
<th>$\bar{t}$</th>
<th>event</th>
<th>future alternatives</th>
<th>future’s entropy</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Big Bang</td>
<td>${ b_1, b_2, b_3, ..., b_n }$</td>
<td>$k_B n \ln 2$</td>
</tr>
<tr>
<td>1</td>
<td>$b_3 \rightarrow 0$</td>
<td>${ b_1, b_2, b_3 := 0, b_4, ..., b_n }$</td>
<td>$k_B (n-1) \ln 2$</td>
</tr>
<tr>
<td>2</td>
<td>$b_1 \rightarrow 1$</td>
<td>${ b_1 := 1, b_2, b_3 := 0, b_4, ..., b_n }$</td>
<td>$k_B (n-2) \ln 2$</td>
</tr>
<tr>
<td></td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
As events occur over time, an entropic power is generated. Furthermore, the second law of thermodynamics imposes that the space quantity \( (F \, d\vec{x}) \) must grow proportionally. To maintain \( dS/d\bar{t} = 0 \), the growth must correspond to \( d\vec{x} = cd\bar{t} \); special relativity. Extending this example to the continuous partition function, we also recover general relativity and dark energy as per the earlier derived equations of states. In the continuous case, we would use the natural bit (the nat, in base \( e \)) to express future possibilities. A continuous event would consume a non-integer quantity of future possibilities.

**But a system cannot decrease its entropy over time without violating the second law of thermodynamics!** A system can decrease its entropy if it is connected to an entropy sink. For example, biological life can reduce its entropy but only at the cost of severely increasing it in its environment. This requires excess energy (Gibbs free energy) and, in the case of Earth, the Sun supplies it. Thus, the power-time conjugate can decrease the entropy as long as it is connected to a sink.

**So, there should be a sink in the universe available to offset the decrease in entropy caused by increasing \( \bar{t} \)?** In the case of time, the sink is the universe itself. The laws of physics that we have derived are in fact the limits required to produce an entropy sink of sufficient size to accommodate a forward direction of time for an observer (we will discuss this more rigorously in a moment in the section on limiting relations).

**Can we calculate the exact future before it occurs?** An observer cannot pre-calculate his exact future before it occurs without increasing the size of the entropy sink. Here we make a distinction between calculating a probable future versus the exact future. Calculating a probable future does not necessarily imply a reduction of entropy within the system, but calculating the exact future requires consuming the entropy of all possible alternative futures. Therefore, an entropy sink is required to offset the reduction. Calculating an exact future is equivalent to causing it.

**Does the second law of thermodynamics need to be corrected for the wider system, which includes future states?** Yes. Time is usually considered to be an independent background to statistical physics and, to our knowledge, statistical physics has not been used with a time quantity before. When we do add time as a thermodynamic quantity to a partition function, a new behavior emerges. Indeed, an observer cannot move into the future unless all alternative futures are ‘closed’. Thus, its time-entropy must decrease when he does. The second law of
thermodynamics is a consequence of the system increasing its space-entropy to offset the reduction in future alternatives as time moves along. Thus, this system follows a general entropy conservation law.

10.5 Limiting relations

With our new interpretation of space as an entropy sink for time, let us immediately prove three limits from first principles: the speed of light, a limiting stiffness, and a limiting volumetric flow rate applicable to the universe. To prove that they are limits, we will consider the assumption that an observer who evolves forward in time must see a growth in the size of its available entropy sink to offset the reduction in future alternatives. The limit occurs when the sink exactly offsets the reduction in entropy attributable to time (in which case \( dS/dt = 0 \)). First, let us see how the power-time formulation implies a limiting speed.

\[
TdS = -Pd\hat{t} + Fd\bar{x}
\]

(10.5.1)

\[
\frac{T}{F} \frac{dS}{dt} = -\frac{P}{F} + \frac{d\bar{x}}{d\hat{t}}
\]

(10.5.2)

To see why this implies a limiting speed, first, consider that the units of this equation are \( \text{length/time} \) and hence are indeed describing a speed. Second, consider the following three cases:

\[
\frac{d\bar{x}}{d\hat{t}} = \frac{P}{F} \implies \frac{dS}{dt} = 0 \quad (10.5.3)
\]

\[
\frac{d\bar{x}}{d\hat{t}} < \frac{P}{F} \implies \frac{dS}{dt} < 0 \quad (10.5.4)
\]

\[
\frac{d\bar{x}}{d\hat{t}} > \frac{P}{F} \implies \frac{dS}{dt} > 0 \quad (10.5.5)
\]

We notice a reversal in the production of entropy at the inflection point where \( dS/dt = 0 \). Therefore, for an observer at rest to evolve forward in time, it must see its entropy sink grow at the speed of \( c := P/F \). Therefore, the entropy sink of an observer moving forward in time must grow at the speed of light.

The following relations each characterize a limiting quantity.
Each relation can easily be obtained from the power-time formulation by posing the other quantities as 0. To show that the quantities are inflection limits, it suffices to notice that they each correspond to a growth of the entropy sink that an observer at rest must see to fuel its forward translation in time.

It is well known that a limiting speed implies special relativity, but what about the other two limits? It is less known, but a maximum stiffness does imply general relativity. In this context, we can interpret space as being very stiff but nonetheless compressible. The maximum volumetric flow rate is associated with dark energy and is responsible for the Hubble horizon—beyond which the flow rate would be exceeded. These are in fact the approaches (in disguise) that we took to derive general relativity and dark energy earlier.

## 11 Fluctuating space-time

What are thermal time and thermal space? Consider the thermodynamic quantities $t$ and $x$ of the power-time formulation. Their average value is given by the standard relations (from 6.2.10):

<table>
<thead>
<tr>
<th>quantity</th>
<th>average</th>
</tr>
</thead>
<tbody>
<tr>
<td>thermal-time</td>
<td>$t$</td>
</tr>
<tr>
<td>thermal-space</td>
<td>$x$</td>
</tr>
</tbody>
</table>

Furthermore, as thermal-time and thermal-space are thermodynamic averages, they will undergo fluctuations (from 6.2.10):

<table>
<thead>
<tr>
<th>quantity</th>
<th>fluctuation</th>
</tr>
</thead>
<tbody>
<tr>
<td>thermal-time</td>
<td>$(\Delta t)^2$</td>
</tr>
<tr>
<td>thermal-space</td>
<td>$(\Delta x)^2$</td>
</tr>
</tbody>
</table>
Using the original argument made by Einstein in 1905, which led to the derivation of Brownian motion, we argue here that fluctuations of the $t$ and $x$ variables produce a universal Brownian motion along the axis themselves. What does a thermal space-time with fluctuations look like? The consequences of such are nothing to be feared; indeed, we will shortly show that Brownian motion over $\mathcal{X}$ will produce the Schrödinger equation (section 11.1) and that Brownian motion over both $\mathcal{X}$ and $\mathcal{I}$ will produce the Dirac equation (section 11.2).

Remark. We also show an alternative way to derive these equations from the Feynman path integral formulation (section 12). But the derivation from thermal fluctuation is so direct and intellectually pleasing that we will do it nonetheless. This derivation gives a very natural explanation for the Schrödinger equation.

To derive these equations from thermal fluctuations, we import the mathematical proofs of the field of Stochastic mechanics. The derivation of the Schrödinger equation from Brownian motion was done by Nelson\textsuperscript{40} and reviewed by the same author some 46 years later\textsuperscript{41}. The field was created to offer an alternative (non-quantum) derivation of the Schrödinger equation. Particles were imagined to be punctual, to undergo a type of universal Brownian motion through all space; this resulted, with no additional assumptions, in their behavior corresponding to the Schrödinger equation. The theory agrees observationally with the usual wave interpretation of particles. Thus, the argument was: haha! Particles are punctual!

Here, however, we do not have to agree with the initial motivation of the derivation. Indeed, the math still holds regardless of the initial motivations. We adopt a more "software engineering" view of the physical reality. The particle, whatever it may be, provides a public interface to the observer (these are the allowed measurements), whereas its inner workings are private. As only the public interface is available, the observer ought to develop a picture of the particle using only this interface and to ignore its private content —thus, the wave interpretation is the best explanation.

To reconcile the fields, rather than suggesting that Brownian motion applies to the particle, we are instead suggesting that any positional or time information undergoes a "Dirac equation-like diffusion" so as to make positional or time information perishable over time. To illustrate, we can imagine placing a mark at a position in space. After a certain time, the Brownian motion will diffuse the position of the marker at any number of possible locations until its actual position is measured again. Instead of being punctual, the marker could be continuous and weighted and the same diffusion-
like behavior will be observed. This Brownian motion would universally apply to the axis itself. This is not a claim that a particle is punctual.

At the end of his review, Nelson goes on to list what he claims to be a series of failures of Stochastic mechanics. The word failure is a bit strong, as the problems have no observational consequences — they only occur under the hood. Nelson states: "The non-locality of stochastic mechanics conspires to bring the records into agreement." The problems are interpretative; how can two correlated particles separated by space communicate faster than light to agree with local measurements? This problem is because of the assumption that the particles exist punctually. Again, reinterpreting the derivation in terms of fluctuation of the axis of time and space which encodes position (rather than as fluctuations in the position of punctual particles) solves these interpretation issues.

**Remark.** That the fluctuations occur on the axis is directly supported by equation 11.0.3 and 11.0.4. We can now import the math of stochastic mechanics into our construction to derive the equations.

### 11.1 Schrödinger equation

Here, we will offer a sketch and refer to their respective authors for the more rigorous treatment. The derivation of the Schrödinger equation from Brownian motion was done by Nelson.\(^{42}\)

Nelson first considers the Langevin equation,

\[
\begin{align*}
\frac{d}{dt}[x(t)] &= v(t) dt \\
\frac{d}{dt}[v(t)] &= -\frac{\gamma}{m} v(t) dt + \frac{1}{m} W(t) dt
\end{align*}
\]

, which describes a particle in a fluid undergoing a Brownian motion as a result of the random collisions with the water molecules. Here \(W(t)\) is a noise term responsible for the Brownian motion and \(v(t)\) is a viscosity term specific to the properties of the fluid.

Nelson replaces the acceleration \(\frac{d[v(t)]}{dt}\) by \(F/m\) (from \(F = ma\)). Then, he is able to show that the Langevin equation in gradient form becomes:

\[
\nabla \left( \frac{1}{2} \tilde{u}^2 + D \nabla \cdot \tilde{u} \right) = \frac{1}{m} \nabla V
\]

where \(D := \hbar/(2m)\) is the diffusion coefficient, where \(\tilde{F} = -\nabla V\), where \(\tilde{u} = v \nabla \ln \rho\) and \(\rho\) is the probability density of \(x(t)\). As this is a
The world is an autology derived from all tautologies.

Eliminating the gradients on each side and simplifying the constants, Nelson obtains:

$$\frac{m}{2} \ddot{\vec{u}}^2 + \frac{\hbar}{2} \nabla \cdot \ddot{\vec{u}} = V - E \tag{11.1.4}$$

where $E$ is the arbitrary integration constant. Nelson then converts this equation to a linear equation via a change of variable applied to the term $\ddot{\vec{u}}^2$. Posing,

$$\ddot{\vec{u}} = \frac{\hbar}{m} \frac{1}{\psi} \nabla \psi \tag{11.1.5}$$

Nelson obtains

$$\left[ -\frac{\hbar^2}{2m} \nabla^2 + V - E \right] \psi = 0 \tag{11.1.6}$$

which is the time-independent Schrödinger’s equation. The time-dependent Schrödinger’s equation is recovered as per the usual replacement $\psi := e^{R+iS}$. Finally, Nelson obtains:

$$\frac{i\hbar}{\partial t}\psi(x,t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(x,t) \right] \psi(x,t) \tag{11.1.7}$$

, which is the time-dependent Schrödinger’s equation.

11.2 Dirac equation

We recently used the entropic force $TdS = Fdx$ and the Unruh temperature to recover $F = ma$. Then, we used $0 = -Pdt + Fdx$ to recover special relativity. Finally, we showed that a Brownian motion resulting from the thermal fluctuations of $x$ recovers the Schrödinger equation as a thermo-statistical analog to $F = ma$. Of course, the natural question to ask is this: will the thermal fluctuations of both $t$ and $x$ be enough to recover the Dirac equation as a thermo-statistical analog to special relativity? The answer is yes!

Similarly to the stochastic-mechanical derivation of the Schrödinger equation, other authors previously derived the Dirac equation from universal Brownian motion\footnote{D McKeon and G N. Ord. Time reversal in stochastic processes and the dirac equation. Physical review letters, 69:3-4, 08 1992}. In the original stochastic-mechanical derivation, the origin of such universal Brownian motion is ambiguous and is, at best, imported as a hypothesis. Thus, the benefit of our construction is to provide a thermal source of such universal Brownian motion. Hence, the derivation of the Dirac equation and the
The Schrödinger equation by these authors can nicely be imported into our thermodynamic construction.

The derivation of the Dirac equation was noticed by studying random walk effects that were applicable to telegraphic communication. McKeon and Ord propose a random walk model in space and in time which, once applied to the telegraph equations, produces the Dirac equation. We provide a sketch of the proof here and refer to the authors’ paper for the rigorous treatment. Starting from the equation for a random walk in space, the authors obtain:

\[ P_\pm(x, t + \Delta t) = (1 - a \Delta t)P_\pm(x \mp \Delta x, t) + a \Delta t P_\pm(x \mp \Delta x, t) \quad (11.2.1) \]

Afterward, the authors extend this equation with a random walk in time and obtain:

\[ F_\pm(x, t) = (1 - a_L \Delta t - a_R \Delta t)F_\pm(x \mp \Delta x, t) + \\
\quad a_{LR} \Delta t B_\pm(x \mp \Delta x, t + \Delta t) + a_{RL} \Delta t F_\pm(x \pm \Delta x, t - \Delta t) \quad (11.2.2) \]

where \( F_\pm(x, t) \) is the probability distribution to go forward in time and \( B_\pm(x, t) \) the probability distribution to go backward in time. They then introduce a causality condition such that \( F_\pm(x, t) \) and \( B_\pm(x, t) \) only depends on probabilities from the past.

\[ F_\pm(x, t) = B_\mp(x \pm \Delta x, t + \Delta t) \quad (11.2.3) \]

From equation (11.2.2) and (11.2.3), they get:

\[ B_\pm(x, t) = (1 - a_L \Delta t - a_R \Delta t)B_\pm(x \mp \Delta x, t + \Delta t) + \\
\quad a_{LR} \Delta t B_\pm(x \mp \Delta x, t + \Delta t) + a_{RL} \Delta t F_\pm(x \mp \Delta x, t - \Delta t) \quad (11.2.4) \]

In the limit \( \Delta x, \Delta t \to 0 \) and with \( \Delta x = v \Delta t \), they get:

\[ \pm v \frac{\partial F_\pm}{\partial x} + \frac{\partial F_\pm}{\partial t} = a_{LR} (-F_\pm + B_\pm) + a_{RL} (-F_\pm + F_\mp) \quad (11.2.5) \]
\[ \pm v \frac{\partial B_\pm}{\partial x} + \frac{\partial B_\pm}{\partial t} = a_{LR} (-B_\mp + F_\mp) + a_{RL} (-B_\mp + B_\pm) \quad (11.2.6) \]

Posing these changes of variables,

\[ A_\pm = (F_\pm - B_\pm) \exp[(a_L + a_R)t] \quad (11.2.7) \]
\[ \lambda := -a_L + a_R \quad (11.2.8) \]
then \ref{eq:11.2.6} becomes

\[ v \frac{\partial A_\pm}{\partial x} \pm \frac{\partial A_\pm}{\partial t} = \lambda A_\pm \]  \hspace{0.5cm} (11.2.9)\]

Finally, they pose \( v = c, \lambda = mc^2/\hbar \) and \( \psi = F(A_+, A_-) \), and they get

\[ i\hbar \frac{\partial \psi}{\partial t} = mc^2 \sigma_y \psi - i\hbar \sigma_z \frac{\partial \psi}{\partial x} \]  \hspace{0.5cm} (11.2.10)\]

which is the Dirac equation in 1+1 space-time.

\section{Feynman path integral formulation of QM}

The connection between classical statistical physics and quantum field theory is well established\cite{McCoy}. In classical statistical physics, we have:

\[ \overline{O}_j = \frac{1}{Z} \sum_{q \in Q} O_j e^{-E(q)/kT} \]  \hspace{0.5cm} (12.0.1)\]

and in quantum field theory, we have:

\[ \overline{O}_j = \frac{1}{Z_E} \int [dq] O_j e^{-S_E(q)/\hbar} \]  \hspace{0.5cm} (12.0.2)\]

This is the Feynman path integral formulation of quantum field theory. The constructions are reciprocal; the thermal fluctuations of the first one are the quantum fluctuations of the second one. The Euclidean-space representation (above) can be connected to the Lorentzian representation via a Wick rotation \( t \rightarrow it \).

The partition function for analytical facts (without temperature) can be formulated as a quantum field theory quite directly. We start with the analytical-fact representation:

\[ Z(D, W)_{\text{analytical-fact}} = \sum_{q \in Q} e^{-[Dx(q)-W_t(q)]} \]  \hspace{0.5cm} (12.0.3)\]

Then, we give units to \( D \) and \( W \) to map them to a physical quantity and we introduce a new constant \( (\hbar) \) to cancel out the units in the exponential. As the algorithmic quantity of \( D \) and \( W \) are now mapped to physical quantities, we will rename them to \( E \) and \( p \), respectively.

\[ Z_{\text{quantum-mechanical}} = \sum_{q \in Q} e^{-\frac{1}{\pi}[px(q)-E_t(q)]} \]  \hspace{0.5cm} (12.0.4)\]

\cite{McCoy} Barry M McCoy. The connection between statistical mechanics and quantum field theory. arXiv preprint hep-th/9403084, 1994
The units are:

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Type</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>Energy</td>
<td>J</td>
</tr>
<tr>
<td>$p$</td>
<td>Momentum</td>
<td>$J s/m$</td>
</tr>
</tbody>
</table>

Then, posing $S_E(q) = Et(q) - px(q)$

$$Z = \sum_{q \in Q} e^{-S_E(q)/\hbar}$$  \hspace{1cm} (12.0.7)

This is the discrete regime. In it, we obtain a formulation of lattice QFT within the formalism of the Feynman path integral. We can also pose the smoothness approximation $q :\rightarrow L$ (see macroscopic limit section 9). In this case, we get:

$$Z = \int_{Q} e^{-S_E(q)/\hbar} dq$$  \hspace{1cm} (12.0.8)

where the average value of each observable $O_j$ is given by:

$$\overline{O}_j = \frac{1}{Z} \int O_j e^{-S_E(q)/\hbar} dq$$  \hspace{1cm} (12.0.9)

Here, the action $S_E$ is an space-time event function dependent of both $x$ and $t$. In the Feynman path integral, $S_E$ connects to the Lagrangian as follows:

$$S[x, \dot{x}] = \int L[x(t), \dot{x}(t)] dt$$  \hspace{1cm} (12.0.10)

We can check that our derivation is reasonable by deriving $S_E$ by $dt$ and checking if it is a Lagrangian:

$$\frac{d}{dt} S_E = \frac{d}{dt} (Et(q) - px(q))$$  \hspace{1cm} (12.0.11)

$$= E - p\dot{x}(q)$$  \hspace{1cm} (12.0.12)

In this interpretation, quantum fluctuations replace the thermal fluctuations of statistical physics. The usual interpretation of the path integral applies to understanding the actual world; incompatible paths over the micro-states interfere destructively and compatible ones interfere constructively, etc. From this result, we infer a few hints as to how to unify gravitation with quantum field theory.

**Hints:**
• The previous results (special relativity, general relativity, dark energy, the arrow of time, etc) are imported into the path integral formulation here.

• As gravitation is derived as an entropic and emergent law, it may not make any sense to attempt to quantize it.

• As gravitation is produced from the second term of the Taylor expansion of $l(q)$, it may simply vanish at very small scales (instead of diverging to infinity).

• Gravitation is an entropic effect created by the macroscopic entropy associated with the degrees of freedom of a quantum field system.

13 Conclusion

The starting point of the auto-tauto-universal program is the removal of all possible philosophical vector of attacks towards itself. This is done by removing all statements which could carry logical doubt. Starting from first-order logic then stripping it of its deductive apparatus (axioms and rules of inference) we obtain a purely tautological system (e.i. an indubitable system). Second, when we list all of its tautologies we notice that the system is universal. Thus, it must be a complete (universal) and indubitable (tautological) description of the world. From this discovery, we adjust our understanding of the world to match its implications. Essentially, it means that all facts will now be analytical and that the actual world is an emergent behavior over the set of all analytical facts.

The theory of the emergence of properties from facts is provided by feasible mathematics. By applying feasible mathematics to the set of all tautological statements of Tauto-universal logic, we obtain an autological theory of physics.

$$\text{feasible-mathematics}(\text{tautological} \land \text{universal}) \implies \text{autological} \quad (13.0.1)$$

An autological physical theory has the same properties as reality. Thus, studying it using a formal meta-theory (such as feasible mathematics) should be equivalent to using mathematics to understand the universe. This, is indeed what we find. Explicitly, we consider the objective properties of facts (statement-length and proof-length) and avoid "poetic" properties (such as how interesting a fact might be to us). Doing so, the natural description of these facts is as an ensemble $Z$ of feasible mathematics.
We find that when Z describes such a system of analytical facts, the familiar laws of physics are recovered as emergent from the statistical ensemble; this also includes the notion of space and time. The world that is actual can be tentatively understood as an emergent average over the set of all possible facts of reality for some fixed resources associated with a purely statement-length and proof-length description for each fact (analytical description). This interpretation is directly mappable to a thermodynamic system from which we understand and derive the laws of physics from.

Understanding the world as a purely emergent behavior holds several philosophical advantages. The construction provides a possible mean to explain the origins of the laws of physics as per John Wheeler’s suggestion of law without law (or as order from disorder). Indeed, the obscure origin of the Dirac and Schrödinger equations is now clearly shown to be a result of statistical fluctuations applicable to x and t. Second, the laws of inertia, general relativity, and dark energy are simply the result of taking the Taylor expansion of an arbitrary space-encoding function. Third, as these laws are derived from the general equation of state of the system, the laws of physics do not need to be invoked as a ‘special case’. In the present construction, the laws of physics are a consequence of the mere fact that the world can be described fully and only using pure logic; hence, the ‘axiomatic-load’ of the construction is minimal.

The construction allows a possible explanation of the arrow of time. Indeed, moving into the future requires a negative power. A possible cause of negative power is closing future alternatives, which works towards reducing the entropy over time. To preserve the second law of thermodynamics, an entropy sink must be grown as time moves forward to offset said entropy reduction. Thus, the passage of time is heavily connected to the size of the entropy sink. The minimal growth rate requirements of this entropy sink are precisely the limits required to derive special relativity, general relativity, and dark energy. Therefore, we conclude that the entropy sink spawns the observable universe. The second law of thermodynamics understood as an increase in entropy over time is only half the truth. The second law is perceived in the entropy sink while the larger system, made to include future possibilities, has a constant entropy. In this system, future possibilities are consumed as time moves forward.

References


