A proof of the Riemann Hypothesis

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Abstract: This paper investigates the characteristics of the power series representation of the Riemann Xi function. A detailed investigation of the behaviour of the zeros of the real part of the power series and the behaviour of the curve, combined with a substitution of polar coordinates in the power series and in the definition of the critical strip (leading to a critical area), and the relationship with the zeros of the imaginary part of the power series leads to the conclusion that the Riemann Xi function only has real zeros.

Introduction

This paper is a preprint of a paper submitted to The IET Journal of Engineering. If accepted, the copy of record will be available at the IET Digital Library. This paper addresses one of the key unresolved questions arising from Riemann's original 1859 paper regarding the distribution of prime numbers ('Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse'[1] - translation in Edwards [2]) - the truth or otherwise of the Riemann Hypothesis ('One finds in fact about this many real roots within these bounds and it is very likely that all of the roots are real' - referring to the roots of the Riemann Xi function).

This paper starts from the power series representation of the Xi function ('This function is finite for all values of t and can be developed as a power series in tt which converges very rapidly')(Reimann)[1] and (Edwards)[3].

Section 1 defines the power series that is being investigated, examines the key points of the behaviour of the series coefficients and examines the shape of the real part curve when there is no imaginary component.

Section 2 deals with the effects of a change of coordinates to polar coordinates (both on the function and on the shape of the critical strip).
Section 3 investigates the paths of the zeros of both real and imaginary parts of the Xi function.

Section 4 develops the implications of the earlier investigations and the change of coordinates, leading to the conclusions (the proof of the Riemann Hypothesis) in Section 5.

1 Power Series

1.1 Original Equation

Riemann's original equation in his paper (Riemann)[1]:

ξ(t)=4$\int_1^\infty (d/dx(x^{3/2}\psi'(x)))x^{-1/4}cos(\frac{t}{2}logx)dx$

where $\psi(x) = \sum_{m=1}^{\infty} e^{-m^2\pi x}$

To avoid ξ-Ξ confusion, the equation from Edwards[3] is used:

ξ(s)=4$\int_1^\infty (d/dx(x^{3/2}\psi'(x)))x^{-1/4}cosh(\frac{1}{2}(s-\frac{1}{2})logx)dx$

This leads to (Edwards)[3]:

$\xi(s) = \sum_{n=0}^{\infty} a_{2n}(s-\frac{1}{2})^{2n}$

where a_{2n}=4$\int_1^\infty (d/dx(x^{3/2}\psi'(x)))x^{-1/4}\frac{logx^{2n}}{2^{2n}(2n)!})dx$

Now, if t=(a+bi), then (s-$\frac{1}{2}$) = it = (ai-b), and:

ξ(s)=$\sum_{n=0}^{\infty} a_{2n}(ai-b)^{2n}$

Due to the fact that all a_{2n} are positive (Edwards p41)[4], it immediately follows from the above that if a=0, there are no real zeros of the function and if b=0 then there are potentially many real zeros (depending on the actual values of a_{2n}).

It is important to note at this point that it has been proven that

ξ(s)=$\sum_{n=0}^{\infty} a_{2n}(ai-b)^{2n}$

(a polynomial in tt) has been proven to converge as the coefficients decrease rapidly; this result is necessary in the convergence of the product representation (Hadamard) [5].

1.2 Coefficients

In appendix A Table 1 shows the values for the first 50 coefficients (a_{2n}). Note that they are monotonically strictly decreasing and rapidly decreasing (necessary for rapid convergence - this will continue for all the coefficients). As |(ai-b)|increases, then the number of terms in the series expansion needed for convergence of the result increases. This is consistent with the values of the coefficients.

Figures 1 and 2 illustrate this.

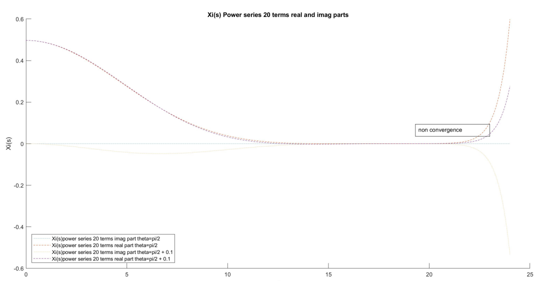

Fig. 1: $\xi(s)$ First 20 terms in power series.

1.3 Real Curve Shape

Looking at ξ(s)=$\sum_{n=0}^{\infty} a_{2n}(ai-b)^{2n}$ in more detail:

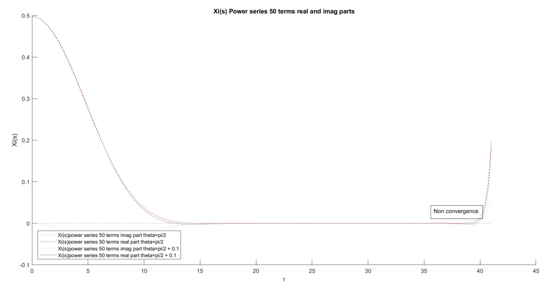

Fig. 2: $\xi(s)$ First 50 terms in power series. Note greater r before non convergence.

Firstly setting b=0 to give ξ(s)=$\sum_{n=0}^{\infty} a_{2n}(ai)^{2n}$

Secondly investigating some of the properties of the terms of the Riemann definition of ξ(s) (Edwards P16)[6] for $s = (\frac{1}{2} + ri)$; this is the equivalent of varying a and setting b=0 in the power series (we will see below in the polar coordinates section why r is used as the variable):

ξ(s) = $\Pi(\frac{s}{2})(s-1)\pi^{\frac{-s}{2}}\zeta(s)$

a) The $\Pi(\frac{s}{2})$ term, where Π is the factorial function. The magnitude of the factorial function for real numbers of increasing size increases rapidly.

However, the behaviour for complex numbers with a fixed real part and an imaginary part of increasing size is very different - the magnitude of the function decreases very rapidly with increasing imaginary number size.

In addition, it is oscillatory for both real and complex components.

This behaviour can be seen by investigating the product representation of the factorial function for the complex number s as known to Euler [8]:

$$\Pi(s) = \lim_{N\to\infty} \frac{N!}{(s+1)(s+2)...(s+N)}(N+1)^s$$

For s=(a+bi), where |a|<1, then, for any N, as b increases in magnitude both real and imaginary components of the $(N+1)^s$ term are oscillatory with a constant magnitude of oscillation, while the denominator increases rapidly in magnitude.
In the limit, this leads to an oscillating function of rapidly decreasing magnitude.

See Figure 3 and Figure 4 for illustrations.

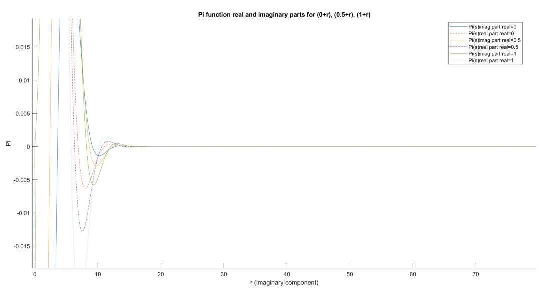

Fig. 3: $\Pi(\frac{1}{2} + ri)$, r<18

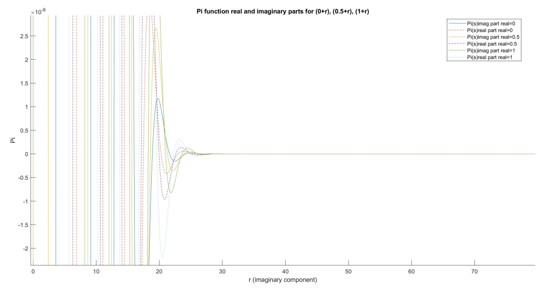

Fig. 4: $\Pi(\frac{1}{2} + ri)$, r<30 Note rapid decrease in magnitude

b) (s-1) The magnitude of this (non oscillatory) term increases slowly with the magnitude of s.

c) $\pi^{\frac{-s}{2}}$. The real and imaginary components of this term oscillate with a fixed magnitude of oscillation.

d) $\zeta(s)$. The real and imaginary components of this term oscillate with a very slowly increasing magnitude (see Figure 5 for illustration).

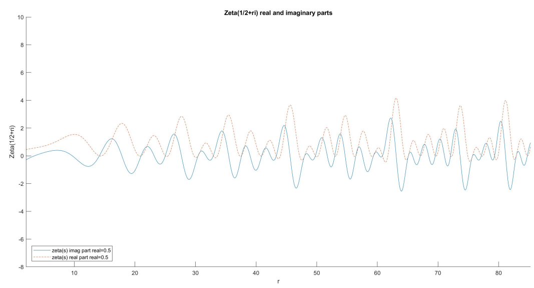

Fig. 5: $\zeta(\frac{1}{2} + ri)$

Looking at the product of the individual terms, this means that the curve is oscillatory with decreasing magnitude of oscillation as r increases.

In addition, since we know that for b=0 then there is no imaginary element of the function, the function tends to zero in the limit (as expected).

This reduction in magnitude can be seen in the curves with the actual a_{2n} below. Figure 6 and Figure 7 show 2 sections of ξ(s) with b=0.

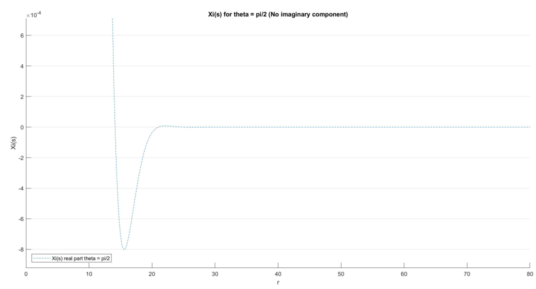

Fig. 6: $\xi(s)$ No imaginary component, r<25

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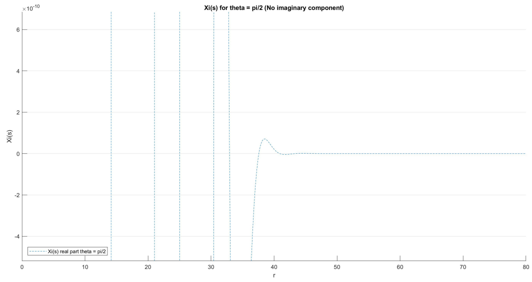

Fig. 7: $\xi(s)$ No imaginary component, r$<$ 40 Note rapid decrease in magnitude

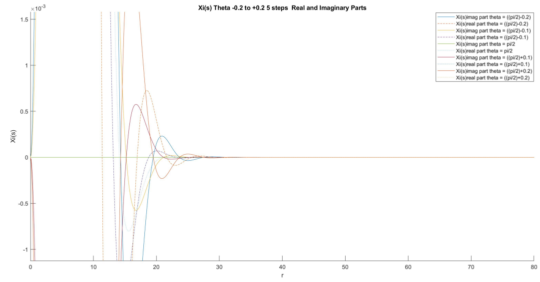

Fig. 9: $\xi(s)$ with varying θ and larger r.

2 Polar Coordinates

2.1 Substitution

Using de Moivre's Theorem (Heading p115 [7]) $(ai-b)^{2n}$ can be rewritten as $r^{2n}(cos\theta + isin\theta)^{2n}$ and expanded as $r^{2n}cos2n\theta + ir^{2n}sin2n\theta$, taking r to range from 0 to ∞ and θ to range from $\frac{\pi}{2}$ to π for the most relevant quadrant. The structure of the expression (see below) means that the π to 2π half is a reflection of the 0 to π half. The behaviour of the expression is markedly different for $r \leq 1$. From this point, I will consider only $r > 1$ (since we know that there are no relevant zeros for $r \leq 1$).

2.2 Complete Expression

The above results in: ξ(s)=$\sum_{n=0}^{\infty} a_{2n}r^{2n}(cos\theta + isin\theta)^{2n}$

=$a_0 + a_2r^2cos2\theta + a_4r^4cos4\theta + a_6r^6cos6\theta + a_8r^8cos8\theta$...

$+i(a_2r^2sin2\theta + a_4r^4sin4\theta + a_6r^6sin6\theta + a_8r^8sin8\theta$...)

Both real and imaginary parts of the expression are single valued for each r, θ combination. Both real and imaginary parts have a period of π. For $\theta = \frac{\pi}{2}$ the expression is equal to ξ(s)=$\sum_{n=0}^{\infty} a_{2n}(ai)^{2n}$. For the actual values of a_{2n} the expression does have multiple real zeros.

Figure 8 and Figure 9 show the variation of the function with variation in θ.

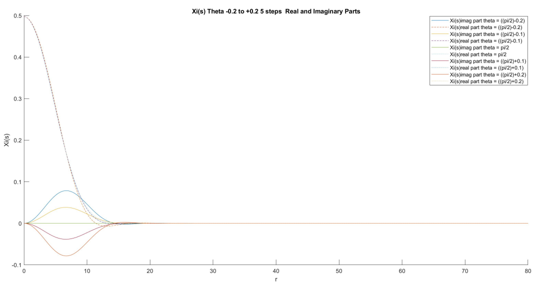

Fig. 8: $\xi(s)$ with varying θ

2.3 Critical Strip to Critical Area

The polar coordinate substitution is very interesting here. The critical strip (between $\frac{1}{2} +/- \frac{1}{2}$) changes to $rcos\theta$=$+/- \frac{1}{2}$.

This means that as r increases, the width of the critical area reduces as $cos\theta$ reduces and so θ reduces, leading to useful limits that can be exploited.

3 Paths of Zeros

3.1 Real Part Zeros

Using $\theta = (\frac{\pi}{2} + \epsilon)$:

cos2θ = $cos(2(\frac{\pi}{2} + \epsilon))$ = cosπcos2ϵ - sinπsin2ϵ = - cos2ϵ and $cos(2(\frac{\pi}{2} - \epsilon))$ = cosπcos2ϵ + sinπsin2ϵ = - cos2ϵ

Similar expressions can be generated for cos2nθ for all values of n with similar results (except alternating signs).

This means that the path of the function $a_0 + a_2r^2cos2\theta + a_4r^4cos4\theta + a_6r^6cos6\theta + a_8r^8cos8\theta$...=0 is reflected across $\theta = \frac{\pi}{2}$ for varying r.

This equation describes a family of curves.

No function in the family intersects any other function in the family (the function is single valued for each r,θ combination).

For the actual values of a_{2n}, it appears that each function in the family will pass through $\theta = \frac{\pi}{2}$.

When $\theta \neq \frac{\pi}{2}$, then we know that the function extends from the zeros on the $\theta = \frac{\pi}{2}$ line.

If a function does not pass through $\theta = \frac{\pi}{2}$, then it will have 2 reflected branches on either side of $\theta = \frac{\pi}{2}$ (non-intersecting with any other of the family of functions).

3.2 Imaginary Part Zeros

Using $\theta = (\frac{\pi}{2} + \epsilon)$:

sin2θ = $sin(2(\frac{\pi}{2} + \epsilon))$ = sinπcos2ϵ + cosπsin2ϵ = - sin2ϵ and $sin(2(\frac{\pi}{2} - \epsilon))$ = sinπcos2ϵ - cosπsin2ϵ = + sin2ϵ

Similar expressions can be generated for sin2nθ for all values of n with similar results (except alternating signs).

This means that the path of the function $a_2r^2sin2\theta + a_4r^4sin4\theta + a_6r^6sin6\theta + a_8r^8sin8\theta$...=0 is reflected across $\theta = \frac{\pi}{2}$ for varying r.

This equation describes a family of curves.

No function in the family intersects any other function in the family (the function is single valued for each r,θ combination).

For the actual values of a_{2n} it appears that each function in the family will pass through $\theta = \frac{\pi}{2}$.

When $\theta = \frac{\pi}{2}$, then we know that the function is identically zero.

If a function does not pass through $\theta = \frac{\pi}{2}$, then it will have 2 reflected branches on either side of $\theta = \frac{\pi}{2}$ (non-intersecting with any other of the family of functions).

The real part and imaginary part have the same number of pairs of zeros (the imaginary part has an additional zero at r=0).

See Figure 10 for an illustration of the paths of real and imaginary zeros for the actual values of a_{2n} in the same graph.
Note that the paths of the zeros do not intersect in these samples (as shown above), except where the imaginary function is identically zero.

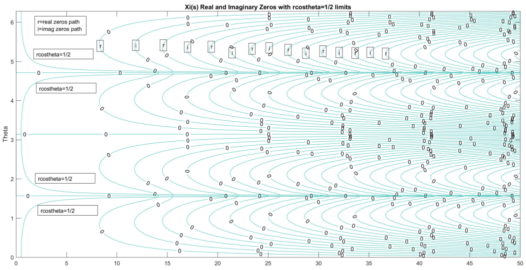

Fig. 10: $\xi(s)$ real and imaginary part zeros.

3.2 (Real + Imaginary) and (Real - Imaginary) Part Zeros

The complete function will be zero when both real and imaginary expressions are equal to each other and both zero. Thus we are looking for common zeros of these two expressions:

$a_0 + a_2r^2cos2\theta + a_4r^4cos4\theta + a_6r^6cos6\theta + a_8r^8cos8\theta$...=0 (1)
and $a_2r^2sin2\theta + a_4r^4sin4\theta + a_6r^6sin6\theta + a_8r^8sin8\theta$...=0 (2)

It is useful to investigate the combined expressions ((1)+(2)) and ((1)-(2)). If and only if they are simultaneously zero then the complete function is zero.

Starting with ((1)+(2)):
$a_0 + a_2r^2cos2\theta + a_4r^4cos4\theta + a_6r^6cos6\theta + a_8r^8cos8\theta$...
+ $a_2r^2sin2\theta + a_4r^4sin4\theta + a_6r^6sin6\theta + a_8r^8sin8\theta$...=0

Differentiating with respect to r:

$a_2(2rcos2\theta + r^2(-2sin2\theta)d\theta/dr)$
$+ a_4(4r^3cos4\theta + r^4(-4sin4\theta)d\theta/dr)$
$+ a_6(6r^5cos6\theta + r^6(-6sin6\theta)d\theta/dr) + ...$
$+ a_2(2rsin2\theta + r^2(2cos\theta 2)d\theta/dr)$
$+ a_4(4r^3sin4\theta + r^4(4cos4\theta)d\theta/dr)$
$+ a^6(6r^5sin6\theta + r^6(6cos6\theta)d\theta/dr) + ... = 0$

$d\theta/dr = (a_2(2rcos2\theta + 2rsin2\theta)$
$+ a_4(4r^3cos4\theta + 4r^3sin4\theta)$
$+ a_6(6r^5cos6\theta + 6r^5sin6\theta) + ...)/(a_4(r^22sin2\theta - r^22cos2\theta)$
$+ a_4(r^44sin4\theta - r^44cos4\theta)$
$+ a_6(r^66sin6\theta - r^66cos6\theta) + ...)$

=$(1/r)(a_2(2r^2cos2\theta + 2r^2sin2\theta) + a_4(4r^4cos4\theta + 4r^4sin4\theta)$
$+ a_6(6r^6cos6\theta + 6r^6sin6\theta) + ...)/(a_2(r^22sin2\theta - r^22cos2\theta)$
$+ a_4(r^44sin4\theta - r^44cos4\theta) + a_6(r^66sin6\theta - r^66cos6\theta) + ...) -$
(3)

Similarly starting with ((1)-(2)):

$a_0 + a_2r^2cos2\theta + a_4r^4cos4\theta + a_6r^6cos6\theta + a_8r^8cos8\theta$...-
($a_2r^2sin2\theta + a_4r^4sin4\theta + a_6r^6sin6\theta + a_8r^8sin8\theta$...)=0

Differentiating with respect to r:

$a_2(2rcos2\theta + r^2(-2sin2\theta)d\theta/dr)$
$+ a_4(4r^3cos4\theta + r^4(-4sin4\theta)d\theta/dr)$
$+ a_6(6r^5cos6\theta + r^6(-6sin6\theta)d\theta/dr) + ...$
$- (a_2(2rsin2\theta + r^2(2cos\theta 2)d\theta/dr)$
$+ a_4(4r^3sin4\theta + r^4(4cos4\theta)d\theta/dr)$
$+ a^6(6r^5sin6\theta + r^6(6cos6\theta)d\theta/dr) + ...) = 0$

$d\theta/dr = (a_2(2rcos2\theta - 2rsin2\theta) + a_4(4r^3cos4\theta - 4r^3sin4\theta)$
$+ a_6(6r^5cos6\theta - 6r^5sin6\theta) + ...)/(a_4(r^22sin2\theta + r^22cos2\theta)$
$+ a_4(r^44sin4\theta + r^44cos4\theta) + a_6(r^66sin6\theta + r^66cos6\theta) + ...)$

=$(1/r)(a_2(2r^2cos2\theta - 2r^2sin2\theta) + a_4(4r^4cos4\theta - 4r^4sin4\theta)$
$+ a_6(6r^6cos6\theta - 6r^6sin6\theta) + ...)/(a_2(r^22sin2\theta + r^22cos2\theta) +$
$a_4(r^44sin4\theta + r^44cos4\theta)$
$+ a_6(r^66sin6\theta + r^66cos6\theta) + ...) - (4)$

Reusing: cos2$\theta = cos(2(\frac{\pi}{2} + \epsilon))$ = cosπcos2ϵ - sinπsin2ϵ = - cos2ϵ
and $cos(2(\frac{\pi}{2} - \epsilon))$ = cosπcos2ϵ + sinπsin2ϵ = - cos2ϵ
sin2$\theta = sin(2(\frac{\pi}{2} + \epsilon))$ = sinπcos2ϵ + cosπsin2ϵ = - sin2ϵ and $sin(2(\frac{\pi}{2} - \epsilon))$ = sinπcos2ϵ - cosπsin2ϵ = + sin2ϵ

Similar expressions can be generated for sin2nθ and cos2nθ for all values of n with similar results (except alternating signs).

The derivative expression (3) for ϵ is the negative of (4) for - ϵ and that for (4) for ϵ is the negative of (3) for - ϵ - that is, the derivative expressions (3)and (4) are reflected through $\theta = \frac{\pi}{2}$ and if they cross $\theta = \frac{\pi}{2}$ then there will be a coincident pair of real zeros on $\theta = \frac{\pi}{2}$.

If they do not cross $\theta = \frac{\pi}{2}$, then there will be a reflected pair of imaginary zeros tracing reflected paths.

In addition, as the functions are single valued for each r, θ combination then there are no intersections with any other of the same family of functions.

This means that there will be no additional complete function zeros generated - each pair of imaginary part zeros will only coincide with one pair of real part zeros.

One can also see from the above expressions that as r increases, the derivative of each function tends to 0 (1/r tends to 0 as r becomes large) (i.e one would expect to see an increasing number of almost parallel, almost horizontal functions as r increases).

See Figure 11 for for illustrations of the paths of the ((1)+(2)) and ((1)-(2)) zeros for the actual values of a_{2n}.

4 Development and Implications

4.1 Assumed Function Φ

Assuming a function Φ=$\sum_{n=0}^{\infty} d_{2n}(ri)^{2n}$ with only real zeros and $d_0 = a_0$

This function will also be oscillatory with rapidly decreasing magnitude of oscillation.

Expanding: Φ=$\sum_{n=0}^{\infty} d_{2n}(ri)^{2n}$=$d_0 - d_2r^2 + d_4r^4 - d_6r^6 + d_8r^8$...

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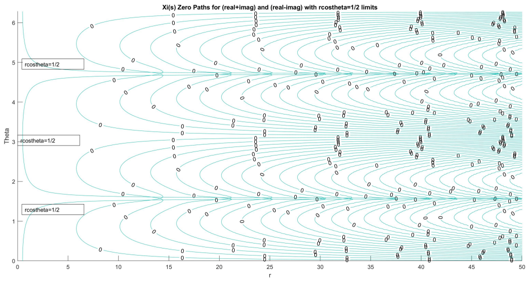

Fig. 11: $\xi(s)$ (real + imaginary) and (real - imaginary) zero paths.

Investigating $\Phi = d_0 + d_2r^2cos2\theta + d_4r^4cos4\theta + d_6r^6cos6\theta + d_8r^8cos8\theta...$, which has only real zeros at $\theta = \frac{\pi}{2}$

For this section, keep in mind the critical area, with a boundary defined by $rcos\theta$=+/$-\frac{1}{2}$.

Looking at the behaviour of the zeros of the function as we vary the coefficients of the expression:

Firstly d_0:

As a real constant added to the sum of the variable components of the real expression, this has the effect of moving the sum of the variable components vertically.

When the sum of the variable components is negative and varies in magnitude about d_0 for varying r, then there are real zeros.

If not, then there will not be real zeros.

In section 1.3 it was shown that the curve is oscillatory with ever decreasing magnitude as r increases.

This means that if d_0 varies by any amount δ (however small), then there will be a value of r above which there will be no more real zeros (ie only imaginary zeros - once δ is larger than the magnitude of the function).

Also, considering the $rcos\theta = \frac{1}{2}$ limit, then at some point (for greater r), then the distance of the imaginary zeros from the $\theta = \frac{\pi}{2}$ line will be greater than that limit (ie zeros outside the critical area) and will continue to be greater than that limit.

See Figures 12 and 13, for illustrations if we vary a_0.

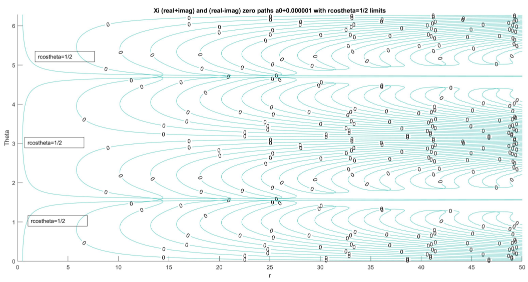

Fig. 12: $\xi(s)$ + 1E-6 (real + imaginary) and (real - imaginary) zeros.

Secondly d_{2n}:

If we vary a single coefficient d_{2n} by any amount δ (however small), then we are adding (or subtracting) $\delta r^{2n}cos2n\theta$ to (or from) the

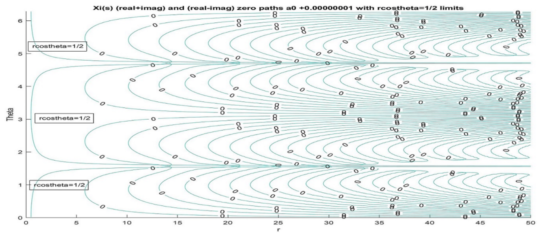

Fig. 13: $\xi(s)$ + 1E-8 (real + imaginary) and (real - imaginary) zeros.

expression.

When $\theta = \frac{\pi}{2}$, leading to adding (or subtracting) δr^{2n}, we can see that this has the same effect as varying d_0 , but with a continually increasing deviation as r increases - the key point being that it leads to zeros outside the critical area.

This result holds if we vary multiple d_{2n} - there will be a value of r above which there will be an increasing deviation of the imaginary zeros from the $\theta = \frac{\pi}{2}$ line, at some point exceeding the $rcos\theta = \frac{1}{2}$ limit and thus zeros outside the critical area.

Varying θ has the same effect as varying all the d_{2n} (except d_0), leading to zeros outside the critical area.

See Figures 14 and 15 for illustrations if we vary a_{2n}.

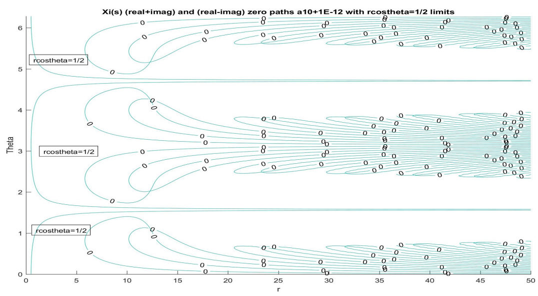

Fig. 14: $\xi(s)$ + 1E-12(r^{10}) (real + imaginary) and (real - imaginary) zeros.

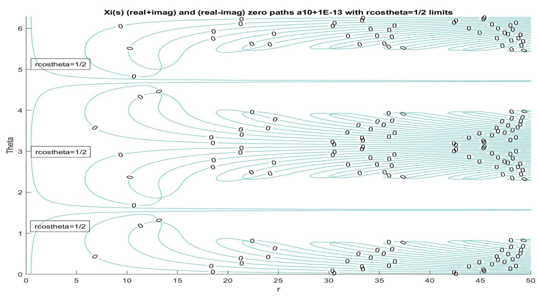

Fig. 15: $\xi(s)$ + 1E-13(r^{10}) (real + imaginary) and (real - imaginary) zeros.

From these results for $\theta = \frac{\pi}{2}$ (equivalent to s= ($\frac{1}{2}$+it)), combined with the facts that 1) a_0 = d_0, 2) the two expressions have the same structure and 3) ξ(s) has many real zeros and no zeros outside the critical area, we can conclude that Φ and ξ(s) have the same coefficients and so are the same function. For clarity, Φ and the real part of ξ(s) have all real zeros.

5 Conclusions

Known previously - there can be only zeros of the complete function where zeros of the real part and zeros of the imaginary part coincide.

Known previously - ξ(s) does not have zeros outside the critical area/critical strip.

In Section 4 it was shown that the real part of ξ(s) has all real zeros and that varying θ (i.e adding an imaginary component to the power series in polar coordinates; equivalent to having a real part of s $\neq \frac{1}{2}$ using the original coordinates) results in zeros outside the critical area/critical strip.

In Section 3 it was shown that there are no additional zeros of the complete function due to the coincidence of imaginary zeros from the real and imaginary parts of ξ(s).

Combining these conclusions, all of the roots of the Riemann Xi function (where s=($\frac{1}{2} + ti$)) are real - QED.

1 References

1 Riemann, B.: ‘Gesammelte Werke.’(Teubner, Leipzig, 1892; reprinted by Dover Books, New York, 1953.) p145.
2 Edwards, H.M.: ‘Riemann’s Zeta Function’(Dover Publications, 2001) p299
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All figures created in MATLAB.

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Appendix A - Coefficients

Table 1 The First 50 a_{2n} Coefficients of ξ(s) calculated numerically with MATLAB.

Coefficients	Values
a0	0.497120778
a2	0.011485972
a4	0.000123452
a6	8.32355E-07
a8	3.99223E-09
a10	1.4616E-11
a12	4.27454E-14
a14	1.03096E-16
a16	2.09977E-19
a18	3.67814E-22
a20	5.62286E-25
a22	7.59176E-28
a24	9.14334E-31
a26	9.90611E-34
a28	9.72469E-37
a30	8.7046E-40
a32	7.14349E-43
a34	5.40097E-46
a36	3.77845E-49
a38	2.45541E-52
a40	1.48738E-55
a42	8.42529E-59
a44	4.4758E-62
a46	2.23578E-65
a48	1.05272E-68
a50	4.68274E-72
a52	1.9719E-75
a54	7.87603E-79
a56	2.9891E-82
a58	1.07972E-85
a60	3.71788E-89
a62	1.22216E-92
a64	3.84066E-96
a66	1.1553E-99
a68	3.3305E-103
a70	9.2123E-107
a72	2.4475E-110
a74	6.2524E-114
a76	1.5372E-117
a78	3.641E-121
a80	8.3152E-125
a82	1.8325E-128
a84	3.9005E-132
a86	8.0242E-136
a88	1.5967E-139
a90	3.0751E-143
a92	5.7365E-147
a94	1.0371E-150
a96	1.8185E-154
a98	3.0939E-158